**Recall** :-

$S$ - sample space - Countable / finite
$\mathcal{F}$ - Events $\equiv \mathcal{P}(S)$ power set of $S$
$\mathbb{P}$ - probability $\mathbb{P}: \mathcal{F} \to [0,1]$

- $\mathbb{P}(S) = 1$
- $\{E_k\}_{k \geq 1}$ $E_k \subseteq S$ $E_\ell \cap E_j = \phi$ $\ell \neq j$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

$S =$ outcome of $n$-tosses of a coin $\equiv \{H, T\}^n$
"Coin biased Heads - $p$"

$$\mathbb{P}(\{\omega_1, \ldots, \omega_n\}) = p^{\#\{j : \omega_j = H\}} (1-p)^{\#\{j : \omega_j = T\}}$$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

**Example:** $n = 5$ $A = \{3 \text{ Heads in } 5 \text{ tosses}\}$

$$\mathbb{P}(A) = \sum_{\omega \in A} p^3 (1-p)^2$$
$$= |A| \, p^3 (1-p)^2$$
$$= {}^5C_3 \, p^3 (1-p)^2$$

**Random variables** : $X : S \to T$

**Example** : $X = \#$ of heads in $n$-tosses of a biased $p$-coin

$X \sim$ Binomial$(n, p)$ :- Range$(X) = \{0, 1, 2, \ldots, n\}$

$$\mathbb{P}(X = k) = {}^nC_k \, p^k (1-p)^{n-k}$$

- $E[X]$ - expected value of $X$

$$= \sum_{k \in T} k \, \underbrace{\mathbb{P}(X=k)}$$

where the random variable was centered

$$= \sum_{k \in T} k \, f_X(k)$$

p.m.f of $X$

- $SD[X] = \sqrt{Var[X]}$ - Variance of $X$

Spread of the random variable.

$$Var[X] := E(X - E[X])^2$$

De Moivre's C.L.T. $X_n \sim \text{Binomial}(n, p)$ $p =$ constant

$$\mathbb{P}\left(a \le \frac{X_n - np}{\sqrt{np(1-p)}} \le b\right) \longrightarrow \int_a^b \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy$$

as $n \to \infty$

Poisson Approximation $Y_n \sim \text{Binomial}(n, \frac{\lambda}{n})$

$$\mathbb{P}(Y_n = k) \longrightarrow \frac{e^{-\lambda} \lambda^k}{k!} \quad \forall k \in \{0, 1, \ldots\}$$

Geometric $(p)$ $X =$ # of trials required for first success in independent $\underbrace{\text{Bernoulli}(p)}_{\text{Binomial}(1,p)}$

$\boxed{\text{Discuss soon}}$

HyperGeometric $(N, m, k)$ - $X = \ldots$ Exercise

Uniform $\{1, \ldots, n\}$ — $X = \ldots$

Example: Choose a number "uniformly" from $(0,1)$

i.e $\mathbb{P}(\{x\}) = p \equiv$ one wish $\forall x \in (0,1)$.

Is such a Probability Possible ?

$$E = \{ \tfrac{1}{n} : n \geq 2 \} \subseteq (0,1)$$

$$= \bigcup_{k \geq 2} \{1/k\}$$

Axiom 2 of Probability

$$\mathbb{P}(E) = \sum_{k=1}^{\infty} \mathbb{P}(\{1/k\})$$

($p > 0$ - Series diverges) $\quad = p + p + \cdots$

$\Rightarrow$ Force :- $p = 0$

Is the below Contradiction

$$\mathbb{P}(S) = \mathbb{P}\Big(\bigcup_{x \in (0,1)} \{x\}\Big)$$

Axiom 2 only works for countable disjoint unions

not ⓧ $\quad$ "=" $\displaystyle\sum_{x \in (0,1)} \mathbb{P}(\{x\})$ $\quad$ uncountable

$$= 0$$

Challenge: $S = (0,1)$ $\quad \mathcal{F} \equiv$ Create Event space

$\mathbb{P} \equiv$ "ensures every number is equally likely to be chosen"

$\mathbb{P}$ - Axiom 2 $\quad$ & $\quad \mathbb{P}(S) = 1. \equiv \{ \mathbb{P}(\{x\}) = 0$

## Equally likely re formulation


$\mathbb{P}([\frac{1}{4}, \frac{3}{4}])$ = "length of interval $[\frac{1}{4}, \frac{3}{4}]$"

$= \frac{1}{2}$

wish:

$A \subseteq (0,1)$  $\mathbb{P}(A)$ = "length of A"

Difficulty – cannot define lengths for all $A \subseteq (0,1)$,

– only do for "Borel sets"

Typical Event A :– intervals, complement, unions

## Density function :– $f : \mathbb{R} \to \mathbb{R}$

(i) $f(x) \geq 0$

(ii) $f$ is piecewise continuous

(iii) $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

Contains intervals
– unions
– complement

check book for precise def.

## Probability on $\mathbb{R}$ and Event $\equiv \mathcal{B} \neq \mathcal{P}(\mathbb{R})$

Define: $\mathbb{P}(A) = \int_A f(x)\, dx \quad \forall A \in \mathcal{B}$

[Theorem 5.1.5] ← · $\mathbb{P}(S) = \int_{-\infty}^{\infty} f(x)dx = 1$

- Ex - Axiom 2 ✓

Ex: $\mathbb{P}(\{a\}) = \int_a^a f(x)dx = 0$

$a < b \quad a, b \in \mathbb{R}$

## Revisit Example: $U(a,b) \equiv$ uniform on $(a,b)$

Wish – Choose a number "uniformly" from $(a,b)$

Define:

$$f : \mathbb{R} \to \mathbb{R}$$

$$\left.\begin{array}{l} f \geq 0 \\ f \text{ is p.c.} \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{array}\right\} \equiv \quad f(x) = \begin{cases} 0 & x \notin (a,b) \\ \frac{1}{b-a} & x \in (a,b) \end{cases}$$

and

$$\mathbb{P}(A) = \int_A f(x)\,dx$$

Example:

$a = 0, \quad b = 1 \qquad A = [1/4, 3/4]$

$$\Rightarrow \quad \mathbb{P}(A) = \int_{1/4}^{3/4} f(x)\,dx = \int_{1/4}^{3/4} \frac{1}{1-0}\,dx$$

Wish ✓

$$= 3/4 - 1/4 = 1/2$$

## Continuous Random Variable:

$X : S \to \mathbb{R}$ is a continuous random variable if it has density $f$ such that

$$\mathbb{P}(X \in A) = \int_A f(x)\,dx$$

$f = f_X(\cdot) \equiv$ probability density function of $X$.

$\sigma \neq 0, \ \mu \in \mathbb{R}$

$X \sim$ Normal $(\mu, \sigma^2)$ – Distribution on $\mathbb{R}$

$$\mathbb{P}(X \in A) = \int_A f(x)\, dx \quad \text{where}$$

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \qquad x \in \mathbb{R}$$

Ex: Check $f$ is a density function

De Moivre's Central limit theorem : $\mu = 0, \ \sigma = 1$