Announcements :-

• Postpone the midterm — Nov 5th
— online — During class)

(write on paper & take scan & submit pdf) — (R-program) (- PDF or R-code text)

(I)   (II)

[Thin attendance] • Thursday — office hour
7pm - 8pm

• Don't accept Late submissions — { Drop one lowest H.w. Score for final H.w Score

Recall

- Continuous random variable

$X : S \to \mathbb{R}$ if $\exists$ $f_X : \mathbb{R} \to [0, \infty)$

p.d.f such that

[Distribution of X] • $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

[Distribution function] • $F(x) := \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$

$F \equiv F_X : \mathbb{R} \to [0,1]$

F - continuous & $F'(x) = f_X(x)$

at all continuity points of $f_X(\cdot)$

Examples :-

∘ $X \sim \text{Uniform}(a,b)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

· $X \sim \text{Normal}(\mu, \sigma^2)$

$$f_X(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \qquad x \in \mathbb{R}.$$

Exponential Random Variable - $X \sim \text{Exp}(\lambda)$

observed in experiments

- Radio active isotopes decay to its stable form.
- $N(0)$ - amount of radioactive material at time 0.

  $N(t)$ - amount of radioactive material at time t. (not decayed)

$$\frac{N(t)}{N(0)} \simeq e^{-\lambda t} \quad \text{for some } \lambda > 0.$$

X = time taken by a randomly chosen radio active atom to decay to its stable form.

Is it possible to define such a random variable?

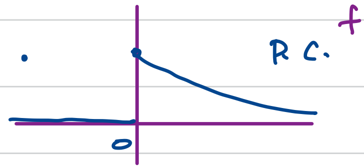
$$\mathbb{P}(X \geq t) = e^{-\lambda t}$$

<u>Definition</u> :- $X \sim \text{Exp}(\lambda)$ if it has a p.d.f $f_X : \mathbb{R} \to [0,\infty)$ given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Checks:-

a [p.d.f] • $f_X(\cdot) \geq 0$



• $\int_{-\infty}^{\infty} f_X(x)\,dx := \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \left(-e^{-\lambda x}\right)\Big|_{0}^{\infty} = 1$

• $\mathbb{P}(X \geq t) = \int_{t}^{\infty} f_X(x)\,dx$

[Model is correct]

$$= \begin{cases} 0 & t < 0 \\ \int_{t}^{\infty} \lambda e^{-\lambda x}\,dx & t \geq 0 \end{cases}$$

$$= \begin{cases} 0 & t < 0 \\ e^{-\lambda t} & t \geq 0 \end{cases} \quad — (*)$$

Another Experiment : Waiting time at a Counter $= X$

- Cinema sales ticket
- Bus stop
- Server to process
- ...

[Memoryless Property]

$X \sim \mathcal{E}xp(\lambda)$

$$\mathbb{P}(X > t+s \mid X > t) \qquad t \geqslant 0, \; s \geqslant 0$$

$$= \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > t)}$$

$$= \frac{\cancel{e^{-\lambda(t+s)}}}{\cancel{e^{-\lambda t}}}$$

$$= e^{-\lambda s} = \mathbb{P}(X > s)$$

Expectation & Variance for Continuous Random variables

- X is a continuous random variable with p.d.f $f_X : \mathbb{R} \to (0, \infty)$

- $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$\boxed{\text{if integral is well defined}}$

Contrast - Discrete

$$E[X] = \sum_{t \in \text{Range}(X)} t\, P(X=t)$$

if $E[X] < \infty$ then

- $$Var[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx$$

Interpretation :- $E[X] \equiv$ measure of centre of the distribution of X

$SD[X] = \sqrt{Var[X]}$

$\equiv$ measure of spread of the distribution of X

• $X \sim \text{Uniform}(a,b)$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx = \int_a^b x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \left. \frac{x^3}{2} \right)_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{a+b}{2}.$$

• $\underline{X \sim \text{Normal}(\mu, \sigma^2)}$

$$E[X] = \int_{-\infty}^{\infty} x \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \, dx$$

[$\underline{\text{Ex}}$: integral exists]

$u = \frac{x-\mu}{\sigma}$; $du = \frac{dx}{\sigma}$

$$= \int_{-\infty}^{\infty} \frac{(\sigma u+\mu)\, e^{-\frac{u^2}{2}}}{\sqrt{2\pi}\,\cancel{\sigma}}\,\cancel{\sigma}\, du$$

$\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$ is p.d.f of normal(0,1)

$$= \int_{-\infty}^{\infty} \frac{\sigma u\, e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\, du + \mu \int_{-\infty}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

$$= \sigma \int_{-\infty}^{\infty} \frac{u\, e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\, du + \mu \cdot 1$$

$\frac{u\, e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$ is an odd

$$= \sigma \cdot 0 + \mu$$

$$\Rightarrow \quad E[X] = \mu \qquad \square$$

Ex:- $\quad Var[X] := \int_{-\infty}^{\infty} \frac{(x-\mu)^2\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dx$

$$= \cdots = \sigma^2$$

$X \sim \mathcal{E}xp(\lambda)$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^{\infty} x\, \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} x\, e^{-\lambda x}\, dx$$

$$\overset{\text{Ex}}{=} \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-\lambda x}}{\lambda}\, dx \right]$$

[Integration by parts]

$$\text{Ex.} = \lambda \left[0 + \frac{1}{\lambda^2} \right]$$

$$= \frac{1}{\lambda}$$