Recall:- X, Y - two continuous random variables

- $f: \mathbb{R}^2 \to [0,\infty)$; $f(x,y) \equiv$ p.c. & $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

joint p.d.f.

$$\mathbb{P}((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

joint distribution function:

$$F(a,b) = \mathbb{P}(X \le a, Y \le b)$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

Marginal density

- $f_X(a) = \dfrac{\partial}{\partial a} F(a,b)$

- $f_Y(b) = \dfrac{\partial}{\partial b} F(a,b)$

Examples: $D \subseteq \mathbb{R}^2$ with finite area

5.4.3. $f(x,y) = \begin{cases} \dfrac{1}{|D|} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$ $(X,Y) \sim \text{Uniform}(D)$

5.4.5. $f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

- Marginal density — Independence

5.4.1 Marginal density

(X,Y) are random variables & have a joint probability density function $f : \mathbb{R}^2 \to [0,\infty)$

- $\mathbb{P}(X \le a) = \mathbb{P}(X \le a, -\infty < Y < \infty)$

$$= \int_{-\infty}^{a} \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\, dy \right]}_{g(x)} dx$$

$g : \mathbb{R} \to [0,\infty)$ given by $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$\Rightarrow \quad \mathbb{P}(X \le a) = \int_{-\infty}^{a} g(x)\, dx$$

$$\Rightarrow \quad f_X(a) = g(a) \quad - \left[g\text{ - continuous if} \right]$$

$$= \int_{-\infty}^{\infty} f(a,y)\, dy \quad \forall a \in \mathbb{R}$$

Marginal density

Similarly:

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\, dx \quad \forall b \in \mathbb{R}$$

5.4-2 Independence

X, Y are two random variables

Recall: They are independent if

$$P(X\in A, Y\in B) = P(X\in A)\, P(Y\in B)$$

$$\Updownarrow$$

$$\mathbb{P}(X\le x, Y\le y) = P(X\le x)\, P(Y\le y)$$

i.e.

$$F(x,y) = F_X(x)\, F_Y(y) \quad \text{— } (\ddagger)$$

$$\forall x,y\in\mathbb{R}$$

If (X,Y) are independent then $(\ddagger) \Rightarrow$

$$f(x,y) = f_X(x)\, f_Y(y) \quad \forall x,y\in\mathbb{R}$$

[f, f_X, f — continuous; differentiate in each variable]

Conversely: let X have pdf $f_X(\cdot)$, Y have pdf $f_Y(\cdot)$. Suppose $f(\cdot,\cdot) = f_X(\cdot)f_Y(\cdot)$ is joint density

$$\mathbb{P}(X\in A, Y\in B) = \int_B\int_A f(x,y)\,dx\,dy = \int_B\int_A f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_A f_X(x)\,dx \cdot \int_B f_Y(y)\,dy = \mathbb{P}(X\in A, Y\in B)$$

for all events A & B.

$\Rightarrow X, Y$ are independent. ▯

Remark:

X, Y are independent continuous random variables $\Longleftrightarrow$ (joint density of (X,Y) = marginal density of X $\times$ marginal density of Y)

Example 5.4.3 Contd.

$D = (0,1) \times (3,5)$ — Ex $(X,Y) \sim \text{Uniform}(D)$

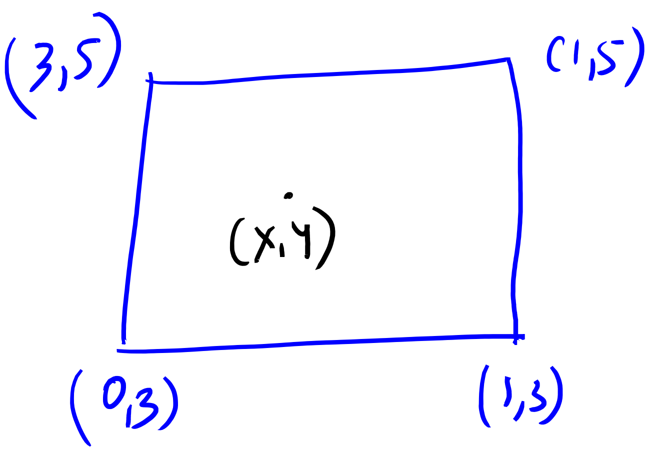
$$f(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \; ; \; f_Y(y) = \begin{cases} 1/2 & y \in (3,5) \\ 0 & \text{otherwise} \end{cases}$$

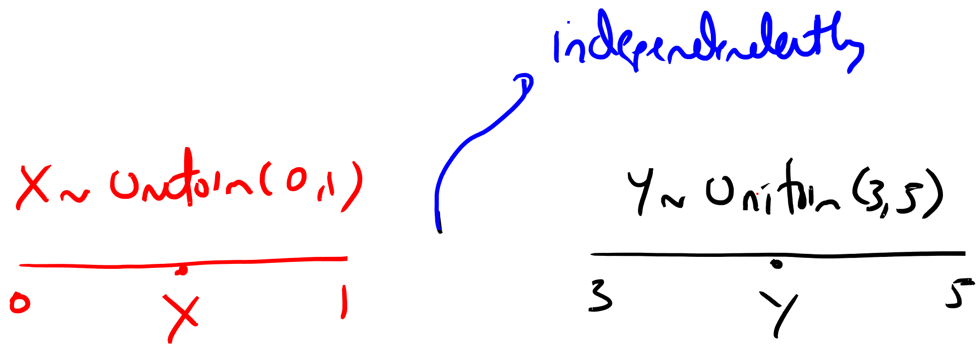
$$\Rightarrow f(x,y) = f_X(x)\, f_Y(y)$$

$X \sim \text{Uniform}(0,1)$
$Y \sim \text{Uniform}(3,5)$

$\Rightarrow (X,Y)$ are independent



$\equiv$



$(X,Y) \sim \text{Uniform}(D)$

Is this always true ?

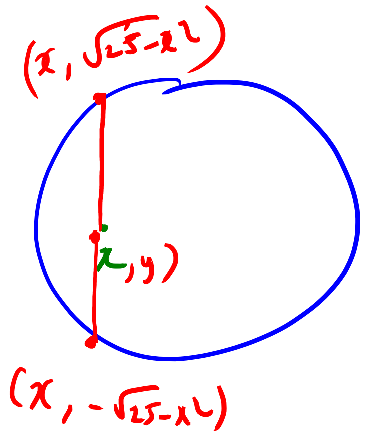
$$D = \{ (x,y) : x^2 + y^2 < 25 \}$$

$$|D| = 25\pi \qquad (X,Y) \sim \text{Uniform}(D)$$

$$f(x,y) = \begin{cases} \dfrac{1}{25\pi} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

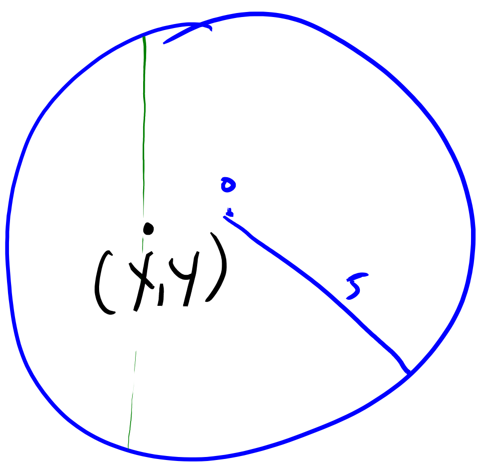
Ex:-

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \begin{cases} \displaystyle\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} f(x,y)\, dy & -5 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$



$$= \begin{cases} \dfrac{2}{25\pi}\sqrt{25-x^2} & -5 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Symmetry: $f_Y(y) = \begin{cases} \dfrac{2}{25\pi}\sqrt{25-y^2}, & -5 < y < 5 \\ 0 & \text{otherwise} \end{cases}$



$(X,Y) \sim \text{Uniform}(D)$

(X,Y) are

$\not\equiv$

not independent

$$f_X(0) = \frac{2}{25\pi}$$

$$f_Y(0) = 2/25\pi$$

$$f(0,0) = \frac{1}{25\pi} \neq f_X(0)\, f_Y(0)$$

5.4.3 Conditional density

X, Y - are two continuous random variables

$f(\cdot,\cdot)$ - joint density of X and Y

$f_X(\cdot)$ - marginal density of X

$f_Y(\cdot)$ - marginal density of Y.

Independence $\iff$ $f(a,b) = f_X(a) f_Y(b)$ $\forall a,b \in \mathbb{R}$

Suppose: A - event and $P(A) > 0$. For any event B

$Q(X \in B) := P(X \in B \mid A) \equiv$ Conditional distribution of X given the event A.

- In above case; $A = \{Y = b\}$ for some $b \in \mathbb{R}$.

  Y - continuous $\mathbb{P}(A) = 0$.

- Suppose $f_Y(b) > 0$ - given. $\equiv$ What is the implication of this in

$$\mathbb{P}(X \in [3,4] \mid Y = b) \quad ?$$

Heuristic Calculation: $b \in \mathbb{R}$, $f_Y(b) > 0$, $f_Y(\cdot)$ - Continuous

- There is $n \in \mathbb{N}$ $f_Y(y) > 0$ $\forall y \in (b - \frac{1}{n}, b + \frac{1}{n})$

$$\Rightarrow \mathbb{P}(Y \in [b - \tfrac{1}{n}, b + \tfrac{1}{n}]) = \int_{b-\frac{1}{n}}^{b+1/n} f_Y(y)\,dy > 0$$

- $A^n = (b - \frac{1}{n} < Y < b + \frac{1}{n})$ has $\mathbb{P}(A^n) > 0$

$$\mathbb{P}(X \in [3,4] \mid A^{(n)}) = \frac{\mathbb{P}(X \in [3,4], A^n)}{\mathbb{P}(A^n)}$$

$$= \frac{\mathbb{P}(X \in [3,4], b - \frac{1}{n} < Y < b + \frac{1}{n})}{\mathbb{P}(b - \frac{1}{n} < Y < b + \frac{1}{n})}$$

$$= \frac{\int_3^4 \left[\int_{b-1/n}^{b+1/n} f(x,y)\,dy \right] dx}{\int_{b-1/n}^{b+1/n} f_Y(y)\,dy}$$

$n \to \infty$. $A^n = \left(b - \frac{1}{n} < Y < b + \frac{1}{n} \right)$

$\longrightarrow \quad (Y = b) = A$

. $2n \int_{b-1/n}^{b+1/n} f_Y(y)\, dy \longrightarrow f_Y(b)$

. $2n \int_{b-1/n}^{b+1/n} f(x, y)\, dy \longrightarrow f(x, b)$

Calculus results

. $\mathbb{P}\left(X \in [3,4] \mid Y = b \right)$ ________

"="

$$\lim_{n \to \infty} \mathbb{P}\left(X \in [3,4] \mid b - \tfrac{1}{n} < Y < b + \tfrac{1}{n} \right)$$

$$= \lim_{n \to \infty} \frac{\int_3^4 \left[2n \int_{b-1/n}^{b+1/n} f(x, y)\, dy \right] dx}{2n \int_{b-1/n}^{b+1/n} f_Y(y)\, dy}$$

$$= \frac{\int_3^4 f(x, b)\, dx}{f_Y(b)} = \int_3^4 \frac{f(x, b)}{f_Y(b)}\, dx$$

Definition 5.4.11. Let (X,Y) be random variables having joint density f. Let the marginal density of Y be $f_Y(\cdot)$. Suppose $b \in \mathbb{R}$ such that $f_Y(b) > 0$. and $f_Y(\cdot)$ is continuous at b. Then the conditional density of X given $Y=b$ is given by

$$f_{X|Y=b}(x) = \frac{f(x,b)}{f_Y(b)} \quad \text{for all } x \in \mathbb{R}.$$

Similarly $f_X(\cdot)$ marginal density of X was such that $f_X(a) > 0$ and $f_X(\cdot)$ was continuous at a. Then the conditional density of $Y|X=a$ is given by

$$f_{Y|X=a}(y) = \frac{f(a,y)}{f_X(a)} \quad \forall\ y \in \mathbb{R}$$

$$\mathbb{P}(X \in A \mid Y=b) = \int_A f_{X|Y=b}(x)\,dx$$

$$\mathbb{P}(Y \in B \mid X=a) = \int_B f_{Y|X=a}(y)\,dy$$