Recall:- Functions of random variables.

- $X: S \to T$ ← Random variable; $f: T \to \mathbb{R}$ ← function; $Y = f(X)$ ← random variable. Distribution of Y?

- $\underbrace{X_1, \dots, X_n}$: $S \to T$ ← Random variables; $f: T^n \to \mathbb{R}$ ← function; $Z = f(X_1, \dots, X_n)$ ← random variable. Distribution of Z?

- We illustrated with some examples how to compute distributions of Y and Z

Example 1 :- $X \sim$ Uniform $\{-2,-1,0,1,2\}$ $\quad f: \{-1,0,1\} \to \mathbb{R} \quad f(x) = x^2$

$Y = f(X)$ Range$(Y) = \{0, 1, 4\}$ —

$$\left.\begin{aligned} \mathbb{P}(Y=0) &= \mathbb{P}(X=0) = 1/5 \\ \mathbb{P}(Y=1) &= \mathbb{P}(X=1 \cup X=-1) = \frac{2}{5} \\ \mathbb{P}(Y=4) &= \quad \cdots \quad = 2/5. \end{aligned}\right\} \text{Distribution of } Y$$

Example 2:- $X \sim$ Bernoulli(p) $Y \sim$ Bernoulli(p), independent

$$Z = X + Y$$

$$\mathbb{P}(Z=0) = \mathbb{P}(X=0, Y=0) = (1-p)^2$$

$$\mathbb{P}(Z=1) = \mathbb{P}(X=1, Y=0 \cup X=0, Y=1) = 2p(1-p)$$

$$\mathbb{P}(Z=2) = \mathbb{P}(X=1, Y=1) = p^2$$

$\square$

In general:- Y, X — random variables that took values in $\{0, 1, 2, \ldots\}$ — independent

$Z = X + Y$ Distribution of Z?

- Range$(Z) \equiv \{0, 1, 2, \ldots\}$
- $\mathbb{P}(Z=n) = ?$ for various values of n.

$$\{Z=n\} = \bigcup_{j=0}^{n} \{X=j, Y=n-j\}$$

$$\mathbb{P}(Z=n) = \sum_{j=0}^{n} \mathbb{P}(X=j, Y=n-j)$$

$$= \sum_{j=0}^{n} \mathbb{P}(X=j) \cdot \mathbb{P}(Y=n-j)$$

Prescription of finding distribution of sums of independent random variables

$\underbrace{Z, Y}_{\text{Independent}}, X \in \{0, 1, \dots, n\}$, $\underbrace{Z = X+Y}_{\text{Sum}}$ $\underbrace{f_Z(n) = \sum_{j=0}^{n} f_X(j)\, f_Y(n-j)}_{\text{Convolution of } f_X \text{ and } f_Y.}$

where $f_Z(\cdot)$ p.m.f of Z

$f_X(\cdot)$ pmf of X

$f_Y(\cdot)$ pmf of Y

Example 3.3.4 :- $X \sim \text{Poisson}(\lambda_1)$ & X and Y are independent
$Y \sim \text{Poisson}(\lambda_2)$

(a) $Z = X+Y$ Find distribution of Z.

$\text{Range}(X) = \{0,1,2,\ldots\}$; $f_X(k) = \frac{e^{-\lambda_1}\lambda_1^k}{k!}$

$\text{Range}(Y) = \{0,1,2,\ldots\}$; $f_Y(k) = \frac{e^{-\lambda_2}\lambda_2^k}{k!}$ $k=0,1,2,\ldots$

$Z = X+Y$, For any $n \in \{0,1,2,\ldots\}$

$$f_Z(n) = \sum_{j=0}^{n} f_X(j)\, f_Y(n-j)$$

$$= \sum_{j=0}^{n} \frac{e^{-\lambda_1}\lambda_1^j}{j!}\frac{e^{-\lambda_2}\lambda_2^{n-j}}{n-j!} = e^{-(\lambda_1+\lambda_2)}\sum_{j=0}^{n}\frac{\lambda_1^j\lambda_2^{n-j}}{j!\,n-j!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{j=0}^{n}\frac{\lambda_1^j\lambda_2^{n-j}}{j!\,n-j!}\cdot n! \underset{\text{Binomial expansion}}{=} \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

$Z \sim \text{Poisson}(\lambda_1+\lambda_2)$

Generalise :- Ex: $X_i \sim \text{Poisson}(\lambda_i)$ $i=1,2,\ldots k$, $Z = \sum_{i=1}^{k} X_i$,
$\{X_i\}_{i=1}^{k}$ – independent Then $Z \sim \text{Poisson}\left(\sum_{i=1}^{k}\lambda_i\right)$

(b) $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$ X, Y independent

$Z = X + Y$ (Part (a) $\Rightarrow$ $Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$)

Fix $n \geq 0$: $X \mid Z = n \equiv$ what is its distribution?

If $Z = n$ then $X \in \{0, 1, \ldots, n\}$. ($\because Z = X+Y$, $Y \in \{0, \ldots n\}$)

$k \in \{0, 1, 2, \ldots n\}$

$$\mathbb{P}(X = k \mid Z = n) = \frac{\mathbb{P}(X = k, Z = n)}{\mathbb{P}(Z = n)}$$

$$= \frac{\mathbb{P}(X = k, X + Y = n)}{\mathbb{P}(Z = n)}$$

$\{X = k, X+Y = n\} \equiv \{X = k, Y = n - k\}$

$$= \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(Z = n)}$$

X and Y are independent

$$= \frac{\mathbb{P}(X = k)\, \mathbb{P}(Y = n - k)}{\mathbb{P}(Z = n)}$$

$$= \frac{\dfrac{e^{-\lambda_1} \lambda_1^{k}}{k!} \cdot \dfrac{e^{-\lambda_2} \lambda_2^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!\,(n-k)!} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\therefore \mathbb{P}(X = k \mid Z = n) = \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

$k \in \{0, 1, \ldots n\}$

$$X \mid Z = n \sim \text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

Theorem 3.3.5: let $X_1, \dots, X_n$ be random variables on a single sample space S. let f be a function of n variables, $Z = f(X_1, \dots X_n)$ is well defined on S.

$$\mathbb{P}(f(X_1, \dots X_n) \in B) = \mathbb{P}((X_1, \dots X_n) \in f^{-1}(B))$$

for any event B in the range of f

Proof (Sketch):

$$\{ s \in S \mid f(X_1(s), \dots X_n(s)) \in B \}$$

$$\|$$

$$\{ s \in S \mid (X_1(s), \dots X_n(s)) \in f^{-1}(B) \}$$

$\boxed{\text{Set-Theory fact}}$

- result is immediate

$\square$

3.3.2 : Functions and Independence

If X and Y are independent random variables

$$X : S \to T_1 \quad , \quad g : T_1 \to \mathbb{R}$$

$$Y : S \to T_2 \quad , \quad f : T_2 \to \mathbb{R}$$

$$Z = g(X) \quad , \quad W = f(Y)$$

(intuitively) $\Rightarrow$ Z and W are also independent

Theorem 3.3.6 :- Let $n > 1$ be a positive integer. For $j \in \{1, \dots, n\}$ define a positive integer m_j & Suppose $\{X_{ij}\}_{j \in \{1,\dots,n\}, i \in \{1,\dots,m_j\}}$ be an array of mutually independent random variables. Let f_j be functions such that

$$Y_j = f_j(X_{1j}, \dots, X_{m_j j})$$

are well defined. Then the resulting random variables $Y_1, Y_2, \dots Y_n$ are mutually independent.

Proof:- "Random quantities produced from independent inputs will be independent".

let $B_1, \dots, B_n$ be sets in the ranges of $Y_1 \dots Y_n$

$$\mathbb{P}(Y_1 \in B_1, \dots, Y_n \in B_n)$$

$$= \mathbb{P}(f_1(X_{11}, \dots X_{m_1 1}) \in B_1, \dots, f_n(X_{1n}, \dots X_{m_n n}) \in B_n)$$

$$= \mathbb{P}((X_{11} \dots X_{m_1 1}) \in f_1^{-1}(B_1), \dots, (X_{1n}, \dots X_{m_n n}) \in f_n^{-1}(B_n))$$

$\{X_{ij}\}$ are mutually independent

$$= \prod_{j=1}^{n} \mathbb{P}((X_{j1} \dots X_{m_j j}) \in f_j^{-1}(B_j))$$

$$= \prod_{j=1}^{n} \mathbb{P}(f_j(X_{j1}, \dots X_{m_j j}) \in B_j)$$

$$= \prod_{j=1}^{n} \mathbb{P}(Y_j \in B_j)$$

$\Rightarrow \{Y_1, \dots, Y_n\}$ are mutually independent $\square$