Due: Tuesday December 1st 2020

Problem to be turned in: 1(b), 3, 4

1. Let (X, Y) be random variables whose probability density function is given by $f : \mathbb{R}^2 \to \mathbb{R}$. Find the probability density function of X and probability density function of X in each of the following cases:-
   (a) $f(x, y) = (x + y)$ if $0 \le x \le 1, 0 \le y \le 1$ and 0 otherwise
   (b) $f(x, y) = 2(x + y)$ if $0 \le x \le y \le 1$ and 0 otherwise
   (c) $f(x, y) = 6x^2y$ if $0 \le x \le 1, 0 \le y \le 1$ and 0 otherwise
   (d) $f(x, y) = 15x^2y$ if $0 \le x \le y \le 1$ and 0 otherwise

2. Suppose $g : \mathbb{R} \to \mathbb{R}$ be continuous and a probability density function, such that $g(x) = 0$ when $x \notin [0, 1]$. Let $D \subset \mathbb{R}^2$ be given by

$$D = \{(x, y) : x \in \mathbb{R} \text{ and } 0 \le y \le g(x)\}$$

   Let (X, Y) be uniformly distributed on D. Find the probability density function of X.

3. Let $c > 0$. Suppose that X and Y are random variables with joint probability density

$$f(x, y) = \begin{cases} c(xy + 1) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c.
   (b) Compute the marginal densities $f_X(x)$ and $f_Y(y)$ and the conditional density $f_{X|Y}(x|y)$

4. Suppose X is a uniform random variable in the interval $(0, 1)$ and Y is an independent exponential(2) random variable. Find the distribution of $Z = X + Y$.

5. Sunita makes cuts at two points selected at random on a piece of lumber of length L. Find the distribution of M, the length of the middle piece. What is the expected value of the length of the middle piece?

6. Let

$$f(x, y) = \begin{cases} \eta(x - y)^\gamma & \text{if } 0 \le x < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) For what values of γ can η be chosen so that f be a joint probability density function of X, Y.
   (b) In cases as in (a), what are the values of η ?
   (c) In such cases as in (a) and (b)
      i. Find the marginal densities of X and Y.
      ii. Find the distribution of $X + Y$.

7. Let $D = \{(x, y) : x^2 \le y \le x\}$. A point (X, Y) is chosen uniformly from D. Find the joint probability density function of X and Y.