Recall :- Conditional density

(X,Y) are two continuous random variables

- having joint density f.

$f_Y(\cdot)$ be the marginal density of Y

$b \in \mathbb{R}$, $f_Y(b) > 0$ & is continuous at b

- conditional density of X given $Y=b$ is

$$f_{X|Y=b}(x) = \frac{f(x,b)}{f_Y(b)}$$ [Derivation]

$f_X(\cdot)$ be the marginal density of X

$a \in \mathbb{R}$, $f_X(a) > 0$ & is continuous at a

- conditional density of Y given $X=a$ is

$$f_{Y|X=a}(y) = \frac{f(a,y)}{f_X(a)}$$

Example 5.4.12: Let (X,Y) have joint p.d.f f given by

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)} \qquad x,y \in \mathbb{R}$$

- Marginal of X :-

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)}\,dy$$

$$f_X(x) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(xy + y^2)}{2}}\,dy$$

observe: $y^2 + xy = \left(y - \frac{x}{2}\right)^2 - \frac{x^2}{4}$

$$f_X(x) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2}{2} + \frac{x^2}{8}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y - \frac{x}{2}\right)^2}\,dy}$$

Ex: (Normal p.d.f) $\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y - \frac{x}{2}\right)^2}\,dy = \sqrt{2\pi}$

$$f_X(x) = \frac{\sqrt{3}}{4\pi} e^{-\frac{3x^2}{8}} \sqrt{2\pi} \qquad x \in \mathbb{R}$$

$$= \frac{\sqrt{3}}{\sqrt{4}} \frac{1}{\sqrt{2\pi}} e^{-3x^2/8}, \quad x \in \mathbb{R}$$

$$\Rightarrow \quad X \sim \text{Normal}\left(0, \tfrac{4}{3}\right) \quad \text{(Ex.)}$$

Observe: $f(x,y) = f(y,x) \quad \forall \; x,y \in \mathbb{R}$

$$\Rightarrow \quad f_Y(y) = \frac{\sqrt{3}}{\sqrt{4}} \frac{e^{-3y^2/8}}{\sqrt{2\pi}}, \quad y \in \mathbb{R}$$

$$Y \sim \text{Normal}\,(0, 4/3)$$

Are X and Y independent?

$$- \; f_X(x)\, f_Y(y) = \frac{\sqrt{3}}{\sqrt{4}} \frac{e^{-3x^2/8}}{\sqrt{2\pi}} \cdot \frac{\sqrt{3}}{\sqrt{4}} \frac{e^{-3y^2/8}}{\sqrt{2\pi}}$$

$$= \frac{3}{8\pi} e^{-3x^2/8 - 3y^2/8} \qquad x,y \in \mathbb{R}$$

$$- \; f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)}$$

clearly $f_X(x)\, f_Y(y) \neq f(x,y) \Rightarrow$ Not independent $\square$

Conditional density of Y given $X=x$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \qquad y \in \mathbb{R}$$

$$= \frac{\frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)}}{\frac{\sqrt{3}}{\sqrt{4}} \frac{1}{\sqrt{2\pi}} e^{-3x^2/8}}, \quad y \in \mathbb{R}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2 - xy + \frac{x^2}{4}\right)}, \quad y \in \mathbb{R}$$

$$f_{Y|X=x}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y - \frac{x}{2}\right)^2} \quad y \in \mathbb{R}$$

$$\Rightarrow Y|_{X=x} \sim \text{Normal}\left(\frac{x}{2}, 1\right)$$

Aside

$\rightarrow (X,Y) \sim$ Bivariate normal — we might discuss later.

5.5.1. Distributions of Sums of Independent Random Variables

X and Y are two independent random variables.

$f_X(\cdot)$ be the marginal density of X

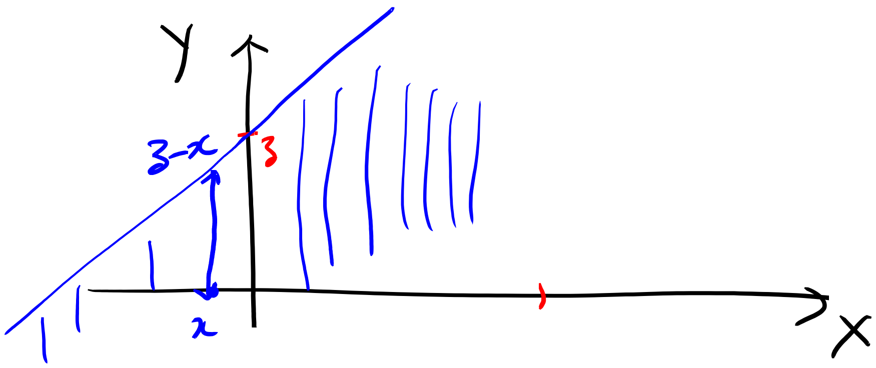
$f_Y(\cdot)$ be the marginal density of Y

$Z = X + Y$ Q:- What is p.d.f of Z?

Step1 :- Find distribution function of Z.

$$F(z) = \mathbb{P}(Z \leq z)$$

$$= \mathbb{P}(X+Y \leq z)$$



$$\{y+x \leq z\}$$

$$F_Z(z) = \int_{-\infty}^{-\infty} \left[\int_{-\infty}^{z-x} f(x,y)\, dy\right] dx$$

X, Y are independent $\Rightarrow$ $f(x,y) = f_X(y)\, f_Y(y)$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x} f_X(x)\, f_Y(y)\, dy\right] dx$$

$$F_Z(z) \overset{Ex}{=} \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} f_X(x) f_Y(u-x)\, dx\right] du$$

Step2 :

$$\Rightarrow \quad f_Z(z) = F_Z'(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

Theorem X, Y are independent random variables with p.d.f $f_X(\cdot)$ and $f_Y(\cdot)$

$Z = X+Y$, has a p.d.f given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$\equiv$ p.d.f of z is the convolution of $f_X(\cdot)$ and $f_Y(\cdot)$

p.d.f of Sum of X and Y

Example 5.5.3: let $\lambda > 0$ $X \sim$ Exponential (λ) - independent -
$Y \sim$ Exponential (λ)

Let $Z = X+Y$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y>0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) \underset{\text{Theorem}}{=} \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\lambda e^{-\lambda x},\ \underline{\underline{x>0}}}\ f_Y(z-x)\, dx$$

$f_Y(z-x) \to \lambda e^{-\lambda(z-x)}$

$z-x>0$

$\underline{\underline{x<z}}$

$$= \int_0^\infty \lambda e^{-\lambda x} f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx$$

$$f_Z(z) = \lambda^2 e^{-\lambda z} \int_0^z dx = \lambda^2 z e^{-\lambda z}, \quad z \geq 0$$

Check:- $f_Z(z) = 0$ otherwise

$\Rightarrow$ $Z \neq$ Exponential distributed

Definition 5.5.5: $X \sim \text{Gamma}(n, \lambda)$ $\lambda > 0$, $n \in \mathbb{N}$

if its p.d.f is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, & x \geq 0. \\ 0, & \text{otherwise} \end{cases}$$

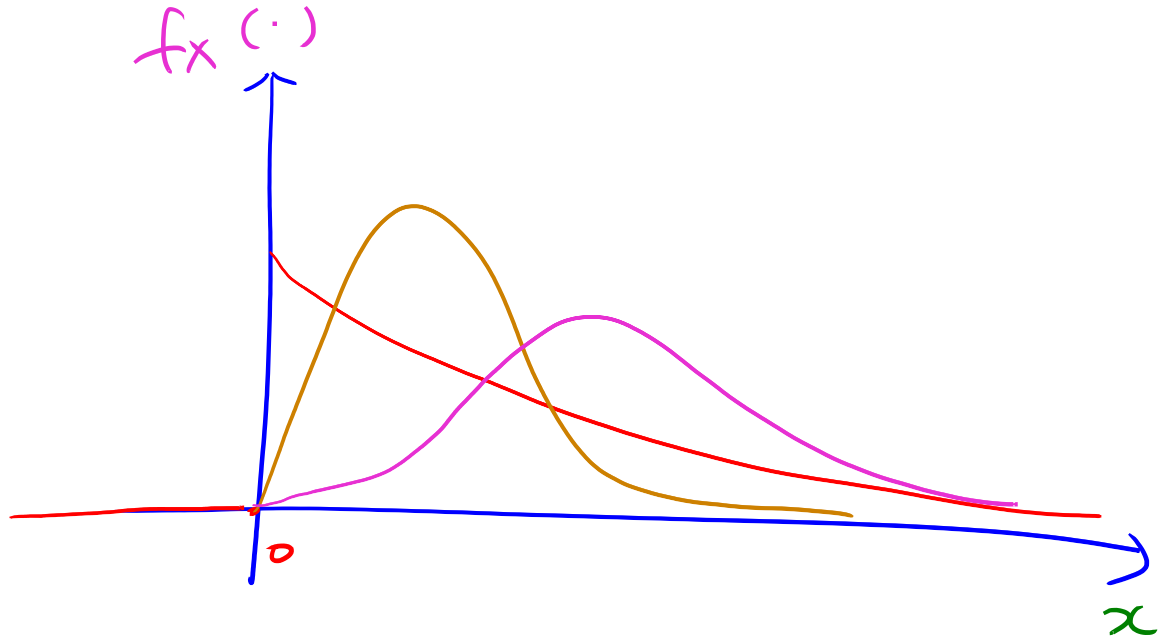
n - shape parameter, λ - rate parameter

Remarks:-

- $n=1$ $X \sim \text{Gamma}(1, \lambda) \equiv \text{Exponential}(\lambda)$
- $n=2$ $Z \sim \text{Gamma}(2, \lambda) = X + Y$ independent, $X \sim \text{Exp}(\lambda)$ $Y \sim \text{Exp}(\lambda)$

Ex: $X_1, X_2, \ldots X_n$ independent Exponential(λ)

$$Z_n = \sum_{i=1}^n X_i \equiv Z_n \sim \text{Gamma}(n, \lambda)$$



$X \sim \text{Gamma}(n, \lambda)$

$n = 1$

$n = 2$

$\frac{\lambda x^3 e^{-\lambda x}}{3!}$: $n = 4$

X, Y are two independent random variables with marginal densities $f_X : \mathbb{R} \to \mathbb{R}$ and $f_Y : \mathbb{R} \to \mathbb{R}$

$$Z = \frac{X}{Y}$$

Q: what is pdf of Z?

Step 1: $$F_Z(z) = \mathbb{P}(Z \le z)$$

$$= \mathbb{P}\left(\frac{X}{Y} \le z\right)$$

$$\cancel{= \mathbb{P}(X \le Yz) \; X}$$

$$= \int\int_{\{(x,y):\ \frac{x}{y} \le z,\ y \ne 0\}} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \iint_{\{(x,y):\ x \le yz,\ y > 0\}} f_X(x)\, f_Y(y)\, dx\, dy + \iint_{\{(x,y):\ x \ge yz,\ y < 0\}} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_0^{\infty} \int_{-\infty}^{yz} f_X(x)\, f_Y(y)\, dx\, dy + \int_{-\infty}^{0} \int_{yz}^{\infty} f_X(x)\, f_Y(y)\, dx\, dy$$

Step 2: $f_Z(z) = F_Z'(z)$

Calculus Ex:- $f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(zy)\, f_Y(y)\, dy$

Example 5.5.9 $X \sim \text{Normal}(0,1)$ independent
$Y \sim \text{Normal}(0,1)$

$$Z = \frac{X}{Y}$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(zy)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} |y|\, \frac{e^{-\frac{z^2 y^2}{2}}}{\sqrt{2\pi}}\, \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{|y|}{2\pi} e^{-\left(\frac{z^2+1}{2}\right) y^2} dy$$

$$= \frac{1}{\pi} \int_{0}^{\infty} y \, e^{-\left(\frac{z^2+1}{2}\right) y^2} dy$$

Change of variable
$u = \left(\frac{z^2+1}{2}\right) y^2$

$$= \frac{1}{\pi(1+z^2)} \int_{0}^{\infty} e^{-u} du$$

$$f_Z(z) = \frac{1}{\pi(1+z^2)} \qquad -\infty < z < \infty$$

$Z \sim$ Cauchy(1)