Recall :-

- S - uncountable set
- $\mathcal{F}$ - (σ-field) - proper event space definition
- $\mathbb{P}$ - Probability

- $f: \mathbb{R} \to \mathbb{R}$ - piecewise continuous, $f(\cdot) \geq 0$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

[Probability density function]

$$\mathbb{P}(A) = \int_A f(x)\,dx$$

- $X: S \to \mathbb{R}$ was a continuous random variable if $\mathbb{P}(X \in A) = \int_A f(x)\,dx$ with $f: \mathbb{R} \to \mathbb{R}$ being a probability density function.

- $\mathbb{P}(X = a) = 0$ [unlike a discrete r.v.]

- Examples:
  - $X \sim \text{Uniform}(a,b)$, $a<b$ $\langle \cdots \rangle$ $f(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & \text{o.w.} \end{cases}$

    choosing a number from (a,b) in an "equally likely" manner

  - $X \sim \mathcal{E}xp(\lambda)$ $\langle \cdots \rangle$ $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{o.w} \end{cases}$

    lifetime of a bulb or a decay time of a radioactive particle

  - $X \sim \text{Normal}(\mu, \sigma^2)$ $\langle \cdots \rangle$ $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

    $X_i \sim \text{Bernoulli}(p)$ independent, $S_n = \sum_{i=1}^{n} X_i$, $S_n \sim \text{Binomial}(n,p)$

    $\mu_n = np$, $\sigma_n = \sqrt{np(1-p)}$

    [Sketch] $\mathbb{P}\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \longrightarrow \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$

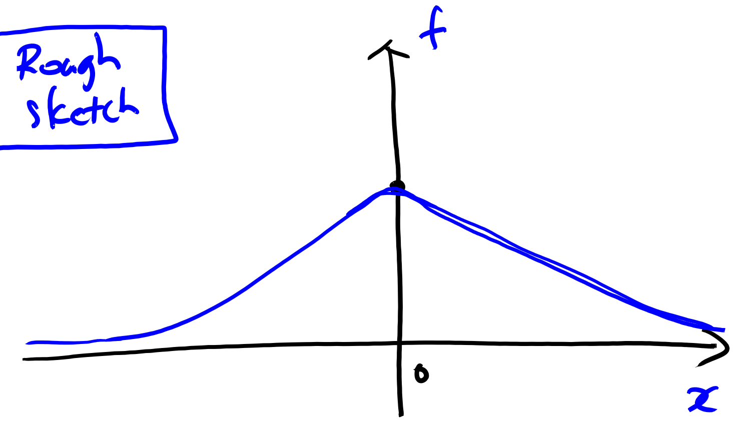
$X \sim$ Normal $(0,1)$ Calculating Probabilities associated to X

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$\to$ no Anti derivative; need to do this numerically.

Properties of Normal $(0,1) \equiv X$

$\boxed{\text{Rough sketch}}$

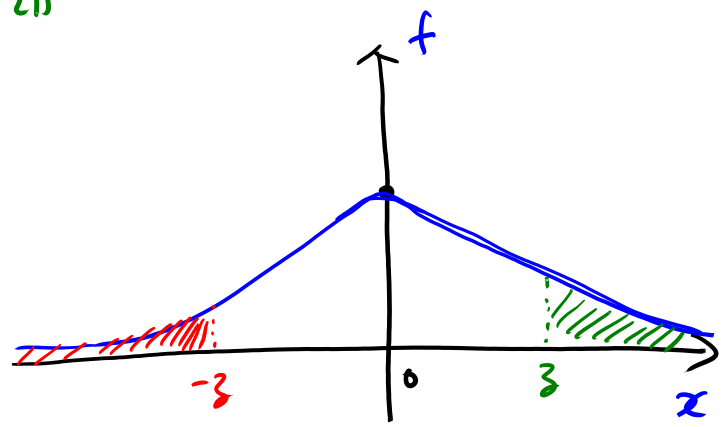


$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \in \mathbb{R}$$

Symmetry. $f(x) = f(-x)$, $\forall x \in \mathbb{R}$

$$\mathbb{P}(X \leq 3) = \int_{-\infty}^{3} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \underset{y=-x}{\overset{\text{c.o.v.}}{=}} \int_{-3}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy = 1 - \int_{-\infty}^{-3} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$$= 1 - \mathbb{P}(X \leq -3) \quad \text{—} (*)$$



Rewrite $(*)$:

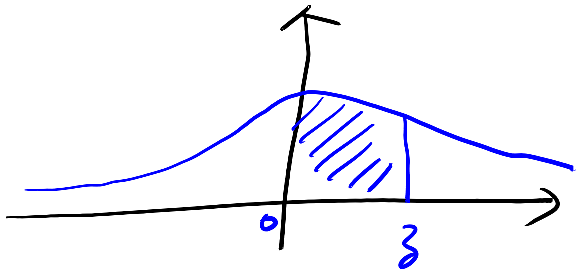
$$1 - \mathbb{P}(X \leq 3) = \mathbb{P}(X \leq -3)$$

$$\int_{3}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy = \mathbb{P}(X > 3) = \mathbb{P}(X \leq -3) = \int_{-\infty}^{-3} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

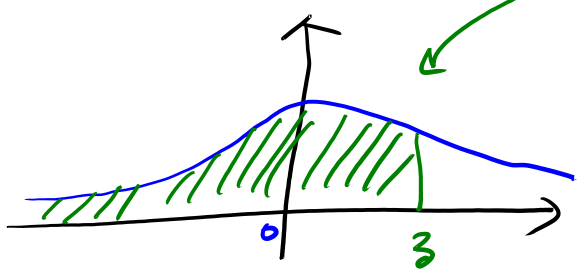
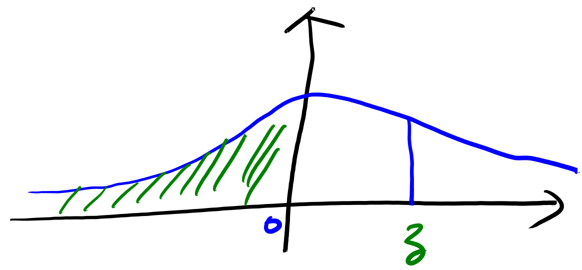
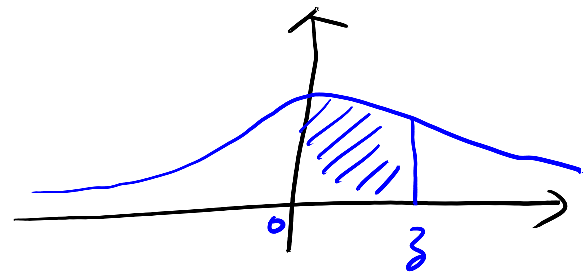
$$\mathbb{P}(X > 3) \;\text{|||}\; \mathbb{P}(X \geq 3)$$

Numerically: $z \geq 0$ — **Compute** $\equiv \displaystyle\int_0^z \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$

- $[0,z]$
- can be done by Riemann approximation or other sophisticated methods



$z \in \mathbb{R}$: $\quad z > 0 \implies \displaystyle \mathbb{P}(X \leq z) = \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$

 $=$  $+$ 

$$= \int_{-\infty}^{0} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy + \int_0^z \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

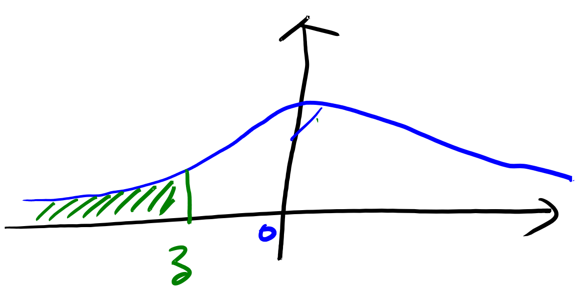
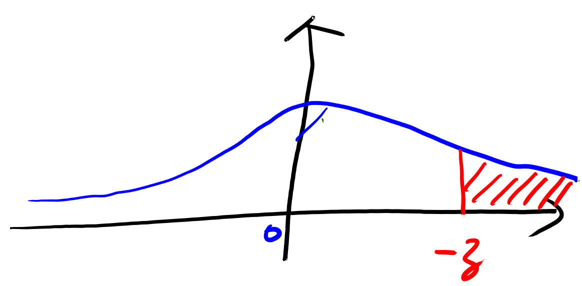
(Numerically)

$$= \mathbb{P}(X \leq 0) + \cdots$$

$$\underset{\text{Ex}}{=} \frac{1}{2} + \cdots$$

$\boxed{\begin{array}{l} (*)\ \ \mathbb{P}(X \leq 0) = \mathbb{P}(X \geq 0) \\ \Leftarrow\ \ \mathbb{P}(X \geq 0) + \mathbb{P}(X \leq 0) = 1 \end{array}}$

$z < 0$: $\displaystyle \implies \mathbb{P}(X \leq z) = \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$

 $\underset{(*)}{\overset{\text{Rewrite}}{=}}$  $\displaystyle = \int_{-z}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$

$$= 1 - \int_{-\infty}^{-z} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy = 1 - \left(\frac{1}{2} + \int_0^{-z} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy\right) = \frac{1}{2} - \int_0^{-z} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

$z \in \mathbb{R}$

$$\mathbb{P}(X \le z) = \begin{cases} \frac{1}{2} + \int_0^z \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy & z > 0 \\ \frac{1}{2} - \int_{-\infty}^{-z} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy & z \le 0 \end{cases}$$

- Numerically compute

$z > 0$; $\int_0^z \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \equiv$ Normal Tables (Available in Literature)

Extend : Normal (μ, σ^2) :

$Y \sim$ Normal (μ, σ^2)

$$\mathbb{P}(Y \le y) = \int_{-\infty}^{y} \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dz$$

Substitute $x = \frac{z-\mu}{\sigma}$; $dx = \frac{1}{\sigma} dz$

$$= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}\,\cancel{\sigma}}\, \cancel{\sigma}\, dx$$

$$= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = \mathbb{P}\left(X \le \frac{y-\mu}{\sigma}\right)$$

where $X \sim$ Normal $(0,1)$

- Use normal tables to compute Probabilities of Y.

Change of Variable :- [Transformation]

- illustrate methodology via Examples
  - via distribution functions of random variables
  - pmf / p.d.f $\Longleftrightarrow$ $\boxed{\text{distribution functions}}$

    Discrete   continuous

- Consider Continuous random variables

Example 5.3.1 $X \sim \text{Uniform}(0,1)$ $\quad Y = X^2$ - p.d.f of Y?

$$f_X(x) = \begin{cases} \frac{1}{1-0} & 0 < x < 1 \\ 0 & \text{o.w} \end{cases} = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w} \end{cases} \quad (+)$$

Step 1: Compute distribution function of X.

$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(y)\,dy =_{Ex.} \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \quad (++)$$

Step 2:- Compute distribution function of Y

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) \equiv ?$$

$Y = X^2 \Rightarrow 0 < X < 1 \Rightarrow Y \in (0,1)$

- $y < 0$ $\quad F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = 0$ — (*)

$\left(x^2 \geq 0 \text{ so } \mathbb{P}(x^2 \le y) = 0 \right)$

- $y \geq 0$ $\quad F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$

$(\sqrt{y} > 0)$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \overset{(+)}{=} \int_0^{\sqrt{y}} f_X(x)\,dx$$

$$= \int_{-\infty}^{0} 0 \cdot dx + \int_0^{\sqrt{y}} f_X(x)\,dx$$

$$\overset{(+)}{=} \int_{-\infty}^{\sqrt{y}} f_X(x)\,dx = F_X(\sqrt{y})$$

$$\overset{(++)}{=} \begin{cases} \sqrt{y} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases} \quad \text{— (**)}$$

From (*) & (**)

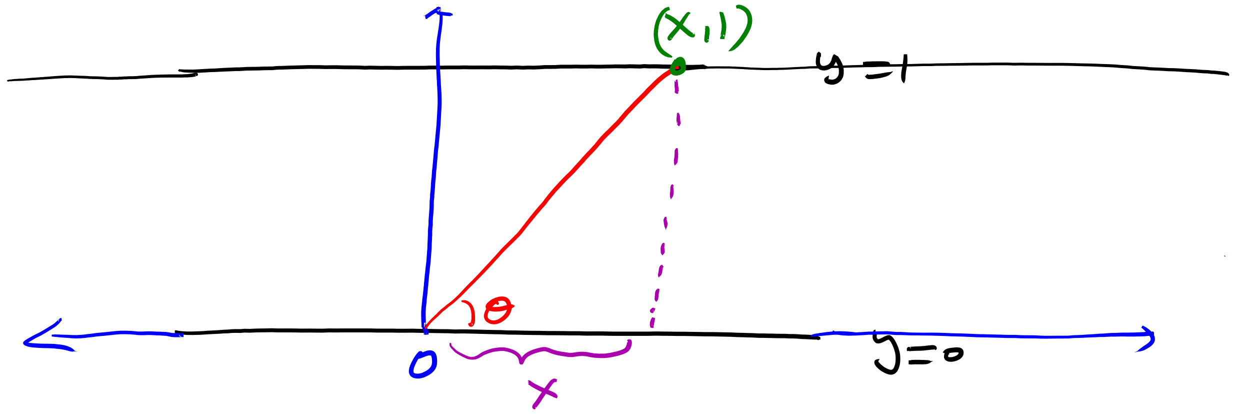
$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ \sqrt{y} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

step3:- Compute p.d.f of Y. $\quad [\, F_Y'(y) = f(y) \text{ — (+++)} \,]$

$$f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & y \geq 1 \end{cases} \qquad \left[\frac{d(y^{1/2})}{dy} = \frac{1}{2\sqrt{y}} \right]$$

$\square$

Example 5.3.4 :-



Choose an angle :- $\theta \sim$ uniformly $(0,\pi)$

draw a line segment from $(0,0)$ to line $y=1$

: Say point hit by segment $= (x,1)$

Find the p.d.f of X [$X \in \mathbb{R}$ - "continuous r.v."]

Observe :- $X = \tan\left(\frac{\pi}{2} - \theta\right)$

Step 1 :- Find distribution function of θ.

$\theta \sim$ uniform $(0,\pi)$

$$F_\Theta(y) = \begin{cases} 0 & y \le 0 \\ y/\pi & 0 < y < \pi \\ 1 & y \ge \pi \end{cases} \quad - \quad (\dagger)$$

Step 2 :- Find distribution function of X

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \mathbb{P}\left(\tan\left(\frac{\pi}{2} - \theta\right) \le x\right)$$

$$= \mathbb{P}\left(\frac{\pi}{2} - \theta \le \arctan(x)\right)$$

$$= \mathbb{P}\left(\theta \geq \frac{\pi}{2} - \arctan(x)\right)$$

$$= 1 - \mathbb{P}\left(\theta < \frac{\pi}{2} - \arctan(x)\right)$$

$$= 1 - F_\theta\left(\underbrace{\frac{\pi}{2} - \arctan(x)}\right)$$

$$\boxed{x \in \mathbb{R} \Rightarrow \frac{\pi}{2} - \arctan(x) \in (0,\pi)} \Rightarrow$$

$$= 1 - \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan(x)\right)$$

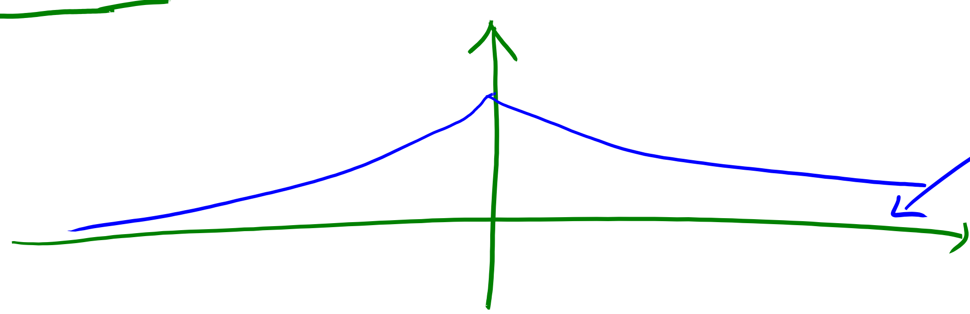
$\neq$

$$= \frac{1}{2} + \frac{\arctan(x)}{\pi} \quad , \quad x \in \mathbb{R}$$

Step3:- Compute p.d.f of X $\quad [\ F_X'(\cdot) = f_X(\cdot)\ ]$

$$\Rightarrow \quad f_X(x) = \frac{1}{\pi}\,\frac{1}{1+x^2} \qquad x \in \mathbb{R}$$

$\boxed{X \sim \text{Cauchy}(0,1)}$ - Cauchy distribution



decays is not like normal