Recall :

Set up So far

S - sample countable set
$\mathcal{F}$ - events $\equiv \mathcal{P}(S)$ any subset of S
$\mathbb{P} : \mathcal{F} \to [0,1]$, $\mathbb{P}(S) = 1$ & $\{E_i\}_{i=1}^{\infty}$ $E_k \cap E_j = \phi$ then
$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

$X : S \to \mathbb{R}$ - random variables
$t \in Range(X)$ $f_X(t) := \mathbb{P}(X = t)$ } distribution of X.

S - not Countable. $S = (0,1)$ e.g.

Q: choose a number in $(0,1)$ in a "uniform" manner?

Discrete / Countable setting : Uniform $\{1, \dots, n\}$ - every outcome in $\{1, \dots, n\}$ was equally likely

- Take the same approach. $\mathbb{P}(\{x\}) = p$ (say) $0 < p < 1$

- $x \in (0,1)$ require

Probability of a single outcome should be zero

$E = \{ \frac{1}{2}, \frac{1}{3}, \dots \}$

$$\Rightarrow \mathbb{P}(E) = \mathbb{P}\left(\bigcup_{k=2}^{\infty} \{1/k\}\right)$$
$$= \sum_{k=2}^{\infty} \mathbb{P}(\{1/k\})$$
$$= p + p + p + \dots$$

diverge unless $p = 0$

$\boxed{\text{forces } p = 0}$

• Require: $\mathbb{P}(S) = 1$

Naive: $\mathbb{P}(S) = \mathbb{P}(\bigcup_{s \in S} \{s\})$

not part of axioms as S - uncountable

$$= \sum_{s \in S} \mathbb{P}(\{s\}) = 0 + 0 + \cdots$$

$$= 0$$

not a contradiction

Interpret "uniform" as $E = [\frac{1}{4}, \frac{3}{4}] \equiv$ "$\frac{1}{2}$ length of $(0,1)$"

So we may say want: $\mathbb{P}(E) = \frac{1}{2}$.

Declare: $\mathbb{P}(A) =$ "length of A in $(0,1)$". for any $A \subseteq S$?

Q: What is length of A

- A: [Measure Theory that you have done]

- Complicated to do outside A - being interval or countable disjoint union of intervals.

- There are $A \subseteq S$ for which length of A cannot be defined.

$\Rightarrow$ New Definition for Events. i.e. to take a collection of subsets of S for which we can define "length" $\longrightarrow$ Probability $\longrightarrow$ "uniform".

Definition 5.1.1. : If S - sample space then σ-field $\mathcal{F}$ is a collection of subsets of S such that

(i) $S \in \mathcal{F}$

(ii) $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

(iii) $\{E_k\}_{k=1}^{\infty}$ - $E_k \in \mathcal{F}$ then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$.

An event will be any of of σ-field $\mathcal{F}$.

- Borel σ-field: Smallest σ-field in $\mathbb{R}$ that contains all intervals. [- length for any element in the Borel σ-field] (one can define)

Definition 5.1.2. let S - be a sample space. let $\mathcal{F}$ be a σ-field of S. A "probability" is $\mathbb{P} : \mathcal{F} \to [0,1]$ such that

(i) $P(S) = 1$

(ii) $E_1, E_2 \ldots$ are a countable collection of disjoint events in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

The triplet $(S, \mathcal{F}, \mathbb{P})$:- Probability space

- Existence of $\mathbb{P}$; of $\mathcal{F}$ in $\mathbb{R}$

Event A in $S = \mathbb{R}$

$A \equiv$ finite / Countable unions of intervals

5.1.1 Probability densities on $\mathbb{R}$

Primary method — to define Probability on $\mathbb{R}$ is via "density function".

Example 5.1.3. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For any event A, $\boxed{\mathbb{P}(A) := \int_A f(x)\,dx}$

- $A = [a,b] \subseteq [0,1]$, $\mathbb{P}(A) = \int_a^b 1\,dx = b-a$

- $\boxed{\text{For disjoint union intervals}}$ $A = [\frac{1}{5}, \frac{2}{5}] \cup [\frac{3}{5}, \frac{4}{5}]$

$$\mathbb{P}(A) = \int_{1/5}^{2/5} 1\,dx + \int_{3/5}^{4/5} 1\,dx = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$[3,4] \not\subseteq [0,1]$ $\quad \mathbb{P}([3,4]) = \int_3^4 f(x)\,dx = \int_3^4 0\,dx = 0.$

- $\mathbb{P}(\cdot)$ assigns 0 probability to any event outside $[0,1]$

- $\mathbb{P}(\{x\}) = \int_x^x f(y)\,dy = 0$, $\mathbb{P}([0,1]) = \int_0^1 1\,dx = 1$

$\square$

Definition 5.1.4: $f : \mathbb{R} \to \mathbb{R}$ is called a density function if f satisfies the following:

(i) $f(x) \geq 0$

(ii) f – piecewise continuous and

(iii) $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

Theorem 5.1.5: Let $f : \mathbb{R} \to \mathbb{R}$ be a density function. Define

$$\mathbb{P}(A) = \int_A f(x)\, dx \quad \text{for any event } A \subseteq \mathbb{R}$$

Then $\mathbb{P}$ defines a Probability on $\mathbb{R}$. The function f is called "the density function" for the probability $\mathbb{P}$.

Proof:- • $\displaystyle \mathbb{P}(\mathbb{R}) = \int_{\mathbb{R}} f(x)\, dx$ [f – density]

[Axiom 1 verified] $= 1.$

• $f(x) \geq 0 \;\; \forall x \in \mathbb{R} \implies P(A) = \displaystyle\int_A f(x)\,dx \geq 0$

$A \in \mathcal{F}$; & $\mathbb{P}(A) \leq \displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

[Sketch] $\{E_k\}_{k=1}^{\infty}$ $E_i \cap E_j = \phi$ $E_k \in \mathcal{F}$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \int_{\bigcup_{k=1}^{\infty} E_k} f(x)\,dx$$

[Ex] $$\overset{\circ}{=} \sum_{k=1}^{\infty} \int_{E_k} f(x)\,dx$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

$\mathbb{P}(\cdot)$ Satistics Axiom (ii)

$\Rightarrow$ $\mathbb{P}(\cdot)$ is a Probability on $\mathbb{R}$ $\square$

Example 5.1.6:

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- f - piecewise continuous
- $f \geq 0$
- $$\int_{\mathbb{R}} f(x)\,dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1^3 - 0^3 = 1$$

$\therefore$ f is a density function. [Definition 5.1.4.]

$$\mathbb{P}(A) = \int_A f(x)\,dx \quad (\text{define } \mathbb{P})$$ – [Theorem 5.1.5 defines Probability]

- $f(\cdot) \equiv 0$ on $(0,1)^c \Rightarrow A$ - event $\subseteq (0,1)^c$

$$\Rightarrow \mathbb{P}(A) = 0.$$

- $[\frac{1}{5}, \frac{2}{5}]$ $\quad \mathbb{P}([\frac{1}{5}, \frac{2}{5}]) = \int_{1/5}^{2/5} 3x^2\,dx = \frac{7}{125}$

$[3/5, 4/5]$ $\quad \mathbb{P}([3/5, \frac{4}{5}]) = \int_{3/5}^{4/5} 3x^2\,dx = \frac{37}{125}$

$\Rightarrow \mathbb{P}(\,)$ is not uniform on $(0,1)$

$\therefore \mathbb{P}(\cdot)$ does not assign equal probabilities to intervals of the same length.

- $\mathbb{P}(\{\frac{2}{5}\}) = \int_{2/5}^{2/5} f(x)\,dx = 0$

$\mathbb{P}(\{1/5\}) = \int_{1/5}^{1/5} f(x)\,dx = 0$

$\rightarrow$ $f(1/5) = \frac{3}{25}$

$f(2/5) = 12/25$

$$\boxed{f(2/5) = 4\, f(1/5)}$$

- $\varepsilon > 0$ $\quad \mathbb{P}([\frac{1}{5} - \varepsilon, \frac{1}{5} + \varepsilon]) = \int_{1/5-\varepsilon}^{1/5+\varepsilon} f(x)\,dx \approx \frac{3}{25} \cdot (2\varepsilon)$

$\mathbb{P}([2/5 - \varepsilon, \frac{2}{5} + \varepsilon]) = \int_{2/5-\varepsilon}^{2/5+\varepsilon} f(x)\,dx \approx \frac{12}{25}(2\varepsilon)$

"$f(\frac{2}{5})$ is 4 times $f(\frac{1}{5})$"

.... interpretation for Probability

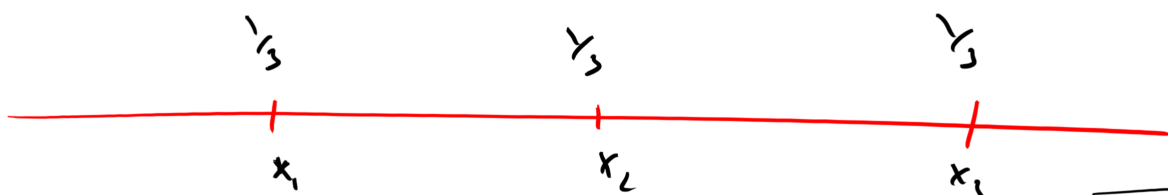
"any small interval around $2/5$" has 4 times
[Same Size]
the probability of "any small interval around $1/5$".
[Same Size]

Q:- $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = c\, e^{-x^2/2}$ $\quad x \in \mathbb{R}$ for some $c > 0$.

Find c : So that $f(\cdot)$ is a density function.

X - discrete

min $E|X-b|$



median - of a Random variable :

m - median

$P(X < m) = P(X \geq m) = \frac{1}{2}$ $\quad \boxed{m=?}$