Multiple Continuous Random Variables

X, Y are continuous random variables have a Joint density function $f : \mathbb{R}^2 \to [0, \infty)$ if

- $f(,)$ is piecewise continuous
- $\int_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1$
- $\mathbb{P}((X,Y) \in A) = \int_A f(x,y)\, dx\, dy$

Joint distribution function $F : \mathbb{R}^2 \to [0,1]$

$$F(x,y) = \mathbb{P}(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$$

Example :- $D \subseteq \mathbb{R}^2$ with positive area.
"choose a point uniformly in D"

$(X,Y) \sim \text{Uniform}(D)$ if its Joint density is given

$$f(x,y) = \begin{cases} \frac{1}{|D|} & \text{if } (x,y) \in D \\ 0 & \text{otherwise.} \end{cases}$$

$|D|$ – area of D

Independence

(X, Y) are continuous random variables having

Joint density — $f(\cdot,\cdot)$

marginal — $f_X(\cdot)$, $f_Y(\cdot)$

p.df of Y ← p.d.f of X ←

(X, Y) are independent iff

Thm 5.4.7 $\quad f(x,y) = f_X(x) f_Y(y) \quad \forall x, y \in \mathbb{R}$

Example (contd.) : $D = [0,1] \times [3,5]$

(X, Y) have joint density

$$f(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

To compute $f_X(\cdot)$

$$\mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y \in \mathbb{R})$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u, v)\, dv\, du$$

$\underline{x < 0}$ $\Rightarrow$ $f(x,y) = 0 \quad \forall\, y \in \mathbb{R}$

$\Rightarrow \mathbb{P}(X \le x) = 0$ ——— (i)

$x > 1 \Rightarrow f(x,y) = 0 \quad \forall\, y \in \mathbb{R}$

$\mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y \in \mathbb{R})$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,v)\, dv\, du$$

$$= \int_{0}^{1} \int_{-\infty}^{\infty} f(u,v)\, dv\, du$$

$f(x,y) > 0$

$\Leftrightarrow$

$(x,y) \in [0,1] \times [3,5]$

$$= \int_{0}^{1} \int_{3}^{5} f(u,v)\, dv\, du$$

$$= \int_{0}^{1} \int_{3}^{5} \frac{1}{2}\, dv\, du$$

$$= \underline{1} \quad \cdots \text{(ii)}$$

$x \in (0,1)$

$$\mathbb{P}(X \le x) = \int_{0}^{x} \int_{3}^{5} \frac{1}{2}\, dv\, du$$

$$= x \quad \cdots \text{(iii)}$$

From (i), (ii) and (iii)

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

$$\Rightarrow \quad f_X(x) = \begin{cases} 0 & x \notin [0,1] \\ 1 & x \in [0,1] \end{cases}$$

$$\Rightarrow \quad X \sim \text{Uniform}[0,1]$$

Remark:

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} \int_{3}^{5} f(u,v)\, dv\, du$$

$$f_X(x) = \frac{d}{dx}\mathbb{P}(X \le x) = \int_{3}^{5} f(x,v)\, dv$$

$$= \begin{cases} 0 & x \notin [0,1] \\ 1 & x \in [0,1] \end{cases}$$

Ex. $Y \sim \text{Uniform}(3,5)$

$$\left[\because \quad f_Y(y) = \int_0^1 f(x,y)\, dx \right]$$

Ex: Verify: $f(x,y) = f_X(x)\, f_Y(y)$

$\therefore$ $X \sim$ Uniform $[0,1]$ and $Y \sim$ Uniform $[3,5]$ independent

$(X,Y) \sim$ Uniform $[0,1] \times [3,5]$

Sums of independent continuous r.v.

X, Y are independent continuous r.v.'s

have joint density

$$f(x,y) = f_X(x)\, f_Y(y)$$

$$Z = X + Y$$

1. $$\mathbb{P}(Z \le z) = \mathbb{P}(X + Y \le z)$$

$$= \iint_{\{(x,y)\,:\, x+y \le z\}} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x,y)\, dx\, dy$$

$$\mathbb{P}(Z \le z) = \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} f_X(x)\, f_Y(u-x)\, dx \right] du$$

$$\begin{cases} f(x,y) = f_X(x)\, f_Y(y) \\ \text{change of variable } u = z-y \end{cases}$$

reorder integrals

$$\Rightarrow \boxed{f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx}$$

Example : $X \sim \mathcal{E}xp(\lambda)$ and independent $Y \sim \mathcal{E}xp(\lambda)$

$$Z = X + Y$$

- $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\cdot \quad f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx \qquad \text{if } z > 0$$

$$\boxed{\begin{aligned} &x > 0 \text{ and} \\ &z - x > 0 \iff x < z \;\&\; z > 0 \end{aligned}}$$

$$= \lambda^2 \int_0^z e^{-\lambda z}\, dx$$

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & \text{if } z > 0 \\ 0 & \text{o.w.} \end{cases}$$

$T_1, T_2, \ldots T_n \ldots$ i.i.d $\mathrm{Exp}(\lambda)$

$\boxed{\text{Inductively}}$

$Z = T_1 + T_2 + \ldots T_n$

$$f_Z(z) = \begin{cases} \dfrac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$Z \sim \mathrm{Gamma}(\underbrace{n}_{\text{shape}}, \underbrace{\lambda}_{\text{rate}})$$

Conditional density of $X \mid Y = b$

X, Y are continuous random variables

$$\mathbb{P}(Y=b) = 0$$

Need to understand $X \mid Y=b = ?$

Step 1 : let $b \in \mathbb{R}$

$f_Y(b) > 0 \implies \mathbb{P}(Y \in (b-\frac{1}{n}, b+\frac{1}{n})) > 0 \quad \forall n \geq 1$

step2 :- $\quad X \mid Y \in \left(b-\frac{1}{n}, b+\frac{1}{n}\right)$

$$\mathbb{P}\left(X \in A \mid Y \in \left(b-\frac{1}{n}, b+\frac{1}{n}\right)\right)$$

$$= \frac{\mathbb{P}\left(X \in A, Y \in \left(b-\frac{1}{n}, b+\frac{1}{n}\right)\right)}{\mathbb{P}\left(Y \in \left(b-\frac{1}{n}, b+\frac{1}{n}\right)\right)}$$

$$= \frac{\int_A \int_{b-1/n}^{b+1/n} f(u, v)\, dv\, du}{\int_{b-1/n}^{b+1/n} f_Y(y)\, dy}$$

step 3: $\quad X \mid Y = b \equiv \text{"}\lim_{n \to \infty}\text{"} X \mid Y \in \left(b-\frac{1}{n}, b+\frac{1}{n}\right)$

$$\frac{\int_A n \int_{b-1/n}^{b+1/n} f(u, v)\, dv\, du}{n \int_{b-1/n}^{b+1/n} f_Y(y)\, dy}$$

Ex: • $n \int_{b-1/n}^{b+1/n} f_Y(s)\, ds \xrightarrow[n\to\infty]{} f_Y(b)$

• $n \int_{b-1/n}^{b+1/n} f(u, v)\, dv \xrightarrow[n\to\infty]{} f(u, b)$

$$\mathbb{P}(X \in A \mid Y = b) = \frac{\int_A f(u, b)\, du}{f_Y(b)}$$

$A = (-\infty, x])$

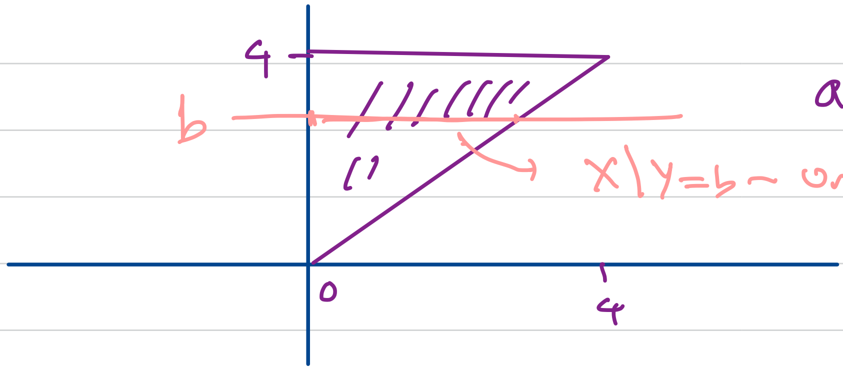
$$\Rightarrow \mathbb{P}(X \le x \mid Y = b) = \frac{\int_{-\infty}^{x} f(u, b)\, du}{f_Y(b)}$$

Differentiating ; $f_Y(b) > 0$ then

$$f_{X \mid Y=b}(x) = \frac{f(x, b)}{f_Y(b)}$$

Conditional density of $X \mid Y = b$

Example : $D = \{ (x,y) \mid 0 < x < y < 4 \}$



$X \mid Y = b \sim \text{uniform}(0,b)$

$\text{area}(D) = \frac{1}{2} 4 \cdot 4 = 8$

$(X,Y) \sim \text{uniform}(D)$

$$f(x,y) = \begin{cases} \frac{1}{|D|} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{8} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases} \quad \text{—(1)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \begin{cases} \int_x^4 \frac{1}{8}\, dy & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{4-x}{8} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases} \quad \text{—(2)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \begin{cases} \int_0^y \frac{1}{8}\,dx & 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{y}{8} & 0 < y < 4 \\ 0 & \text{otherwise} \end{cases} \longrightarrow 3$$

1, 2 & 3 $\equiv$ $f(x,y) \neq f_x(x)\, f_y(y)$

X, Y are NOT independent

$b \in (0,4)$ $\quad X \mid Y=b \equiv ?$

$$f_{X|Y=b}(x) = \frac{f(x,b)}{f_Y(b)}$$

$$= \begin{cases} \dfrac{1/8}{b/8} & 0 < x < b \;\&\; 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y=b}(x) = \begin{cases} \frac{1}{b} & 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

$X \mid Y=b \sim$ Uniform $(0,b)$