Recall :-

S - (uncountable set) / Countable.

$\mathcal{F}$ - $\cdot\ A \in \mathcal{F}$, $A^c \in \mathcal{F}$ $\quad \cdot\ \Omega \in \mathcal{F}$

$\cdot\ \{A_n\}_{n \geq 1}$ $A_n \in \mathcal{F}$ $\Rightarrow$ $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ } Events

$\mathbb{P}: \mathcal{F} \to [0,1]$ ① $P(S) = 1$

② $\{A_i\}_{i \geq 1}$ $A_i \cap A_j = \phi$ $i \neq j$ $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

$\cdot\ f: \mathbb{R} \to [0,\infty)$ - piecewise continuous function } Probability density function.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\cdot\ \mathbb{R}$, $\mathcal{F}$ - smallest σ-field contains intervals

$$\mathbb{P}(A) = \int_A f(x)\,dx.$$ - } defined a Probability on $\mathbb{R}$

Random variables - Examples of density function that are used in applications.

Definition 5.2.1. : let $(S, \mathcal{F}, \mathbb{P})$ be a Probability space let $X: S \to \mathbb{R}$ be a function. Then X is a random variable provided that B is an event in $\mathbb{R}$, then $X^{-1}(B)$ is also an event in $\mathcal{F}$.

Definition 5.2.2: Let $(S, \mathcal{F}, \mathbb{P})$ be a Probability space. Let $X : S \to \mathbb{R}$ be a random variable. X is called a Continuous random variable if there exists a density function $f_X : \mathbb{R} \to \mathbb{R}$ such that for any event A in $\mathbb{R}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx. \quad - \circledast$$

$f_X(\cdot)$ is called the probability density function of X.

Lemma 5.2.3:- Let X be a continuous random variable. Then for any $a \in \mathbb{R}$, $\mathbb{P}(X=a) = 0$.

Proof:-

$$\mathbb{P}(X=a) = \mathbb{P}(a \leq X \leq a)$$

$$= \int_a^a f(x)\, dx = 0$$

$\square$

Remarks:

X - discrete - $\text{Range}(X) - \{x_i\}_{i \geq 1}$; $f_X(x) = \mathbb{P}(X=x)$ p.m.f of X

X - continuous $\quad \text{Range}(X) \subseteq \mathbb{R} \quad \mathbb{P}(X \in A) = \int_A f_X(x)\, dx$

$\mathbb{P}(X=a) = 0 \quad \forall a \in \mathbb{R} \qquad f_X(\cdot)$ p.d.f of X

Distribution Function [Discrete / Continuous worlds]

Definition 5.2.4: If X is a random variable then its distribution function $F : \mathbb{R} \to [0,1]$ is defined by

$$F(x) = \mathbb{P}(X \le x). \quad \text{—— } (*)$$

$F \equiv F_X(\cdot)$ to denote its dependency on X

– X - discrete random variable

• $\text{Range}(X) = \{x_i\}_{i\ge1}$, $f_X(x_i) = \mathbb{P}(X = x_i)$ $i \ge 1$ p.m.f

$\Downarrow$ $\Uparrow$ Ex.

• $F(x) = P(X \le x) = \sum_{i : x_i \le x} \mathbb{P}(X = x_i)$ Distribution function

– X - continuous random variable, with a p.c. density function $f : \mathbb{R} \to [0,\infty)$, then

$$F(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(s)\, ds$$

F.T.o.C: F differentiable :- $F'(x) = f(x)$ $\forall\, x \in \mathbb{R}$ st f is continuous at x.

X - random variable $\longleftrightarrow$ Discrete / Continuous $\longleftrightarrow$ $F_X(x) = \mathbb{P}(X \le x)$ Distribution function

Examples :-

- $S = (a,b)$ $X \sim \text{Uniform}(a,b)$ if p.d.f of X is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}$$

Ex: $\mathbb{P}(X \in (c,d)) = \int_c^d f_X(x)\,dx = \frac{c-d}{b-a}$ $a<c<d<b$

$\propto$ length of (c,d)

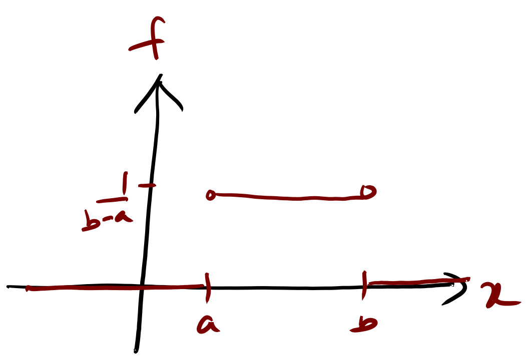
$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$$

- $x \le a$ $f_X(y) = 0 \ \forall\ y \le x \Rightarrow F_X(x) = 0$

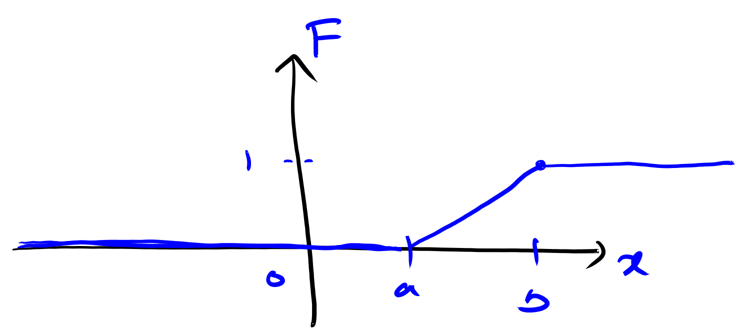
$x \ge b$ $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_a^b f_X(y)\,dy = 1$

- $a<x<b$ $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_a^x f_X(y)\,dy = \int_a^x \frac{1}{b-a}\,dy$

$$= \frac{x-a}{b-a}$$



$\longleftrightarrow$



Example: — Radioactive Isotopes — Decay to a stable form.

$N(t) \equiv$ amount of radioactive material that has not decayed by time t

$$\frac{N(t)}{N(0)} \simeq e^{-\lambda t} \quad \text{for some } \lambda > 0.$$

Model — Probabilistically ..

X — represent the time taken by a randomly chosen "radioactive atom" to decay to its stable form.

[Needs to satisfy]
$$\mathbb{P}(X \geq t) = e^{-\lambda t} \quad \forall t > 0$$

Can we have such an X?

$X \sim Exp(\lambda)$ if X has a p.d.f. given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$X \equiv$ Exponential random variable with parameter λ.

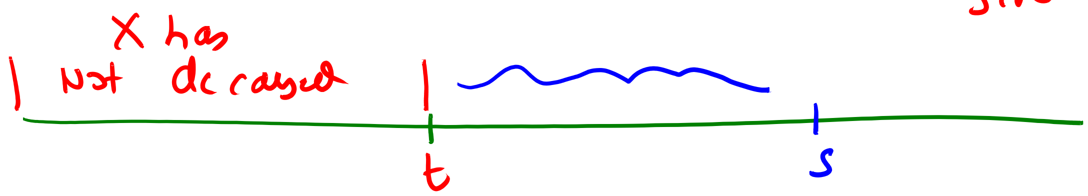
Ex: • $f_X(\cdot)$ = probability density function.

• $F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

$$\boxed{\begin{aligned} &\mathbb{P}(X > t) \\ &= 1 - F_X(t) \\ &= e^{-\lambda t} \end{aligned}}$$

$\cdot \quad \mathbb{P}(X > s+t) = \bar{e}^{\lambda(s+t)} \qquad s,t \geq 0$

$\mathbb{P}(X > s+t \mid X > t) \equiv$ Conditional probability of X - not decayed till $s+t$ given X has not decayed till t



$$= \frac{\mathbb{P}(X > s+t,\ X > t)}{\mathbb{P}(X > t)}$$

$$= \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{\lambda s}$$

$$= \mathbb{P}(X > s)$$

— memoryless property of Exponential.

Ex: - Is there X - discrete with memoryless property & is there a connection of that with exponential random variable?

Normal Random Variable :- (Gaussian Random variable)

Recall : $\cdot\ X_i \sim \text{Bernoulli}(p)$, $i \geq 1$ independent $\quad X_i \in \{0,1\}$; $\mathbb{P}(X_i = 1) = p$

$\cdot\ S_n = \sum_{i=1}^{n} X_i$ then $S_n \sim \text{Binomial}(n, p)$

$\cdot\ p \equiv p_n = \frac{\lambda}{n}$, $\quad \mathbb{P}(S_n = k) \longrightarrow \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, 2, \ldots$

[Poisson (λ)]

Q: If $\underline{0 < p < \frac{1}{2}}$ - $\underline{\text{fixed}}$.

Behaviour of $\underline{\underline{S_n}}$ as $n \to \infty$?

i.e $\mathbb{P}(a \leq S_n < b) = ?$ as $n \to \infty$.

$\underline{a \ \& \ b \text{ don't change with } n?}$

$E[S_n] = np$

$Var[S_n] = np(1-p)$

as n - gets large

$a, b \notin$ "Range (S_n)" $\equiv \left(-\underbrace{\sqrt{np(1-p)}}_{} + \underbrace{np}_{},\ \underbrace{\sqrt{np(1-p)}}_{} + \underbrace{np}_{}\right)$

SD $[S_n]$ $\qquad$ E $[S_n]$

• Meaningful question: $a, b \in \mathbb{R}$ $a < b$

$$\mathbb{P}\left(S_n \in \left(a\sqrt{np(1-p)} + np \, , \, np + b\sqrt{np(1-p)}\right)\right) \to \, ?$$

as $n \to \infty$

i.e. $$\mathbb{P}\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right) \longrightarrow \, ?$$

as $n \to \infty$

$$A_n = \{ k : np + a\sqrt{np(1-p)} \le k < np + b\sqrt{np(1-p)} \}$$

$$\mathbb{P}\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right) = \sum_{k \in A_n} \mathbb{P}(S_n = k)$$

$$= \sum_{k \in A_n} \binom{n}{k} p^k (1-p)^{n-k}$$

• (nontrivial Ex.) $\doteq$ . . . stirlings formula . . . Rieman integration

$$\xrightarrow[n \to \infty]{} \int_a^b \underbrace{\frac{e^{-x^2/2}}{\sqrt{2\pi}}}_{\text{Gaussian density}} dx$$

Central limit Theorem:- The above phenomena is universal - last week's claim

Normal Random Variable

$X \sim \text{Normal}(\mu, \sigma^2)$ $\mu \in \mathbb{R}$ $\sigma > 0$.

If X has the probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; x \in \mathbb{R}$$

In above calculation: $\mu = 0$, $\sigma = 1$

ie $X \sim N(0,1)$

$$\mathbb{P}(a \leq X < b) = \underbrace{\int_a^b \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy}$$

No Anti Derivative ← integral has to be evaluated numerically