Central limit Theorem :

Q: $X_1, X_2, \ldots X_n$ i.i.d , $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\qquad \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \to ?$

$\mu = E X_1$ $\quad \sigma^2 = Var[X_1]$

Definition 8.3.1 A sequence $X_1, X_2, \ldots$ is said to converge in distribution to a random variable X if

$F_{X_n}(x)$ Converges to $F_X(x)$ at every point x for which $F_X(\cdot)$ is continuous.

The following notation

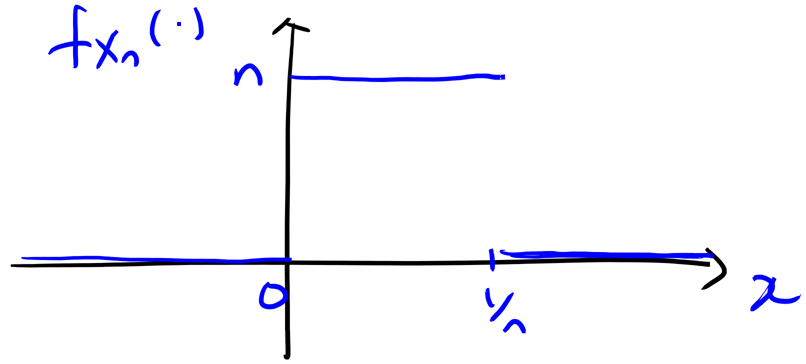
$$X_n \xrightarrow{d} X$$

is typically used to convey that the sequence $X_1, X_2, \ldots$ Converges in distribution to X.

Example 8.3.2 :

$$X_n \sim \text{Uniform}(0, \tfrac{1}{n})$$

ie p.d.f $g_{X_n} \equiv f_{X_n}(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$



Distribution function of X_n

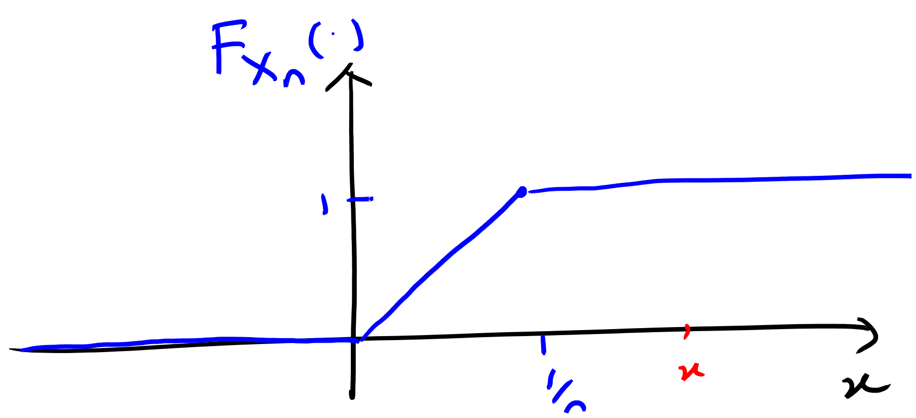
$$F_{X_n}(x) = \int_{-\infty}^{x} f_{X_n}(y)\,dy$$

$x \le 0$, $\quad F_{X_n}(x) = 0$

$0 < x < \frac{1}{n}$, $\quad F_{X_n}(x) = \int_0^x n\,dy = nx$

$x \geq \frac{1}{n}$ $\quad F_{X_n}(x) = \int_0^{1/n} n\,dy = 1$

$$\Rightarrow \quad F_{X_n}(x) = \begin{cases} 0 & x \le 0 \\ nx & 0 < x < \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases}$$



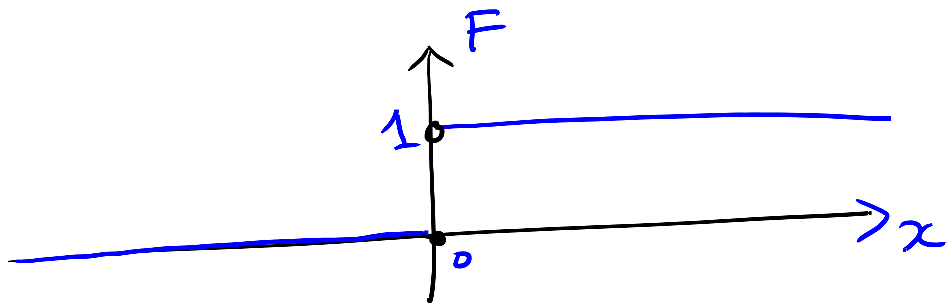
$F_{X_n}(x) \longrightarrow 1 \quad x > 0$ as $n \to \infty$

$F_{X_n}(x) \longrightarrow 0 \quad x \le 0$ as $n \to \infty$

let
$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

$$F_{X_n}(\cdot) \longrightarrow F(\cdot)$$

What random variable X has distribution function F?



let X be a random variable such that

$X = 0$ w.p. 1

$$\Rightarrow \quad F_X(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$F_{X_n}(x) \longrightarrow F_X(x)$$ for all continuity points of $F_X(\,)$

$\therefore \quad X_n \xrightarrow{d} X$

Note :- $P(X_n = 0) = 0 \quad \not\longrightarrow \quad P(X = 0) = 1$

X_n - continuous $\qquad$ X - discrete

Example: $S_1, S_2, \ldots S_n$ are Binomial $(n, p/n)$

$$\forall k \geq 0 \qquad \mathbb{P}(S_n = k) = \binom{n}{k}\left(\frac{p}{n}\right)^k \left(1 - p/n\right)^{n-k}$$

$$\Big\downarrow \text{ as } n \to \infty$$

$$S \sim \text{Poisson}(p) \qquad P(S = k) = \frac{e^{-p} p^k}{k!}$$

$$F_{S_n}(x) = \sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} (p/n)^k (1 - p/n)^{n-k}$$

$$\Big\downarrow \; n \to \infty$$

$$= \sum_{k=0}^{\lfloor x \rfloor} \frac{e^{-p} p^k}{k!}$$

$$= F_S(x) \qquad \forall x \in \mathbb{R}$$

i.e. $\text{Binomial}(n, p/n) \xrightarrow{d} \text{Poisson}(p)$

Definition 8.2.3 A sequence $X_1, X_2, \ldots$ is said to converge in Probability to a random variable X if $\forall \, \varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$$

The following notation

$$X_n \xrightarrow{p} X$$

is typically used to convey that the sequence $X_1, X_2, \ldots$ converges in probability to X.

Ex:- "stronger" notion of convergence than $X_n \xrightarrow{d} X$

Hint:- $X_n \sim \text{uniform}(0, \frac{1}{n})$ $X = 0$ w.p. 1

Example 8.2.4 $X_i \sim$ Uniform $(0,1)$ i.i.d random variables.

$$f_{X_1}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad , \quad E X_1 = 1/2$$

$\varepsilon > 0$

[WLLN] $\mathbb{P}\left(|\bar{X} - \tfrac{1}{2}| > \varepsilon \right) \longrightarrow 0$ as $n \to \infty$

$$\bar{X} \xrightarrow{p} \frac{1}{2}$$

$$\boxed{Y = \frac{1}{2} \text{ w.p. } \frac{1}{2} \qquad \bar{X} \xrightarrow{p} Y}$$

$\forall n \geq 1$, $\quad X_{(n)} = \max(X_1, X_2, \ldots X_n)$

Intuition :- "$X_{(n)} \longrightarrow 1$"

claim :- $\quad X_{(n)} \xrightarrow{p} X \qquad X = 1$ w.p 1

$\varepsilon > 0$ be given.

$\boxed{\text{WLOG } \varepsilon < 1}$

$$\mathbb{P}(|X_{(n)} - X| > \varepsilon)$$

$$= \mathbb{P}(|X_{(n)} - 1| > \varepsilon)$$

$$= \mathbb{P}(X_{(n)} > 1 + \varepsilon \cup X_{(n)} < 1 - \varepsilon)$$

$$= \mathbb{P}(X_{(n)} > 1 + \varepsilon) + \mathbb{P}(X_{(n)} < 1 - \varepsilon)$$

$\mathbb{P}(X_{(n)} > 1 + \varepsilon) = 0$ ie $X_n \sim \text{uniform}(0,1)$

$$+ \mathbb{P}(\max(X_1 \ldots X_n) < 1 - \varepsilon)$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} (X_i < 1 - \varepsilon)\right)$$

$$\overset{\text{independence}}{=} \prod_{i=1}^{n} \mathbb{P}(X_i < 1 - \varepsilon)$$

$$\boxed{F_{X_i}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}} \Rightarrow = \prod_{i=1}^{n} (1 - \varepsilon)$$

$$= (1 - \varepsilon)^n$$

$$\Rightarrow \quad \mathbb{P}\left(|X_{(n)} - X| > \varepsilon \right) \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\therefore \quad X_{(n)} \xrightarrow{p} X \quad \text{as } n \to \infty \qquad \square$$

Definition 8.1.3 $X_1, \ldots X_n, \ldots$ - a sequence of random variables are said to converge w.p.1 or almost everywhere to a random variable X if

$$\mathbb{P}\left(\lim_{n \to \infty} X_n = X \right) = 1.$$

Example. A - event of interest $\mu = \mathbb{P}(X \in A)$

$X_1, X_2, \ldots X_n$ i.i.d with distribution X

Define
$$Y_n = \begin{cases} 1 & \text{if } X_n \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{\# \{X_i \in A\}}{n}$$

SLLN $\quad \bar{Y}_n \xrightarrow{a.e} \mu \quad \text{as } n \to \infty$

WLLN $\quad \bar{Y}_n \xrightarrow{p} \mu \quad \text{as } n \to \infty$

[Relative frequency
↓
Probability]

Remarks: $\{X_n\}_{n\geq 1}$ is a sequence of random variables

- $\underbrace{X_n \xrightarrow{d} X}_{\text{metrizable}}$. $F_{X_n}(x) \longrightarrow F_X(x)$ for all x - cts pt of $F_X(\cdot)$

$\not\Downarrow \qquad \Uparrow$

- $\underbrace{X_n \xrightarrow{p} X}_{\text{metrizable}}$. $\forall \varepsilon > 0 \quad \mathbb{P}(|X_n - X| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

$\not\Downarrow \qquad \Uparrow$

- $\underbrace{X_n \xrightarrow{a.e.} X}_{\text{topology is not metrizable}}$. $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$

Ex.

Moment generating function

let X be a random variable. For $t \in \mathbb{R}$,

define
$$M_X(t) = E[e^{tX}]$$

- M_X - may not be defined for all $t \in \mathbb{R}$

$D = \{ t \in \mathbb{R} \mid M_X(t) < \infty \}$

On D :-

" Differentiate in t "

- $M_X'(t) = \frac{d}{dt} E[e^{tX}]$

D.C.T. Needs to be Established

$= E[\frac{d}{dt} e^{tX}]$

$= E[X e^{tX}]$

- $M_X^{(k)}(t) = E[X^k e^{tX}]$

$$M_X^{(k)}(0) = E[X^k]$$

k th moment of X

- X – random variable

$$m_k = E[X^k] < \infty \qquad k \geq 1$$

$\{m_k\}_{k \geq 1}$ is known

If ~~Carleman condition~~ holds then

the distribution of X is known.

Carleman Condition $\Longleftarrow$ $E[e^{tX}]$ exist in an interval around 0..

$M_X(\cdot)$ is defined around 0 $\Longrightarrow$ X – distribution can be characterized

. $X_1, X_2 .. X_n$ are independent

$$S_n = \sum_{i=1}^{n} X_i$$

when well defined – $M_{S_n}(t) = E\left[e^{t \sum_{i=1}^{n} X_i}\right] =$

$$= E\left[\prod_{i=1}^{n} e^{t X_i}\right]$$

$$= \prod_{i=1}^{n} m_{X_i}(t)$$

• **Theorem 8.3.4** If $X_1, X_2, \ldots$ are a sequence of random variables whose moment generating function $M_{X_n}(\cdot)$ exist around a neighbourhood of 0 & if

$$M_{X_n}(t) \longrightarrow M_X(t)$$

where $M_X(\cdot)$ is moment generating function of a random variable X, then $X_n \xrightarrow{d} X$.

BROAD IDEA of Central limit Theorem Proof

$X_1, \ldots X_n$ i.i.d X

$$\frac{\sqrt{n}\,(\bar{X} - \mu)}{\sigma} = Y_n$$

moment generating function exist

$$M_{Y_n}(t) = E\, e^{Y_n(t)}$$

identically distributed & independence & above

$$= \left(E\left[e^{t\frac{(X-\mu)}{\sigma\sqrt{n}}}\right]\right)^n$$

$$\Big\downarrow\ n \to \infty$$

Taylor approximation

$$= e^{t^2/2} = M_Z(t)$$

$Z \sim \text{Normal}(0,1)$

□