Recall :- X - discrete random variable $(S, \mathcal{F}, \mathbb{P})$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X=t)$$

Converges absolutely $E[X] < \infty$

$\pm\infty$ we say X has infinite expectation

Diverges :- $E[X]$ is not defined.

- $E[c] = c$
- $E[X] < \infty \iff E[|X|] < \infty$
- $E[X+Y] = E[X] + E[Y]$ [Provided both $E[X]$ and $E[Y]$ exist $< \infty$]
- $E[\alpha X] = \alpha E[X]$ for any $\alpha \in \mathbb{R}$, provided $E[X] < \infty$

Ex

Ex. $\Rightarrow$ $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$ $\alpha, \beta \in \mathbb{R}$

provided. $E[X] < \infty$ and $E[Y] < \infty$.

Example 4.1.11 $X \sim \text{Uniform}(\{1,2,3\})$

$Y = 4 - X$ Range$(Y) = \{1,2,3\}$

Ex: $\mathbb{P}(Y=i) = \frac{1}{3}$ $\forall\, i \in \{1,2,3\}$

$Y \sim \text{uniform}\{1,2,3\}$

$$E[X] = \sum_{i=1}^{3} i\, \mathbb{P}(X=i)$$

$$= 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2 = E[Y]$$

$$Z = XY = X(4-X) = 4X - X^2 \in \{3,4\}$$

$$\mathbb{P}(Z=3) = \mathbb{P}(X=1 \cup X=3) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\mathbb{P}(Z=4) = \mathbb{P}(X=4) = \frac{1}{3}$$

Distribution of Z

$$\Rightarrow \quad E[Z] = 3 \cdot \left(\frac{2}{3}\right) + 4\left(\frac{1}{3}\right) = \frac{10}{3}$$

i.e. $E[XY] = \frac{10}{3} \neq E[X]\,E[Y] = 4$

i.e. in general $E[XY] \neq E[X]\,E[Y]$

$\square$

Theorem 4.1.10: Suppose X and Y are discrete random variables, both with finite expected value. and both defined on the same sample space S. If X and Y are independent then $E[XY] = E[X]\,E[Y]$.

Proof:-

$X : S \to \mathcal{U}$ $\quad Y : S \to \mathcal{V}$

$XY : S \to T$ $\quad T = \{ uv \mid u \in \mathcal{U}, v \in \mathcal{V} \}$

$$E[XY] = \sum_{t \in T} t \, \mathbb{P}(XY = t)$$

$$= \sum_{\substack{u \in \mathcal{U} \\ v \in \mathcal{V}}} uv \, \mathbb{P}(X = u, Y = v)$$

X and Y are independent

$$= \sum_{\substack{u \in \mathcal{U} \\ v \in \mathcal{V}}} uv \, \mathbb{P}(X = u) \, \mathbb{P}(Y = v)$$

$$= \left[\sum_{u \in \mathcal{U}} u \, \mathbb{P}(X = u) \right] \left[\sum_{v \in \mathcal{V}} v \, \mathbb{P}(Y = v) \right]$$

$$= E[X] \, E[Y]$$

$\square$

Caution:- Start with $E[X] \cdot E[Y]$ and work backward

- Series converges absolutely & rearrangements okay & independence.

$X \sim \text{Geometric}(p)$: What is $E[X]$?

$\text{Range}(X) = \{1, 2, 3, \ldots\}$ $\qquad$ $\mathbb{P}(X = k) = (1-p)^{k-1} p$, $k \in \text{Range}(X)$.

$$E[X] = \text{``}\sum_{k=1}^{\infty} k \, \mathbb{P}(X=k)\text{''}$$ — Compute this Series.

$$T_n = \sum_{k=1}^{n} k \, \mathbb{P}(X=k) = \sum_{k=1}^{n} k \, p(1-p)^{k-1} = \sum_{k=1}^{n} k(1-(1-p))(1-p)^{k-1}$$

$$= \sum_{k=1}^{n} k(1-p)^{k-1} - \sum_{k=1}^{n} k(1-p)^{k}$$

[Simple algebra] $$= \sum_{k=1}^{n} (1-p)^{k-1} - n(1-p)^n$$

[Geometric Sum] $$= \frac{1-(1-p)^n}{1-(1-p)} - n(1-p)^n$$

$$= \frac{1-(1-p)^n}{p} - n(1-p)^n$$

Facts from Calculus / Analysis :- $\lim_{n \to \infty} (1-p)^n = 0$, $\lim_{n\to\infty} n(1-p)^n = 0$

$T_n \longrightarrow \frac{1}{p}$ as $n \to \infty$.

As $E[X] = \text{``}\sum_{k=1}^{\infty} k \, \mathbb{P}(X=k)\text{''}$

$\therefore$ $E[X] = 1/p$. $\square$

Ex: $X \sim \text{Poisson}(\lambda)$, Find $E[X]$

$X \sim \text{Bernoulli}(p)$:- $E[X] = p$

$X \sim \text{Binomial}(n, p)$:- $E[X] = np$

If we have a random discrete variable $X : S \to T$

$f : T \to \mathbb{R}$

$Y = f(X)$, Y is also a discrete random variable.

$E[Y] = ?$ $\xrightarrow[\text{method}]{}$ Find : - Range (Y)

- $\mathbb{P}(Y = y) = ?$ $\forall y \in \text{Range}(Y)$
- $\sum_{y \in \text{Range}(Y)} y \, \mathbb{P}(Y = y) \equiv E[Y]$

$$\boxed{E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t)}$$

- can we use distribution of X to find $E[Y]$? without computing distribution of Y.

Approach:

$$E[f(X)] = \sum_{u \in \mathcal{U}} u \, \mathbb{P}(f(X) = u) \qquad \text{Range}(f) = \mathcal{U}$$

$$(f(X) = u) = \bigcup_{t \in f^{-1}(u)} (X = t) \quad \text{— Set theory Exercise}$$

Using $\circledast$

$$E[f(X)] = \sum_{u \in \mathcal{U}} u \, \mathbb{P}\Big(\bigcup_{t \in f^{-1}(u)} (X = t)\Big)$$

$$= \sum_{u \in \mathcal{U}} u \sum_{t \in f^{-1}(u)} \mathbb{P}(X = t)$$

$$\Rightarrow \quad = \sum_{u \in \mathcal{U}} \sum_{t \in f^{-1}(u)} u \, \mathbb{P}(X = t)$$

$$= \sum_{u \in \mathcal{U}} \sum_{t \in f^{-1}(u)} f(t) \, \mathbb{P}(X = t)$$

$$\boxed{T = f^{-1}(u)} \quad = \sum_{t \in T} f(t) \, \mathbb{P}(X = t)$$

Theorem 4.1.19: let $X : S \to T$ be a discrete random variable. $f : T \to \mathcal{U}$. Then the expected value of $f(X)$ may be computed as

$$E[f(X)] = \sum_{t \in T} t \, \mathbb{P}(X = t)$$

Theorem 4.1.20: let $X_1, X_2, .. X_n$ be a sequence of random variables all defined on a common Probability space S. Then X_j variables may have different ranges, say T_j

$f : T_1 \times .. T_n \to \mathbb{R}$ be a function.

$$E[f(X_1, .., X_n)] = \sum_{t_1 \in T_1, .., t_n \in T_n} f(t_1, .., t_n) \, \mathbb{P}(X_1 = t_1, .., X_n = t_n)$$

Proof: Ex:

Key: $u \in \text{Range}(f)$

$$\boxed{(t_1, .. t_n) \in f^{-1}(u)} \iff f(t_1 .. t_n) = u$$

Variance and Standard Deviation : Consider 3 r.v. X, Y and Z

- $X = 10$, $\mathbb{P}(X=10) = 1 \rightarrow E[X] = 10 \cdot 1 = 10$
- $\mathbb{P}(Y=9) = \frac{1}{2} = \mathbb{P}(Y=11) \longrightarrow E[Y] = 9\left(\frac{1}{2}\right) + 11\left(\frac{1}{2}\right) = 10$
- $\mathbb{P}(Z=0) = \frac{1}{2} = \mathbb{P}(Z=20) \longrightarrow E[Z] = 0\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right) = 10$

"10" $= E[X] = E[Y] = E[Z]$ – X is off by 1 from Y
10 from Z

– It will be good to quantify how far away a random variable is from its average?

" If $E[X]$ – measures the "Center" of the random variable"

– we would like a notion to define the spread of the random variable.

Definition 4.2.1 : let X be a random variable with finite Expected value. Then the variance of the random variable is written as $Var[X]$ and is defined as

$$Var[X] := E\left[(X - E[X])^2\right]$$

Standard deviation of X – $SD[X]$ and is defined as

$$SD[X] = \sqrt{Var[X]}.$$

· Remarks :- $Var[X]$ – average of the squared distance of X from its expected value

X – has high probability of being far away from $E[X]$ then variance of X will tend to be large

X – is near $E[X]$ with high probability then the variance of X will tend to be small.

· X is measured in say meters then $Var[X]$ – is in $(\text{meters})^2$ and $SD[X]$ is in meters

Standard deviation is considered to measure the true spread of the random variable. ($SD(X)$ = positive square root of $Var[X]$)

Formally : X is a discrete random variable. Suppose $E[X] = 12$ and $SD[X] = 3$

Range (X) = " $9 = 12-3$ " to " $15 = 12+3$ "
"$9 - 15$"

· $E[X] < \infty$ Does not imply that $SD[X] < \infty$, $Var[X] < \infty$

Example 4.2.2 . Roll a die, X – outcome

Range$(X) = \{1, \dots, 6\}$

$$E[X] = \sum_{k=1}^{6} k \, \mathbb{P}(X=k) = \frac{1}{6} \sum_{k=1}^{6} 6 = 3.5$$

$$\mathrm{Var}[X] = \,? = E[\underbrace{(X - E[X])^2}_{f(X)}]$$

$$= \sum_{t \in \{1,\dots 6\}} f(t) \, \mathbb{P}(X=t)$$

$$= \sum_{t=1}^{6} (t-3.5)^2 \frac{1}{6}$$

Sum in reverse order

$$= \frac{(2.5)^2 + (1.5)^2 + (0.5)^2 + (-0.5)^2 + (-1.5)^2 + (-2.5)^2}{6}$$

$$= \frac{35}{12}$$

$$SD[X] = \sqrt{\frac{35}{12}} \simeq 1.71$$

$\boxed{\text{Range}(X) :- \text{"}3.5 - 1.71\text{"} \longleftrightarrow \text{"}3.5 + 1.71\text{"}}$