Basic Concepts in Probability

Question:- What is the likelihood of rain tomorrow?

Ans:- The chance of rain tomorrow is 10%.

Framework:- What all can occur?

Definition:- [Sample Space] A sample space S is a set. The elements of S will be called outcomes and should be viewed as listing of all

— Sample Space —

possibilities that can occur. We will call the process of actually selecting one of these outcomes an "experiment"

Example:- ⓐ Roll a die and note down the outcome of the top face.

$$S = \{1, 2, 3, 4, 5, 6\}$$

ⓑ Toss a coin and note down the outcome of the toss.

$$S = \{\text{Heads}, \text{Tails}\}.$$

ⓒ Foot ball world cup and we are interested in the event that Brazil wins the world cup.

$$S = \{\text{list of all countries playing in the world cup}\}$$

In all above examples we may be interested in a specific outcome or a specific collection of outcomes.

Definition [Temporary Definition] Given a sample S, an event E is any subset of S.

— Event —

Example w.r.t (a) :-

$$E = \{\text{outcome is an odd number}\}$$

$$= \{1, 3, 5\} \subseteq S.$$

Remarks :- • ϕ is an event. [Nothing has happened]

• S is an event [any possibility]

• Specific / Correct definition of event.

[Give later / — Distracting now]

— Probability —

To each event, we want to assign a chance or likelihood or "Probability". This will be a number between 0 and 1.

Intuitive understanding :- Probability of an event E is 0.25 — our interpretation

If we perform the Experiment 100 time E would occur 25 of these times.

A. Kolmogorov - definition of Probability

Definition [Probability Space Axioms]

Let S be a sample space.

Let $\mathcal{F}$ be collection of all events.

A probability is a function

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ such that

① $\mathbb{P}(S) = 1$

② $E_1, E_2, \ldots$ are a countable collection of disjoint events (i.e. $E_i \cap E_j = \phi$ for $i \neq j$) then

$$\mathbb{P}\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\{s \in S \mid s \in E_i \text{ for some } i\}}\Big) = \underbrace{\sum_{j=1}^{\infty} \mathbb{P}(E_j)}_{\text{Series converges}}$$

Exchangeability :-

Experiment :- Sample n balls from a bag without replacement. The bag has M-balls. Let w of them be red. $[n < M]$

$$A_k = \{\text{the } k^{th} \text{ ball drawn is white}\}$$

Show : $\mathbb{P}(A_k) = \dfrac{w}{M}$ $\quad 1 \le k \le n$ $\quad - (*)$

Fix n, w, M

Questions :- • Define S for above Experiment.

• $\mathcal{F} = \mathcal{P}(S)$ • Define $\underbrace{\mathbb{P} : \mathcal{F} \to [0,1]}_{\text{Probability}}$ so that $(*)$ holds.

Question :- Provide an [not previous] example $(S, \mathcal{F}, \mathbb{P})$ and a sequence of events $A_1, A_2, \ldots, A_k$ $\quad k \ge 1$

$\sigma : \{1, \ldots, k\} \to \{1, \ldots k\}$ is a permutation

If $X_i = \mathbb{1}_{A_i}$ then $(X_1, \ldots X_k) \overset{d}{=} (X_{\sigma(1)}, \ldots, X_{\sigma(k)})$