Basic Probability :-

- Background in Probability for many of you.

(8) - Assignment [Comprehensive - 1] :- indicates some proficiency

- Measure Theoretic Probability [Semester 1]

- All of us have "intuitive" understanding of Probability

× ——————— × ——————— ×

Measure Theory class [Done ?]

Ω - nonempty Set [finite, Countable, uncountable]

$\mathcal{F}$ - σ-field or σ-algebras

$A \in \mathcal{F}$ $\quad A \subseteq \Omega$

Satisfies specified rules

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ - Probability

- satisfies specified rules

Measure Theory

Course :- $(\Omega, \mathcal{F}, \mathbb{P})$

- Existence of $\mathbb{P}$
- Construct various $\mathbb{P}$ $\rightsquigarrow$

starting

E.g :- $\Omega = [0,1]$

idea of length of a set $A \subseteq [0,1]$

Many examples :- $\Omega = [0,1]$ $\mathcal{F}$ - Borel σ-algebra

$\mathbb{P} : \mathcal{F} \to [0,1]$

$\mathbb{P}(A) = \int_A f(x)\, dx$ where

$f : [0,1] \to \mathbb{R}$, $f \geq 0$

$\int_0^1 f(x) = 1$

Extend: $\Omega = \mathbb{R}$ ... length .. $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P}(A) = \int_A f(x)\, dx$ $f : \mathbb{R} \to [0,\infty)$

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

dx length measure

Motivation / Applications :

- integration — .. L_p spaces — functional analysis

Revision keys :-

- Monotone Convergence Theorem
- Fatou's lemma
- Dominated Convergence Theorem
- Fubini / Tonelli Theorem

Statement :- Examples ?

identify hypothesis ✗ understand conclusion ✓

- Proof
- Applications , Generalisations

Basic Probability in "Real life"

- statement: There is 10% chance of Rain in Hyderabad on 3-september.

$\underline{Q}$ $\boxed{?}$ $(\Omega, \mathcal{F}, \mathbb{P})$ $\quad A \in \mathcal{F} \quad P(A) = 0.1$?

- statement: There is a fair coin in my pocket. we will toss the coin 5 times. If Heads turns up more than 3 times then I will give all of you 10 points.

- Idea: atleast 3 Heads in 5 tosses

$$P(10 \text{ points}) = {}^5C_3 \left(\tfrac{1}{2}\right)^5 + {}^5C_4 \left(\tfrac{1}{2}\right)^5 + {}^5C_5 \left(\tfrac{1}{2}\right)^5$$

Driving "intuitive" force :- Relative frequency. $\longrightarrow$ Probability as
- Experiment.
- outcomes
- Events
- Probability

$$\frac{\text{\# of times outcome occurs}}{\text{\# of trials.}}$$

End of Course :- Ensure - on same page
- $(\Omega, \mathcal{F}, \mathbb{P})$
- Relative frequency
- $\boxed{\text{Connect the two.}}$

Reading Topics of Interest :-

- Motivation :- keep "interest alive" & save "boredom" from learning interference.

Question 1 :-

M - balls in a bag distinguishable

Equally likely

W - of them are purple

M − W - ... are red.

- choose balls without replacement

A_k = k^{th} ball chosen is purple.

$P(A_1) = ?$ $P(A_2) = ?$, .. $P(A_k) = ?$

$P(A_1) = \frac{W}{M}$ ✓

$P(A_2) = P(A_2 \cap A_1) + P(A_2 \cap A_1^c)$

$= \cdots = \frac{W}{M}$

It is true : $P(A_k) = \frac{W}{M}$.

Exchangeability :-

$M = 10$ $W = 4$

Example of draws :- • • • • • • ✓• • • • •

A_6 complete 4 more draws to empty the bag.

Realisation • • • • • • • • • •

Idea:- $\mathbb{P}($ any particular sequence of red and purple in 10 draws $) = \frac{1}{10!}$

$$\equiv \quad X_i = \begin{cases} 1 & \text{if } i^{th} \text{ ball is purple} \\ 0 & \text{otherwise} \end{cases}$$

Then $(X_1, \ldots, X_{10})$ distribution is invariant under permutation

$$\Rightarrow \quad X_i \overset{d}{=} X_1 \qquad i = 1, \ldots 10$$

$$\Rightarrow \quad E X_1 = E X_i$$

$$\Rightarrow \quad P(A_1) = P(A_i)$$

$$\Rightarrow \quad \frac{w}{m} = \mathbb{P}(A_i) \quad \checkmark \quad \square$$

$\tilde{Q}$:- Think of Experiments where Probability of sequence of events are invariant under permutation.

$\tilde{Q}$: Suppose $(X_1, \ldots X_n)$ $n \geq 1$ $X_i : \Omega \to \mathbb{R}$

$(\Omega, \mathcal{F}, \mathbb{P})$ $\qquad$ Range$(X_i) = \{0, 1\}$

$$\mathbb{P}(X_1 = x_1, \ldots X_n = x_n) = \mathbb{P}(X_1 = x_{\sigma(1)} \ldots X_n = x_{\sigma(n)})$$

$\sigma : \{1 \ldots n\} \longrightarrow \{1, \ldots n\}$ Permutation

$x_i \in \{0, 1\}$

then $X_i \overset{d}{=} X_1 \quad \forall\, i = 2, \ldots, 10$.

— Exchangeable —

$$\mathbb{P}(X_i = y) = \mathbb{P}(X_1 = y) \qquad y \in \{0, 1\}.$$