Recall: $(S, \mathcal{F}, \mathbb{P})$ [Sample Space]

Countable, uncountable

$\underbrace{\text{Countable} \quad \text{uncountable}}_{\text{Experiment}}$

Probability

Event

X: Random variables

Discrete (p.m.f)

Continuous (p.d.f)

$F(x) = \mathbb{P}(X \le x)$

$x \in \mathbb{R}$

- Concept of Independence

- A, B - events $\equiv$ independent $\equiv$

  if occurrence of one event does not affect the probability of occurrence of the other

- Generalize this to random variables

Conditional Probability

Consider the experiment of tossing a fair coin three times with sample space $S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$.

Let A be the event that there are two or more heads. As all outcomes are equally likely

$$\begin{aligned} P(A) &= \frac{|\{hhh, hht, hth, thh\}|}{8} \\ &= . \; 4/8 = 1/2 \end{aligned}$$

Let B be the event that there is a head in the first toss. As above,

$$\begin{aligned} P(B) &= \frac{|\{hhh, hht, hth, htt\}|}{8} \\ &= . \; 4/8 = 1/2 \end{aligned}$$

Now,

$$P(A|B) \longleftarrow \frac{|A \cap B|}{|B|} = . \; \frac{|\{hhh, hht, hth\}|}{|B|} = \frac{3}{4}$$

Conditional Probability

Let S be a sample space with probability P. Let A and B be two events with $P(B) > 0$. Then the conditional probability of A given B written as $P(A|B)$ and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- B has occurred
- $P(A|B) \equiv$ Probability of A given that B has occurred.

Independence

Suppose we toss a coin three times. Then the sample space

$$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}.$$

Let $A \equiv \{$first toss is head$\}$ and $B \equiv \{$ second toss is head$\}$

$$A = \{hhh, hht, hth, htt\} \text{ and } B = \{hhh, hht, thh, tht\}.$$

- Equally likely outcome experiment

- $P(A) = \frac{1}{2}$ $\qquad$ $P(B) = \frac{1}{2}$

- $P(A|B) = \frac{1}{2}$ $\qquad$ $P(A|B^c) = \frac{1}{2}$ (See below)

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|} = \frac{2}{4} = \frac{1}{2}$$

$$P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)} = \dots = \frac{1}{2}$$

$$P(A) = P(A|B) = P(A|B^c)$$

i.e. the occurrence (or non-occurrence) has no effect on the probability of A.

Ex:- $P(B|A^c) = P(B) = P(B|A)$

– A & B are Independent Events –

if

$P(A|B) = P(A) \iff \dfrac{P(A\cap B)}{P(B)} = P(A)$

$\boxed{P(B) > 0}$

$\iff P(A\cap B) = P(A)\,P(B)$

Independence

Definition of Independence of Events :- :

- Two events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

one Equation

- A finite collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2)\ldots P(E_n).$$

2^n equations

whenever E_j is either A_j or A_j^c.

Remark :

$1 \le i \le n$

Mutual

(*) ensures that A_i or A_i^c - occurrence or non-occurrence - Does not affect Probability of A_j $j \neq i$ $1 \le j \le n$

not the # same as

Pairwise $\{A_i, A_j\}_{i \neq j}$ - each pair is independent

(A) $n=2$ $\quad P(A \cap B) = P(A)P(B)$

Ex $\iff P(A \cap B^c) = P(A)P(B^c)$

$\iff P(A^c \cap B) = P(A^c)P(B)$

$\iff P(A^c \cap B^c) = P(A^c)P(B^c)$

$n \ge 3$ - one needs to be careful

[not enough] $\Leftarrow P(A_1 \cap A_2 \cap \cdots A_n) = \prod_{i=1}^{n} P(A_i)$

(B) Its tempting to say

A_1, A_2, A_3 are mutually independent if

Pairwise independent:

A_1, A_2 are independent

A_1, A_3 are independent and

A_2, A_3 are independent

} not enough

See Example 1.4.2 in book.

Independence

Definition of Independence for Random Variables:- :

- Two random variables X and Y are independent if $(X \in A)$ and $(Y \in B)$ are independent for every event A in the range of X and every event B in the range of Y
- A finite collection of random variables $X_1, X_2, \ldots, X_n$ is mutually independent if the sets $(X_j \in A_j)$ are mutually independent for all events A_j in the ranges of the corresponding X_j.
- An arbitrary collection of random variables X_t where $t \in I$ for some index set I is mutually independent if every finite sub-collection is mutually independent.

Dependent Random Variables

Let $X \sim \text{Uniform}(\{1, 2\})$ and let Y be the number of heads in X tosses of a fair coin.

no heads in 1 toss

$$P(Y = 0 | X = 1) = \frac{1}{2}$$

1 head in 1 toss

$$P(Y = 1 | X = 1) = \frac{1}{2}$$

$$Y | X = 1 \sim \text{Bernoulli}\left(\frac{1}{2}\right)$$

Dependent Random Variables

If $X = 2$ then Y is the number of heads in two flips of a fair coin then

0 heads in 2 tosses

$$P(Y = 0|X = 2) = 1/4$$

1 heads in 2 tosses

$$P(Y = 1|X = 2) = 1/2$$

2 heads in 2 tosses

$$P(Y = 2|X = 2) = 1/4$$

Therefore $\underline{Y|X = 2} \sim$ Binomial $(2, \frac{1}{2})$

Dependent Random Variables

Range(X) = $\{1, 2\}$ and Range(Y) = $\{0, 1, 2\}$.
To find the joint distribution of X and Y we must calculate the probabilities of each possibility. In this case the values may be obtained using the definition of conditional probability. For instance,

$$P(X = 1, Y = 0) = P(\underbrace{Y = 0|X = 1}_{\text{0 heads in 1 toss}}) \cdot P(X = 1) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$$

and

$$P(X = 1, Y = 2) = P(\underbrace{Y = 2|X = 1}_{\text{2 heads in 1 toss}}) \cdot P(X = 1) = 0 \cdot \frac{1}{2} = 0$$

Dependent Random Variables

The entire joint distribution $P(X = a, Y = b)$ is described by the following chart.

$P(X=x, Y=y)$	$X = 1$	$X = 2$
$Y = 0$	$1/4$	$1/8$
$Y = 1$	$1/4$	$1/4$
$Y = 2$	0	$1/8$

Dependent Random Variables

$$\frac{P(X=1, Y=0)}{P(Y=0)}$$

There will be three different conditional distributions depending on whether $Y = 0$, $Y = 1$, or $Y = 2$.

Bayes Theorem

$$P(X = 1|Y = 0) = \frac{P(Y = 0|X = 1) \cdot P(X = 1)}{P(Y = 0)}$$

(Ex to do from previous table)

$$= \frac{(1/2)(1/2)}{P(Y=0|X=1)P(X=1) + P(Y=0|X=2)P(X=2)}$$

$$= 2/3$$

$P(X = 2|Y = 0) =$. ...... Ex $= \cdots = 1/3$

Dependent Random Variables

Joint distribution of X and Y

	$X = 1$	$X = 2$	Sum	
$Y = 0$	$1/4$	$1/8$	$3/8$	$P(Y=0)$
$Y = 1$	$1/4$	$1/4$	$1/2$	$P(Y=1)$
$Y = 2$	0	$1/8$	$1/8$	$P(Y=2)$
Sum	$1/2$	$1/2$		
	$P(X=1)$	$P(X=2)$		

$P(X = 1, Y = 0) = 1/4$ while $P(X = 1) \cdot P(Y = 0) = \frac{1}{2} \cdot 3/8 = 3/16$

$3/16 \neq 1/4 \Rightarrow$ X and Y are NOT independent

1st Column sum:

$$\left.\begin{aligned} &\mathbb{P}(X=1, Y=0) \\ &+ \mathbb{P}(X=1, Y=1) \\ &+ \mathbb{P}(X=1, Y=2) \end{aligned}\right\} = \mathbb{P}(X=1)$$

Joint Distribution

If X and Y are discrete random variables, the "joint distribution" of X and Y is the probability Q on pairs of values in the ranges of X and Y defined by

$$Q((a, b)) = P(X = a, Y = b).$$

The definition may be expanded to a finite collection of discrete random variables X_1, X_2, ..., X_n for which the joint distribution of all n variables is the probability defined by

$$Q((a_1, a_2, \ldots, a_n)) = P(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n).$$

Conditional Distribution

Let X be a random variable on a sample space S and let $A \subset S$ be an event such that $P(A) > 0$. Then the probability Q described by

$$Q(B) = P(X \in B | A)$$

is called the "conditional distribution" of X given the event A.

Example: $A = \{X = 1\}$ r.v was Y

$Y | A \sim \text{Bernoulli}(1/2)$

Conditional Expectation and Conditional Variance

Let $X : S \to T$ be a discrete random variable and let $A \subset S$ be an event for which $P(A) > 0$.

The "conditional expected value" of X given A is

$$E[X|A] = \sum_{t \in T} t \cdot P(X = t|A),$$

and the "conditional variance" of X given A is

$$Var[X|A] = E[(X - E[X|A])^2|A].$$

Recall X - Discrete r.v.

Range$(X) = T$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t)$$

$$\mathrm{Var}[X] = E[(X - E[X])^2]$$

Example :-

$X \sim \text{Uniform}(\{1, 2\})$ $\qquad Y \sim \text{Binomial}(X, \frac{1}{2})$

$$E(Y \mid X=1) = \frac{1}{2}$$

$$E(Y \mid X=2) = 1$$

$$E[Y] = 1 \cdot P(Y=1) + 2 \cdot P(Y=2)$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot 1/8$$

$$= 3/4$$