Theorem 1.3.5 : let A be an event in sample with Probability P. let $\{B_i : 1 \le i \le n\}$ be a disjoint collection of events for which $P(B_i) > 0$ for all i.

Assume $A \subseteq \bigcup_{i=1}^{n} B_i$

Suppose $P(B_i)$ and $P(A|B_i)$ are known.

Then $P(A)$ may be computed as

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$$

Proof :- $\{B_i\}_{i=1}^{n}$ are a disjoint sequence of events

$\{A \cap B_i\}_{i=1}^{n}$ are also a disjoint sequence of events

[(2) Thm 1.1.4] $$P(A) = P\left(\bigcup_{i=1}^{n} A \cap B_i\right) = \sum_{i=1}^{n} P(A \cap B_i) \quad \text{—} (*)$$

By def: $$P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)}, \quad \text{for all } i$$

$$\Rightarrow P(A \cap B_i) = P(A|B_i)\, P(B_i)$$

Apply in $(*)$ to get

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i) \qquad \square$$

Example 1.3.6 :-

	Box 1	Box 2	Box 3
Red	4	3	3
Green	3	3	4
Blue	5	2	3

Choose a box at random. From that box choose a ball at random. [Equally likely outcome]

Q: What is the probability that a red ball is chosen?

$R = \{$ that a red ball is chosen $\}$
$B_i = \{$ that Box i is chosen $\}$ for $i = 1, 2, 3$.

$P(R|B_1) = \frac{4}{12}$, $P(R|B_2) = \frac{3}{8}$, $P(R|B_3) = \frac{3}{10}$

$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$.

$R \subseteq \bigcup_{i=1}^{3} B_i$ So apply Theorem 1.3.5

$$P(R) = \sum_{i=1}^{3} P(R|B_i)\, P(B_i)$$

$$= \frac{4}{12} \cdot \frac{1}{3} + \frac{3}{8} \cdot \frac{1}{3} + \frac{3}{10} \cdot \frac{1}{3}$$

$$= \frac{121}{360}$$

D

Given that Red ball is chosen find the probability that it was chosen from Box 3.

ie Find $P(B_3 \mid R)$?

$$P(B_3 \mid R) = \frac{P(B_3 \cap R)}{P(R)}$$

$$= \frac{P(R \cap B_3)}{P(R)} = \frac{P(R \mid B_3)\, P(B_3)}{P(R)}$$

$$= \frac{3/10 \cdot \frac{1}{3}}{121/360} = \frac{36}{121} \simeq 0.298$$

$\therefore$ The conditional probability of that B_3 occurred given that R occurred is $\sim 30\%$.

$$P(B_3 \mid R) = \boxed{\frac{P(R \mid B_3)\, P(B_3)}{P(R)}} = \frac{P(R \mid B_3)\; P(B_3)}{\sum_{i=1}^{3} P(R \mid B_i)\, P(B_i)}$$

Earlier step

Theorem 1.3.11 :- (Bayes Theorem) Suppose A is an event. $\{B_i : 1 \le i \le n\}$ are a collection of events whose union contains A. i.e. $A \subseteq \bigcup_{i=1}^{n} B_i$.

Assume $P(A) > 0$ & $P(B_i) > 0$ $i = 1, \ldots, n$.

Then for any $1 \le i \le n$

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)} \qquad 1 \le i \le n$$

Proof :- Fix $1 \le i \le n$. By definition

$$P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)}$$

$$= \frac{P(B_i \cap A)}{P(A)}$$

definition of conditional Probability

$$= \frac{P(A \mid B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)}$$

Theorem 1.3.5

$\square$

Example 1.3.12 :- Shyam is randomly chosen from the citizens of Hyderabad to test for swine flu.

(i) It is found 95% of people with swine flu test positive for the disease.

(ii) 2% of people without the disease will also test positive.

(iii) 1% of the population has the disease

Suppose Shyam tests positive what is the Probability that Shyam has the disease?

$A = \{$ Shyam has swine flu $\}$

$B = \{$ Shyam tested positive $\}$

(i) $P(B|A) = 0.95$

(ii) $P(B|A^c) = 0.02$

(iii) $P(A) = 0.01$

Understand : $P(A|B) = ?$

Bayes Theorem :- $P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$

(Exercise to check assumptions)

$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(0.99)} = 0.324$$