Recall :- X - a discrete random variable. $E[X] < \infty$

- $X \geq 0$; $c > 0$ $\quad \underbrace{\mathbb{P}(X \geq c)}_{\text{tail Probabilities}} \leq \frac{E[X]}{c}$ [Markov inequality]

- $\mu = E[X] < \infty$, $\sigma = SD[X] = \sqrt{E[(X - E[X])^2]} < \infty$

$k \geq 0$, $\mathbb{P}(\underbrace{|X - \mu| \geq k}_{X \notin (\mu - k\sigma,\ \mu + \sigma k)} \sigma) \leq \frac{1}{k^2}$ [Tschebyschev inequality]

- A - Event $\qquad X | A := Y$ Conditional distribution of X given A

$\mathbb{P}(A) > 0$

- $\mathbb{P}(Y = t)$ or $\mathbb{P}(X = t | A) = \frac{\mathbb{P}(X = t, A)}{\mathbb{P}(A)}$

- Conditional Expectation of X given A

$$E[X|A] = \sum_{t \in Range(X)} t \ \mathbb{P}(X = t | A)$$

- Conditional variance of X given A

$$Var[X|A] = E[(X - E[X|A])^2 | A]$$

$$= \sum_{t \in Range(X)} (t - E[X|A])^2 \ \mathbb{P}(X = t)$$

Example 4.4.3:- Deck of cards — standard 52 cards.

- Cards are drawn, one at a time, randomly with replacement.
- $X \equiv$ # of draws till the first King is drawn.
  $Y \equiv$ # of draws till the first Ace is drawn.

Question:- $E[X \mid Y=3] = ?$

$$\mathbb{L}: \sum_{t \in Range(X)} t \, \mathbb{P}(X=t \mid Y=3)$$

$\underline{Y=3} \equiv$ Ace was drawn for the 1st time is 3rd draw.
[Ace was NOT drawn in 1 and 2

$\mathbb{P}(X=t \mid Y=3) = \mathbb{P}($ King was drawn for 1st time at t^{th} draws $\mid Y=3) = ?$

$$\mathbb{P}(\text{King is drawn in } t^{th} \text{ draw} \mid Y=3) = \begin{cases} \frac{4}{48} & t=1,2 \\ 0 & t=3 \\ 4/52 & t \geq 4 \end{cases}$$

$$\Rightarrow \mathbb{P}(X=t \mid Y=3) = \begin{cases} \left(\frac{44}{48}\right)^{t-1} \cdot \frac{4}{48} & t=1,2 \\ 0 & t=3 \\ \left(\frac{44}{48}\right)^{2} \left(\frac{48}{52}\right)^{t-4} \left(\frac{4}{52}\right) & t \geq 4 \end{cases}$$

$t \geq 4$, $X = t$ $\equiv$ no king in 1 and 2 draws $-\frac{44}{48}$

$=$ at 3rd draw given Ace ✓

$\equiv$ No king for rest of $t-4$ draws and draw

a king in t^{th} draw

$$E[X \mid Y=3] = \sum_{t=1}^{\infty} t \, \mathbb{P}(X=t \mid Y=3)$$

$$= 1 \cdot \frac{4}{48} + 2\left(\frac{44}{48}\right)\frac{4}{48} + \sum_{t=4}^{\infty} t\left(\frac{44}{48}\right)^2\left(\frac{48}{52}\right)^{t-4}\frac{4}{52}$$

$$\underline{Ex} = \frac{985}{72}$$

[Geometric Series $\sum_{m=1}^{\infty} m\, r^m = ?$]

≈ 13.68

- Given Ace appeared for the 1st time in 3rd draw, the conditional expectation of $X \simeq \underline{\underline{13.68}}$.

- $X \stackrel{d}{=} \text{Geometric}(4/52)$; $E(X) = \frac{1}{52/4} = \frac{4}{52} = 13.$ $\square$

[last week(s)]

Example 4.4.7 $Y \sim \text{Uniform}\{1, 2, \dots, n\}$ for some $n \geq 1$

X = # of heads on Y flips of a fair coin.

What is the $E[X]$?

- $\text{Range}(Y) = \{1, \dots, n\}$

$$E(X \mid Y = y) \equiv \checkmark$$

Know how to do.

$$X \mid (Y = y) \stackrel{d}{=} \text{Binomial}\left(y, \tfrac{1}{2}\right).$$

$$\Rightarrow E[X \mid Y = y] = y \cdot \frac{1}{2} = \frac{y}{2}$$

- Define: $g : \{1, \dots, n\} \to \mathbb{R}$ by

$$g(y) := E[X \mid Y = y]$$

$$= \frac{y}{2} \quad \text{[By above computation]}$$

- Q: $E[g(Y)] = ?$

By def: $$E[g(Y)] = \sum_{y \in \text{Range}(Y)} g(y)\, \mathbb{P}(Y = y) = \sum_{y=1}^{n} \frac{y}{2} \cdot \frac{1}{n}$$

$$= \frac{1}{2n} \sum_{y=1}^{n} y = \frac{1}{2n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{4}$$

Claim: $E[X]$

$$\boxed{g(y) = E[X \mid Y=y]}$$

$$E[g(Y)] = \sum_{y \in \text{Range}(Y)} g(y)\, \mathbb{P}(Y=y)$$

$$= \sum_{y=1}^{n} \underbrace{E[X \mid Y=y]}\, \mathbb{P}(Y=y)$$

$$= \sum_{y=1}^{n} \left[\sum_{t \in \text{Range}(X)} t\, \mathbb{P}(X=t \mid Y=y) \right] \cdot \mathbb{P}(Y=y)$$

$$= \sum_{y=1}^{n} \left[\sum_{t \in \text{Range}(X)} t\, \frac{\mathbb{P}(X=t, Y=y)}{\cancel{\mathbb{P}(Y=y)}} \right] \cancel{\mathbb{P}(Y=y)}$$

$$= \sum_{y=1}^{n} \sum_{t \in \text{Range}(X)} t\, \mathbb{P}(X=t, Y=y)$$

$$= \sum_{t \in \text{Range}(X)} t \left[\sum_{y=1}^{n} \mathbb{P}(X=t, Y=y) \right]$$

$$= \sum_{t \in \text{Range}(X)} t\, \mathbb{P}\left(\bigcup_{y=1}^{n} (X=t, Y=y) \right)$$

$$= \sum_{t \in \text{Range}(X)} t\, \mathbb{P}(X=t) = E[X]$$

The previous computation proves the below Theorem

Theorem 4.4.6 : let X and Y be two discrete random variables on the sample space S.

let $g : \text{Range}(Y) \longrightarrow \mathbb{R}$. given by

$$g(y) = E[X \mid Y=y]. \quad \text{Then}$$

$$E[g(Y)] = E[X].$$

Remarks :- $g(y) := E[X|Y=y]$ | $g(Y) \equiv E[X|Y]$

Theorem: $E[E[X|Y]] = E[X]$

Expectation of Y

Conditional Expectation of X given $Y=y$

- $\text{Var}[X] = E[\text{var}[X|Y]] + \text{var}[E[X|Y]]$ – true

$h(y) = \text{Var}[X|Y=y]$ $\quad = E[h(Y)] + \text{Var}(g(Y))$

$g(y) = E[X|Y=y]$

Definition 4.5.1. (Covariance of X and Y)

X and Y be two discrete random variables on a sample space S. Then the "covariance of X and Y" is defined as

$$\mathrm{Cov}[X,Y] = \underset{\substack{\uparrow \\ \text{joint distribution}}}{E}\Big[(X - \overset{\text{over } X}{E[X]})(Y - \overset{\text{over } Y}{E[Y]})\Big]$$

$$\left.\begin{array}{l} X > E[X] \text{ at the same time } Y > E[Y] \\ \quad \& \\ X < E[X] \text{ at the same time } Y < E[Y] \end{array}\right\} \Rightarrow \begin{array}{l} \mathrm{Cov}[X,Y] > 0 \\ \text{[Positively correlated)} \end{array}$$

$$\left.\begin{array}{l} X < E[X] \text{ at the time } Y > E[Y] \\ Y < E[Y] \text{ at the time } X > E[X] \end{array}\right\} \Rightarrow \begin{array}{l} \mathrm{Cov}[X,Y] < 0 \\ \text{(negative correlated)} \end{array}$$

$$E[(X - E[X])(Y - E[Y])] = 0 \iff \mathrm{Cov}[X,Y] = 0 \Rightarrow X \text{ and } Y \text{ uncorrelated}$$

$$\Updownarrow$$

$$E[XY - XE[Y] - E[X]\cdot Y + E[X]E[Y]] = 0$$

$$\Updownarrow$$

$$E[XY] - E[X]E[Y] \cancel{- E[X]E[Y]} \cancel{+ E[X]E[Y]} = 0 \iff E[XY] = E[X]E[Y]$$

Remark: X and Y are independent $\Rightarrow E[XY] = E[X]E[Y]$
(shown earlier)

Theorem: let X and Y be discrete random variables with finite mean for which $E[XY]$ is also finite. Then

$$Cov[X,Y] = E[XY] - E[X]\,E[Y].$$

Ex: Z is another discrete random variable. $\alpha, \beta \in \mathbb{R}$

(a) $Cov(X,Y) = Cov[Y,X]$

(b) $Cov[X, \alpha Y + \beta Z] = \alpha\, Cov[X,Y] + \beta\, Cov[X,Z]$

(c) X and Y are independent with a finite covariance then $Cov[X,Y] = 0$

Example 4.5.6: X, Y with $Range = \{-1, 1\}$ be two random variables

$$\mathbb{P}(X=x, Y=y) \equiv p(x,y)$$

$p(X=x, Y=y)$	$X=1$	$X=-1$	
$Y=1$	0.3	0.2	$\underbrace{0.5}_{}$ $P(Y=1)$
$Y=-1$	0.3	0.2	$\underbrace{0.5}_{}$ $P(Y=-1)$
	$\underbrace{0.6}_{P(X=1)}$	$\underbrace{0.4}_{P(X=-1)}$	

. $Cov[X,Y] = ?$

$$= E(XY) - E(X)\,E[Y]$$

$$E[XY] = \sum_{\substack{x \in Range(X) \\ y \in Range(Y)}} xy\, \mathbb{P}(X=x, Y=y) = 1\cdot 1\,(0.3) + (-1)\,1\,(0.2) + 1(-1)\,(0.3) + (-1)(-1)\,(0.2)$$

$$= 0$$

$$E[X] = \sum_{x \in Range(X)} x \, \mathbb{P}(X=x) = 1\, \mathbb{P}(X=1) + (-1)\, \mathbb{P}(X=-1)$$

$$= 1\,(0.6) + (-1)\,(0.4)$$

$$= 0.2$$

$$E[Y] = \sum_{y \in Range(Y)} y \, \mathbb{P}(Y=y) = 1\,(0.5) + (-1)(0.5) = 0$$

$$\therefore \quad Cov[X,Y] = E[XY] - E[X]\,E[Y] = 0.$$

Question :

[BP] $$E[X|A] := \sum_{t \in T} t \, \mathbb{P}(X=t \mid A)$$

[MT] $$\begin{cases} E[X|A] :\equiv E[X \mid \sigma(A)], \quad \sigma(A) = \{\phi, A, A^c, \Omega\} \\ Y \equiv \sigma(A) \text{ m'ble } \& \\ \quad \int_B Y \, dP = \int_{B} X \, dP \quad \forall \; B \in \sigma(A) \end{cases}$$

Is [BP] $\equiv$ [MT] definitions ?