Recall :- (X,Y) are two continuous random variables

f - Joint density if

$$\mathbb{P}(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$$

f_X Marginal density if

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$f_X(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

f_Y Marginal density if

$$\mathbb{P}(Y \le y) = \int_{-\infty}^{y} f_Y(v)\, dv$$

$$f_Y(v) = \int_{-\infty}^{\infty} f(u,v)\, du$$

$X | Y=y$ - conditional density $f_Y(y) > 0$

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

X, Y independent iff

$$f(x,y) = f_X(x)\, f_Y(y)$$

"Quick" Summary :: Covariance

X - continuous random variable with p.d.f $f_X(\cdot)$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$$

$$\Rightarrow \quad Var[X] = E[(X - E[X])^2]$$

$$SD[X] = \sqrt{Var[X]}$$

(X, Y) - are continuous r.v. with joint density $f(\cdot, \cdot)$

marginal densities - $f_X(\cdot)$ & $f_Y(\cdot)$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \, f(x,y) \, dx \, dy$$

$$Cov[X,Y] = E[(X - E[X])(Y - E[Y])]$$

- Covariance between X & Y

- similar interpretation for discrete r.v. applies here as well.

Markov Inequality & Tschebychev Inequality

X – random variable

$\mu = E[X]$ – Centering of the r.v

$\sigma = SD[X]$ – "Range" of the r.v

$(\mu - \sigma,\ \mu + \sigma)$

Markov's Inequality

continuous

X – r.v supported on non-negative

(i.e $f_X(x) = 0 \quad x < 0$) $\quad c > 0$; $\mu = E[X] < \infty$

$$\mathbb{P}(X \geq c) \leq \mu / c$$

Proof: $\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_{-\infty}^{c} x f_X(x)\, dx + \int_{c}^{\infty} x f_X(x)\, dx$$

$$= \underbrace{\int_{0}^{c} x f_X(x)\, dx}_{\geq 0} + \int_{c}^{\infty} \underset{x \geq c}{x} f_X(x)\, dx$$

$$\Rightarrow \quad \mu \geq 0 + c \underbrace{\int_c^\infty f_X(x)\,dx}$$

$$\Rightarrow \quad \mu \geq c\, P(X \geq c)$$

$$\Rightarrow \quad P(X \geq c) \leq \mu/c \qquad \square$$

Tschebychev's Inequality :- X is continuous r.v

s.t $\mu = E[X] < \infty$ and $\sigma^2 = E(X-\mu)^2 < \infty$

$k \geq 1$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Proof :- $P(|X-\mu| \geq k\sigma)$

$$= P(|X-\mu|^2 \geq k^2\sigma^2)$$

Apply Markov Inequality to $Y = (X-\mu)^2$

$$P(Y \geq k^2\sigma^2) \leq \frac{E[Y]}{k^2\sigma^2}$$

$$\mathbb{P}(|X-\mu| \geq k\sigma) = \mathbb{P}(Y \geq k^2\sigma^2)$$

$$\leq \frac{\mathbb{E}[(X-\mu)^2]}{k^2\sigma^2}$$

$$= \frac{\cancel{\sigma^2}}{k^2\cancel{\sigma^2}}$$

$$= \frac{1}{k^2}$$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \square$$

<u>Remarks</u> :-

- Above inequalities are true for discrete r.v.'s as well.

$X \geq 0$ (Discrete) $\mu = E[X] < \infty$

$$\mathbb{P}(X \geq c) \leq \mu/c$$

X - (Discrete) $\mu = E[X] < \infty$ and

$\sigma^2 = \mathbb{E}(X-\mu)^2 < \infty$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Transformation of random variable

X – Continuous random variable

$g : \mathbb{R} \to \mathbb{R}$ $\quad Y = g(X)$

Q: How to find distribution of Y?

Example : $\quad X \sim \text{Uniform}(0,1)$

$g(x) = x^2$; $\quad Y = X^2$

Step 1 : Find distribution function of Y.

$$F_Y(y) = \mathbb{P}(Y \le y)$$

$$= \mathbb{P}(X^2 \le y) = \begin{cases} 0 & y \le 0 \\ ? & y > 0 \end{cases}$$

$y > 0$

$$\mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$$

(i) $y \in (0,1)$

$$\mathbb{P}(Y \le y) = \mathbb{P}(0 \le X \le \sqrt{y})$$

(Recall)

$$= F_X(\sqrt{y}) = \sqrt{y}$$

(iii) $y \geq 1$ $\quad P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$

$$= P(0 \leq X \leq 1)$$

$$= 1$$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \sqrt{y} & 0 \leq y \leq 1 \\ 1 & y \geq 1 \end{cases}$$

Step 2 : Differentiate (Piecewise) $F_Y(\cdot)$ to get $f_Y(\cdot)$

$$f_Y(y) = F_Y'(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{2}\frac{1}{\sqrt{y}} & 0 < y < 1 \\ 0 & y \geq 1 \end{cases}$$