Recall: $X: S \to \mathbb{R}$ S – uncountable

– continuous random variable

pdf $f_X : \mathbb{R} \to [0,\infty)$

f_X – p.c. $\int_{\mathbb{R}} f_X(x)\,dx = 1$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\,dx$$

$A \in \mathcal{F}$ – event

$$Y = h(X) \equiv \text{Distribution of } Y$$

$F_X(\cdot)$ distribution function of X

$F_Y(\cdot)$ distribution function of Y

$f_Y(\cdot) \equiv F_Y'(\cdot)$

5.4 Multiple Continuous Random Variables

- analysing two continuous random variables – "joint density"
  - analogue – "joint distribution" of discrete random variables. $[\mathbb{P}(X=x, Y=y) \ldots]$

generalised to $n \geq 2$ – focus on 2 first.

Theorem 5.4.1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be non-negative function.

f – piecewise continuous in each variable for which

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

For any event $A \subseteq \mathbb{R}^2$ [Borel sets] define

$$\mathbb{P}(A) = \int_A f(x,y)\,dx\,dy.$$

Then $\mathbb{P}$ is a probability on $\mathbb{R}^2$ & f is called density for $\mathbb{P}$.

Proof: Ex. $\square$

Definition 5.4.2 A pair of random variables (X,Y) is said to have joint density $f(x,y)$ if for every Borel set $A \subseteq \mathbb{R}^2$

$$\mathbb{P}((X,Y) \in A) = \int_A f(x,y)\,dx\,dy.$$

Remark: "Borel set" – for our consideration – It is enough

$$A = (-\infty, a] \times (-\infty, b] \qquad a, b \in \mathbb{R}$$

Joint distribution function of (X,Y) :-

$$F(a,b) \equiv F_{(X,Y)}(a,b) = \mathbb{P}(X \leq a, Y \leq b)$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$$

for all $a, b \in \mathbb{R}$

[$F \to f$
standard multivariate calculus]

Example 5.4.3

$R = (0,1) \times (3,5)$ in $\mathbb{R}^2$

- area of $R = (1-0) \times (5-3) = 2 :\equiv |R|$

- Define: $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$

Ex: f is a joint pdf.

(X,Y) with joint pdf

$$P((X,Y) \in (a,b) \times (c,d)) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

if $(a,b) \times (c,d) \subseteq R$

$$= \frac{1}{2}(d-c)(b-a)$$

"Uniform (R)"

$$= \frac{|(a,b) \times (c,d)|}{|R|}$$

Definition 5.4.4:

Let $D \subseteq \mathbb{R}^2$ be non-empty with positive area. Then $(X,Y) \sim \text{Uniform}(D)$ if it has a joint probability density function given by $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{1}{|D|} & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

(X,Y) - chooses a point uniformly in D

where $|D|$ - denotes the area of D.

Example 5.4.5 (X,Y) have joint density $f:\mathbb{R}^2\to\mathbb{R}$

given by

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

① $f(\cdot,\cdot) \geq 0$, continuous (piecewise)

② $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + yx \Big|_0^1 \right] dy$$

$$= \int_0^1 \left[\frac{1}{2} + y\right] dy$$

$$= \frac{1}{2}y + \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}+\frac{1}{2} = 1$$

$A = (0,\tfrac{1}{2}) \times (0,\tfrac{1}{2})$ [If $(X,Y) \sim \text{Uniform}((0,1)\times(0,1))$ then $\mathbb{P}((X,Y)\in A) = \frac{|A|}{1} = \frac{1}{4}$]

$$\mathbb{P}((X,Y)\in A) = \int_0^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_0^{1/2} [x+y]\,dx\,dy$$

$$\overset{Ex}{=} 1/8$$

Understand Behaviour of X - alone :

$x \in \mathbb{R}$ $\quad \mathbb{P}(X \le x) \equiv$ Calculate distribution function of X

$$\mathbb{P}(X \le x) = \mathbb{P}(X \le x, -\infty < Y < \infty) \qquad (\mathbb{P}(-\infty < Y < \infty) = 1)$$

$$= \mathbb{P}(-\infty < X \le x, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx$$

① $x < 0 \Rightarrow f(x,y) = 0 \ \forall\ y \in \mathbb{R}$

$$\therefore \mathbb{P}(X \le x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} 0 \ dy\, dx = 0$$

② $0 < x < 1 \Rightarrow f(x,y) = x + y$ if $0 < y < 1$ & 0 otherwise

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,y)\, dy\, du$$

$$= \int_{0}^{x} \int_{0}^{1} (u+y)\, dy\, du$$

$$= \int_{0}^{x} \left[uy + \frac{y^2}{2} \Big|_0^1 \right] du = \int_{0}^{x} \left(u + \frac{1}{2}\right) du$$

$$= \frac{u^2}{2} + \frac{u}{2} \Big|_0^x = \frac{x^2}{2} + \frac{x}{2}$$

$x \geq 1 \qquad \mathbb{P}(X \leq x) = 1$ [Ex.]

$$\equiv \quad f_X(x) = \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$