Recall

- Independence
  - A, B events $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

    " occurrence of A or B does not influence the probability of B or A respectively "

$n \geq 2$

- $A_1, \dots, A_n$ are mutually independent if

2ⁿ equations $\Big\{$ $\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} \mathbb{P}(E_i)$ where $E_i = A_i^c$ or A_i

- X, Y are two random variables. Then they are independent if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

A, B are any two events

- discrete ; reduction : independent iff

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\,\mathbb{P}(Y = y)$$

$x \in \mathrm{Range}(X)$
$y \in \mathrm{Range}(Y)$

Recall:

X, Y are two discrete r.v.

- Conditional distribution of $X \mid Y=y$

$$\{\mathbb{P}(X=x \mid Y=y)\}_{x \in \text{Range}(X)}$$

- Conditional Expectation of $X \mid Y=y$

$$E[X \mid Y=y] = \sum_{x \in \text{Range}(X)} x \, \mathbb{P}(X=x \mid Y=y)$$

- Conditional Variance of $X \mid Y=y$

$$E\left[\left(X - E[X \mid Y=y]\right)^2 \,\middle|\, Y=y\right] \equiv Var[X \mid Y=y]$$

Recall: Joint distribution - Discrete random variables

- $X_1, X_2, \dots, X_n$ - Discrete random variables

Joint distribution of $X_1, \dots, X_n$ is given by

$$\left\{ \mathbb{P}(X_1 = x_1, \dots, X_n = x_n) \right\}_{\substack{x_i \in Range(X_i) \\ 1 \le i \le n}}$$

- $\underline{n=2}$

column sum

	$Y=y_1$	$\dots$	$Y=y_n$
$X=x_1$			
$\vdots$		$\cdot \leftarrow \mathbb{P}(X=x_i, Y=y_j)$	
$X=x_m$			

Row sum

$P(X=x_i)$

$\mathbb{P}(Y=y_1)$

$\mathbb{P}(X=x_i \mid Y=y_j)$ & $\mathbb{P}(Y=y_j \mid X=x_i)$

Multinomial

$k=2$ — Corresponds to Binomial

I.I.D :- Independent and identically distributed

Each trial or r.v. has no effect on others

They have the same distribution.

- Suppose we perform n i.i.d. trials each of which has k different possible outcomes.

$\left.\begin{matrix} 1 \\ \vdots \\ j \\ \vdots \\ k \end{matrix}\right\}$ outcomes

n trials × × × × . . ×

- For $j = 1, 2, \ldots, k$, let p_j represent the probability any given trial results in the j^{th} outcome and let X_j represent the number of the n trials that result in the j^{th} outcome.

$$X_j = \sum_{m=1}^{n} 1(\text{Trial } m \text{ has outcome } j)$$

- The joint distribution of all of the random variables X_1, X_2, $\ldots$, X_k is called a "multinomial distribution".

Multinomial - Joint distribution of $(X_1, X_2, \ldots, X_n)$

- $0 \le x_j \le n$, $1 \le j \le k$ & $\sum_{j=1}^{n} x_j = n$

- Let $B(x_1, x_2, \ldots, x_k) = \{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\}$. Then,

$$P(B(x_1, x_2, \ldots, x_k)) = \sum_{\omega \in B(x_1, x_2, \ldots, x_k)} P(\{\omega\})$$

- Each $\omega \in B(x_1, x_2, \ldots, x_k)$

$$P(\{\omega\}) = \prod_{j=1}^{k} (p_j)^{x_j}$$

and

$$|B(x_1, x_2, \ldots, x_k)| = \frac{n!}{x_1! \, x_2! \, \ldots \, x_k!}$$

Ex ≡ { # of ways of allocating n balls into k boxes. such that x_j of them are in box j

Multinomial

The joint distribution of $X_1, X_2, \ldots X_k$ is given by

$$P(X_1 = x_1, \ldots, X_k = x_k) = \begin{cases} \frac{n!}{x_1!\, x_2! \ldots x_k!} \prod_{j=1}^{k} p_j^{x_j} & \text{if } x_j \in \{0, 1, \ldots, n\} \\ & \text{and } \sum_{j=1}^{k} x_j = n \\ 0 & \text{otherwise} \end{cases}$$

Q :- How to work with independent random variables?

Sums of Random Variables – Independent

Toss a coin I | Toss a coin II | $\equiv$ # of heads in I & II

$X = \begin{cases} 1 & \text{if H occurs} \\ 0 & \text{if T occurs} \end{cases}$ $\qquad$ $Y = \begin{cases} 1 & \text{if H occurs} \\ 0 & \text{if T occurs} \end{cases}$ $\qquad$ $Z = X + Y$

Let $X, Y \sim \text{Bernoulli}(p)$ be two independent random variables.

$Z = X + Y$, $Z \sim ?$.

Range $(X) = \{0, 1\}$

Range $(Y) = \{0, 1\}$

$$\boxed{\text{Range}(Z) = \{0, 1, 2\}}$$

$$\mathbb{P}(Z = 0) = \mathbb{P}(X + Y = 0) = \mathbb{P}(X = 0, Y = 0)$$

$$\overset{\text{independence}}{=} \mathbb{P}(X = 0)\,\mathbb{P}(Y = 0)$$

$$= (1-p)(1-p)$$

$$= (1-p)^2$$

$$\mathbb{P}(Z=1) = \mathbb{P}(X+Y=1)$$

$$= \mathbb{P}(\underbrace{X=0, Y=1 \ \cup \ X=1, Y=0}_{\text{mutually Exclusive}})$$

$$= \underbrace{\mathbb{P}(X=0, Y=1)}_{} + \underbrace{\mathbb{P}(X=1, Y=0)}_{}$$

Independence

$$= \mathbb{P}(X=0)\,\mathbb{P}(Y=1) + \mathbb{P}(X=1)\,\mathbb{P}(Y=0)$$

$$= (1-p)\,p + p\,(1-p)$$

$$= 2p(1-p)$$

$$\mathbb{P}(Z=2) = \mathbb{P}(X+Y=2)$$

$$= \mathbb{P}(X=1, Y=1)$$

$$(\text{Independence}) = \mathbb{P}(X=1)\,\mathbb{P}(Y=1)$$

$$= p\ p$$

$$= p^2$$

We have shown

$$\mathbb{P}(Z=k) = \begin{cases} (1-p)^2 & k=0 \\ 2p(1-p) & k=1 \\ p^2 & k=2 \end{cases}$$

$$\mathbb{P}(Z=k) = {}^{2}C_{k}\, p^k (1-p)^{2-k} \qquad k=0,1,2$$

$$\Rightarrow \quad Z \sim \text{Binomial}(2, p)$$

Ex:- Compare above computation with how we found

$$\mathbb{P}(k \text{ successes in } n \text{ trials})$$

$$|||$$

$$\mathbb{P}(\{x_1, \ldots x_n\}) = p^{\#\{j : x_j = H\}} (1-p)^{\#\{j : x_j = T\}}$$

$$\Downarrow$$

$$^{n}C_{k}\, p^k (1-p)^{n-k}$$

Sums of Random Variables

Let $X, Y \sim \text{Bernoulli}(p)$ be two independent random variables.

$Z = X + Y$, $Z \sim$?.

Expectation of product

X and Y are independent, then $E[XY] =$

Covariance

The "covariance of X and Y" is defined as
$Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$.

Covariance-Facts

$Cov[X, Y] = Cov[Y, X]$;

$Cov[X, aY + bZ] = a \cdot Cov[X, Y] + b \cdot Cov[X, Z]$;

$Cov[aX + bY, Z] = a \cdot Cov[X, Z] + b \cdot Cov[Y, Z]$; and

If X and Y are independent with a finite covariance, then $Cov[X, Y] = 0$.

$Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]$.