Recall :-

Sample Space — S any set — all possible outcomes

Event — $E \subseteq S$ — Temporary

Definition 1.1.3 $(S, \mathcal{F})$ $\mathcal{F}$ - set of all events

A probability $P : \mathcal{F} \longrightarrow [0,1]$

① $P(S) = 1$

② $\{E_k\}_{k=1}^{\infty}$ $E_k \in \mathcal{F}$ $E_i \cap E_j = \phi$ $i \neq j$ then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.1.2 Basic Properties

Theorem 1.14: :- P be a probability on S

① $P(\phi) = 0$

② $n \geq 1$, $E_1, \dots E_n$, such $E_i \cap E_j = \phi$ $i \neq j$ then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{j=1}^{n} P(E_j)$$

③ $E \subseteq F$ then $P(E) \leq P(F)$

④ $E \subseteq F$ then $P(F \setminus E) = P(F) - P(E)$

⑤ E^c – complement of E then $P(E^c) = 1 - P(E)$.

⑥ E and F are two events then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Proof :- $E_m = \phi \quad \forall\, m \geq 1$ Trivially $\{E_m\}_{m\geq 1}$ are disjoint sequence of events. So from Axiom (2) in Definition 1.1.3

$$P\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}\Big) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} P(\phi)$$

Convergent series

$$= P(\phi)$$

$0 \leq P(\phi) \leq 1$

$\therefore$ $P(\phi) = 0$ — shows (1) □

For (2), Given $E_1, \ldots E_n$, $n \geq 1$ a finite sequence of disjoint events. Def $E_m = \phi \quad \forall\, m \geq n+1$.

Then $\{E_k\}_{k\geq 1}$ form a sequence of disjoint events

So Axiom (2) Definition 1.1.3 :-

$$P\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}\Big) = \sum_{i=1}^{\infty} P(E_i)$$

$E_m = \phi$
$\forall m \geq n+1$ $\Rightarrow$ $\bigcup_{i=1}^{n} E_i$ $\qquad \sum_{i=1}^{n} P(E_i)$ (1) $P(\phi)=0$

$$\Rightarrow \quad P\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} P(E_i)$$

So that shows (2). □

To show (3) :- Take $E_1 = E \qquad E_2 = F \setminus E$
Apply (2) (with observation $F = E \cup F \setminus E$) □

To show (4) :- Take $E_1 = E$ and $E_2 = F \setminus E$

(2) $\Rightarrow$ $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$$P(F) = P(E) + P(F \setminus E)$$

$$P(F \setminus E) = P(F) - P(E)$$

□

To show (5) :- $F = S$ in (4)
$S / E = E^c$

(4) $\Rightarrow$ $P(E^c) = P(S) - P(E)$

Axiom 1 $\Rightarrow$ $P(E^c) = 1 - P(E)$ □

To show (6) :- $E_1 = E \qquad E_2 = F \setminus E$
$E_1 \cup E_2 = E \cup F$

(2) $\Rightarrow$ $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$\Rightarrow P(E \cup F) = P(E) + P(F \setminus E)$ — (*)

Observe: $F \setminus E \subseteq F$ & $F \setminus F \setminus E = E \cap F$

(4) $\Rightarrow$ $P(F) - P(F \setminus E) = P(E \cap F)$ — (**)

Substitute $P(F/E)$ from (**) into (*)

(**) for $P(F \setminus E)$

$\therefore$ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ □

Example 1.1.5 : A coin is considered to be fair if both outcomes in a toss are equally likely.

$$S = \{H, T\}$$

$$E = \{H\}, \quad F = \{T\} \qquad E \cap F = \phi$$

Suppose $p = P(E) = P(F)$ Both outcomes Equally likely

⑥ - Theorem 1.1.4.

$$1 = P(S) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Axiom 1

Definition 1.1.3

$$= p + p - 0$$ ① Theorem 1.1.4. $P(\phi) = 0$

$$\Rightarrow 1 = 2p \Rightarrow p = 1/2$$

$\therefore$ Heads has a 50% chance of occurring in a toss

$\square$

Equally like Outcomes

S = a finite set $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ for some $n \geq 1$

Observation

If P is a Probability on S and

characterise P ⟵ $P(\{\omega_i\})$ for all $i = 1, \ldots, n$.

$$\Rightarrow \quad E \subseteq S \qquad P(E) = \sum_{\omega \in E} P(\{\omega\}) \quad \text{— ② in Theorem 1.1.4.}$$

In such cases it is enough to know Probability of each outcome to characterise Probability of any event. — $\ast$

The $\ast$ also holds if S is a Countable set.

Assignment of a Probability to each outcome is called