<u>Recall</u> :- · S - any set, $|S| < \infty$, $S = \mathbb{N}, \ldots$
$S = \mathbb{R}$, $(0,1) \ldots$

· $\mathcal{F}$ - event in general is not $\neq \mathcal{P}(S)$
$S = \mathbb{R}$, $\mathcal{F} \equiv$ " all intervals, complements & unions "

Probability density function

$f : \mathbb{R} \to [0, \infty)$

- f is p.c.
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

<u>Motivation</u>

$S = (0,1)$

How to choose a number in $(0,1)$ "uniformly"?

$\mathbb{P} : \mathcal{F} \to [0,1]$

$$\mathbb{P}(A) = \int_A f(x)\,dx$$

<u>Ex</u>:: verify $\mathbb{P}$ - indeed is a Probability

① - $\mathbb{P}(S) = 1$

② $\{E_j\}_{j=1}$ $E_j \cap E_k = \phi$ then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

<u>De Moivre's Central limit Theorem</u>

.. $\Rightarrow$

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \quad x \in \mathbb{R}$$

<u>Ex</u> - f is indeed p.d.f.

- $\mathbb{P}(A) = \int_A f(x)\,dx$ $\cdots\!>$ Normal $(0,1)$

Continuous Random Variables

$(S, \mathcal{F}, \mathbb{P})$ — Probability space

A random variable $X : S \to \mathbb{R}$ is called **Continuous Random Variables** if

- There is a probability density function

$$f_X : \mathbb{R} \to [0, \infty)$$

- Any event A in $\mathbb{R}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx$$

Note: — X is Discrete ; Range$(X) \equiv$ Countable

Probability mass function $f(t) = \mathbb{P}(X = t)$ $\quad t \in$ Range(x)

Contrast to discrete world :

X is continuous random variable and $f_X(\cdot)$ is p.d.f. of X then

$$\mathbb{P}(X = a) = \mathbb{P}(X \in [a, a])$$

$$= \int_a^a f_X(x)\, dx$$

$$= 0 \quad !$$

Distribution Functions of X

let X be any random variable

(Discrete   Continuous)

$F_X : \mathbb{R} \to [0,1]$ (called the distribution function of X)

is given by

$$F_X(x) = \mathbb{P}(X \le x) \quad \forall x \in \mathbb{R}.$$

Discrete

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \sum_{\substack{t \in \text{Range}(X) \\ t \le x}} \mathbb{P}(X = t)$$

Continuous (f_X)

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \mathbb{P}(X \in (-\infty, x])$$

$$= \int_{-\infty}^{x} f_X(y)\, dy$$

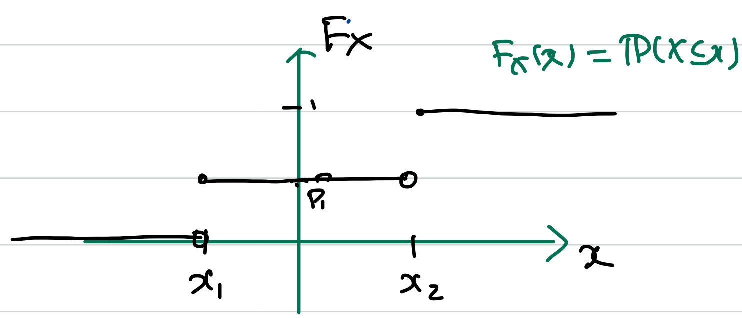
Example :- Discrete

$\text{Range}(X) = \{x_1, x_2\}$

$\mathbb{P}(X = x_1) = p_1$

$\mathbb{P}(X = x_2) = p_2$

$p_1 + p_2 = 1$



Note : Behaviour of $F_X(\cdot)$

Continuous : $F_X(\cdot)$ is continuous $\Leftarrow$ $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$

- $F_X'(x) = f_X(x)$ at all continuity point of $f_X(\cdot)$

Examples of Continuous random variables

- $X \sim \text{Uniform}(a,b)$; $a<b$ $a,b \in \mathbb{R}$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$$

- $x \le a$ $\Rightarrow f_X(y) = 0 \quad y \le x$

$$\Rightarrow \mathbb{P}(X \le x) = \int_{-\infty}^{x} 0 \cdot dy = 0$$

- $a < x \le b$

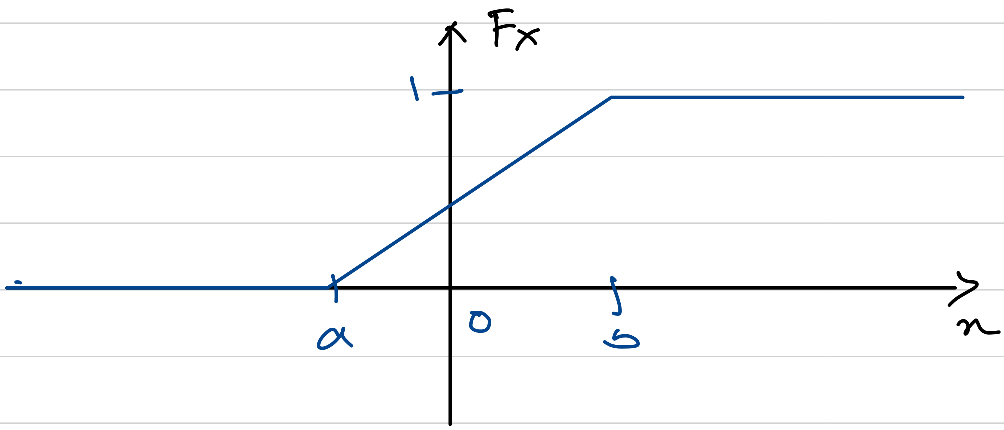
$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$$

$$= \int_{-\infty}^{a} f_X(y)\,dy + \int_{a}^{x} f_X(y)\,dy$$

$$= 0 + \int_a^x \frac{1}{b-a}\,dy$$

$$= \frac{x-a}{b-a}$$

(Ex.) $\underline{x > b}$ $\quad P(X \le x) = 1$



$\underline{X \sim \text{Normal}(0,1)}$

$$f_X(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \qquad y \in \mathbb{R}$$

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

- no clear anti derivative
- computational - numerically

$$f(y) = 1/\sqrt{2\pi}\, \exp(-y^2/2).$$

• Symmetric around 0.

$\Rightarrow$ area of Brown = area of green.

$$\mathbb{P}(X \le -2) \underset{Ex}{=} 1 - \mathbb{P}(X \le 2) \quad - (*)$$

- $\mathbb{P}(X \le 0) = \frac{1}{2} = \mathbb{P}(X \ge 0)$

• Computationally : $\mathbb{P}(X \le z)$ for $z \ge 0$ is enough to do. and use $(*)$ for $z < 0$.

$Y \sim \text{Normal}(\mu, \sigma^2)$

Y has p.d.f

$$f_Y(z) = \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}, \quad z \in \mathbb{R}$$

(Ex.) $$\mathbb{P}(Y \le y) = \int_{-\infty}^{y} f_Y(z)\, dz$$

$$= \int_{-\infty}^{y} \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dz$$

$u = \frac{z-\mu}{\sigma}$ $$= \mathbb{P}\left(X \le \frac{y-\mu}{\sigma}\right)$$

Computational Tool :-

	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.0	0.500	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564	0.571
0.2	0.579	0.587	0.595	0.603	0.610	0.618	0.626	0.633	0.641	0.648
0.4	0.655	0.663	0.670	0.677	0.684	0.691	0.698	0.705	0.712	0.719
0.6	0.726	0.732	0.739	0.745	0.752	0.758	0.764	0.770	0.776	0.782
0.8	0.788	0.794	0.800	0.805	0.811	0.816	0.821	0.826	0.831	0.836
1.0	0.841	0.846	0.851	0.855	0.860	0.864	0.869	0.873	0.877	0.881
1.2	0.885	0.889	0.893	0.896	0.900	0.903	0.907	0.910	0.913	0.916
1.4	0.919	0.922	0.925	0.928	0.931	0.933	0.936	0.938	0.941	0.943
1.6	0.945	0.947	0.949	0.952	0.954	0.955	0.957	0.959	0.961	0.962
1.8	0.964	0.966	0.967	0.969	0.970	0.971	0.973	0.974	0.975	0.976
2.0	0.977	0.978	0.979	0.980	0.981	0.982	0.983	0.984	0.985	0.985

Table 5.1: Table of Normal$(0,1)$ probabilities. For $X \sim$ Normal$(0,1)$, the table gives values of $P(X \leq z)$ for various values of z between 0 and 2.18 upto three digits. The value of z for each entry is obtained by adding the corresponding row and column labels.

$X \sim \text{Normal}(0,1)$

$$\mathbb{P}(-1 \leq X \leq 1) = \mathbb{P}(X \leq 1) - \mathbb{P}(X \leq -1)$$

$$\overset{(*)}{=} \mathbb{P}(X \leq 1) - (1 - \mathbb{P}(X \leq 1))$$

$$= 2\,\mathbb{P}(X \leq 1) - 1$$

(Normal Table)

$$\simeq 2(0.841) - 1$$

$$= 0.682$$