Recall: · sampling with replacement — Bernoulli trials, independence

· Sampling without replacement: $m < r < N$ Hypergeometric (N, r, m)

$S = \{0, 1, \ldots m\}$

$$\mathbb{P}(\{k\}) = \frac{\binom{r}{k}\binom{N-r}{m-k}}{\binom{N}{m}} = \frac{\frac{r!}{k!}\,\frac{N-r!}{m-k!}\,\frac{1}{r-k!}\cdot\frac{1}{N-r-(m-k)!}}{\frac{N!}{m!\,(N-m)!}}$$

$$\mathbb{P}(\{k\}) = \frac{m!}{k!\,(m-k)!}\;\underbrace{\frac{r(r-1)(r-2)\ldots(r-(k-1))}{N(N-1)\ldots(N-(k-1))}}\;\underbrace{\frac{(N-r)(N-r-1)\ldots(N-r-(m-1-k))}{(N-k)\ldots\ldots(N-(m-1))}}_{\leq \; \vdots \; \leq}$$

$p = \frac{r}{N}$, $p_1 = \frac{r-k}{N-k}$, $p_2 = \frac{r-k}{N-m}$

$$\binom{m}{k} p_1^k (1-p_2)^{m-k} \leq \mathbb{P}(\{k\}) \leq \binom{m}{k} p^k (1-p_1)^{m-k}$$

$\boxed{\text{Theorem } 2.3.3}$ — proof of this is done above.

Conclusion:- $\boxed{m << N, \; m <<< r}$ $\Rightarrow$ $p_1 \simeq p_2 \simeq p$

$$\mathbb{P}(\{k\}) \simeq {}^mC_k\, p^k (1-p)^{m-k}$$

$\Rightarrow$ Sampling with replacement "$\simeq$" Sampling without replacement

3 Discrete Random Variables :-

- Various Experiments — giving rise to Bernoulli trials
- Each time we created a new sample space, for a new distribution.

- Can we do all this on one sample space.

To do this :- Define functions on the space that describe the output of interest

— Random variables —

Example 3.1.1 :

Suppose a coin is flipped 3 times.

Q1: # of heads in 3 tosses ?

Q2: Trial at which 1st head shows up?

$$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$$

$X : S \to \{0,1,2,3\}$ by $X(s) = \#$ of heads in s

$Y : S \to \{1,2,3, \text{None}\}$ by $Y(s) =$ Trial at which head appears in s

s	X(s)	Y(s)
hhh	3	1
hht	2	1
hth	2	1
htt	1	1
thh	2	2
tht	1	2
tth	1	3
ttt	0	None

$$\mathbb{P}(2 \text{ heads in } 3 \text{ tosses}) = \mathbb{P}(X^{-1}(\{2\})) = \mathbb{P}(\{hht, hth, thh\})$$
$$= 3/8$$

$\mathbb{P}($ first head occurs in 1^{st} toss$) = \mathbb{P}(Y^{-1}(\{1\}))$

$= \mathbb{P}(\{hhh, hht, hth, htt\})$

$= 4/8$

Similarly one can calculate:

- $\mathbb{P}(X=0) = \frac{1}{8}$, $\mathbb{P}(X=1) = 3/8$, $P(X=2) = 3/8$, $\mathbb{P}(X=3) = \frac{1}{8}$
- $\mathbb{P}(Y=1) = \frac{1}{2}$, $\mathbb{P}(Y=2) = 1/4$, $\mathbb{P}(Y=3) = 1/8$, $\mathbb{P}(Y=\text{None}) = \frac{1}{8}$

Give the complete "distribution of X and Y."

– they describe how X and Y distribute the probabilities on their respective ranges.

S – finite or countable.

Theorem 3.1.2 Let S be a sample space with probability $\mathbb{P}$ and let $X: S \to T$ be a function. Then X – generates a probability $\mathbb{Q}$ on T given by

$$\mathbb{Q}(B) = \mathbb{P}(X^{-1}(B))$$

The probability $\mathbb{Q}$ is called the "distribution of X", since it describes how X distributes the probability from S onto T.

Proof:- T, Events $\equiv$ subsets of T $:= \mathcal{F}$ $\quad B \in \mathcal{F}$

$$\varphi(B) = \mathbb{P}(X^{-1}(B))$$

$\boxed{\begin{array}{l} X : S \to T \\ \mathbb{P} - \text{Probability on } S \end{array}}$

Verify the two Axioms of Probability

① $\varphi(T) = 1$ $\quad$ Now $X^{-1}(T) = \{s \in S \mid X(s) \in T\} = S$

Proof: $\varphi(T) = \mathbb{P}(X^{-1}(T)) = \mathbb{P}(S) = 1$

② $B_1, B_2, \ldots, B_n, \ldots$ be a sequence of disjoint events in T

$$\varphi\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \varphi(B_i)$$

Proof:- $\varphi\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right)$

(*) $X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) \underset{Ex}{=} \bigcup_{i=1}^{\infty} X^{-1}(B_i)$, $\quad$ Set Theory exercise

$$\varphi\left(\bigcup_{i=1}^{\infty} B_i\right) \underset{(*)}{=} \mathbb{P}\left(\bigcup_{i=1}^{\infty} X^{-1}(B_i)\right)$$

(**) $\{B_i\}_{i=1}^{\infty}$ are disjoint & X is a function events in T $\underset{Ex}{\Longrightarrow}$ $\{X^{-1}(B_i)\}_{i=1}^{\infty}$ are also disjoint events in S

Since $\mathbb{P}$ is a Probability on S

$$\varphi\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} X^{-1}(B_i)\right) \underset{(**)}{=} \sum_{i=1}^{\infty} \mathbb{P}(X^{-1}(B_i))$$

$$= \sum_{i=1}^{\infty} \varphi(B_i)$$

$\Rightarrow$ φ is a Probability on T. $\square$

Definition 3.1.5 : A discrete random variable is a function $X : S \to T$ where S is a sample space equipped with a Probability $\mathbb{P}$ and T is a countable (or finite) subset of real numbers.

Define: $f_X : T \to [0,1]$

$$f_X(t) = \mathbb{P}(X=t)$$

$f_X(\cdot)$ is called the Probability mass function of X.

Then, $A \subseteq T$

$$\mathbb{P}(X \in A) = \sum_{t \in A} \mathbb{P}(X=t) = \sum_{t \in A} f_X(t)$$

Example : Say $X \sim$ Bernoulli(p)

$X : S \to \{0,1\}$ $\mathbb{P}$ is a Probability on S

$$\mathbb{P}(X=0) = 1-p \quad \& \quad \mathbb{P}(X=1) = p.$$

$$\text{or} \quad f_X(0) = 1-p \quad f_X(1) = p$$

Definition 3.1.6 let $X : S \to T$ and $Y : S \to T$ be discrete random variables. we say X and Y have equal distribution provided
(same)

$$\mathbb{P}(X=t) = \mathbb{P}(Y=t) \quad \text{for all } t \in T.$$

Common distributions

Definition 3.1.7 :

(a) $X \sim \text{Uniform}\{1, \dots, n\}$ for some $n \geq 1$ if X is a discrete random variable such that

$$\mathbb{P}(X = k) = \frac{1}{n} \qquad 1 \leq k \leq n$$

we say X is a uniform random variable on the set $\{1, \dots, n\}$.

(b) $X \sim \text{Bernoulli}(p)$ for some $0 \leq p \leq 1$ if X is a discrete random variable such that

$$\mathbb{P}(X = 0) = 1 - p \quad \text{and} \quad \mathbb{P}(X = 1) = p$$

we say X is a Bernoulli random variable with parameter p.

(c) $X \sim \text{Binomial}(n, p)$ $0 \leq p \leq 1$, $n \geq 1$

X is a discrete random variable such that

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

3.2.1 Independent Random variables : $X : S \to T_1$, $Y : S \to T_2$

Definition 3.2.1 : Two random variables X and Y are independent if $\{X \in A\}$ and $\{Y \in B\}$ are independent for every event $A^{\subseteq T_1}$ in the range of X and every event $B_{\subseteq T_2}$ in the range of Y.

Equivalently :- • X and Y are independent

if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$ for all $A \subseteq T_1$, $B \subseteq T_2$

• (Exercise) X and Y are independent if

$$\mathbb{P}(X=t, Y=s) = \mathbb{P}(X=t)\,\mathbb{P}(Y=s) \quad \forall t \in T_1, \ \forall s \in T_2$$