Recall :

$\mathbb{P} : \mathcal{F} \to [0,1]$; $\mathbb{P}(S) = 1$

$A_1, \ldots A_n, \ldots$: $A_i \cap A_j = \phi$, $i \neq j$

$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$

- $(S, \mathcal{F}, \mathbb{P}) \equiv$ Probability space
  - S ↳ set of all possible outcomes
  - $\mathcal{F}$ ↳ Events - all subsets of S

- $X : S \to T$, S - Countable. Probability mass function $f_X(t) = \mathbb{P}(X = t)$

- $X \sim$ Binomial (n, p) $\quad X = \#$ of success in n "independent" trials

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, 2, \ldots, n$$

Expected value

Q : On an average how many success will there be in n $\underbrace{\text{independent Bernoulli } (p) \text{ trials}}$?

Bernoulli (p) trials
- n - trials ; Each trial has 2 outcomes: success (p), failure ($1-p$)
- Outcome of one trial does not affect the outcome of any other trial.

$\tilde{Q}$: $X \sim$ Binomial (n, p)

What is the average value of X ?

Example :- – Roll a Die

– Possible outcomes $= \{1, 2, 3, 4, 5, 6\}$

– what is the average value that shows up?

$$\frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$

Rewrite

$$1\left(\tfrac{1}{6}\right) + 2\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right) + 4\left(\tfrac{1}{6}\right) + 5\left(\tfrac{1}{6}\right) + 6\left(\tfrac{1}{6}\right)$$

Probabilities with which each outcome happens

weighted average

$x_1 \ldots x_n$:

$$\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

$w_i = 1/6$

$$\sum_{i=1}^{6} w_i = 1$$

Expectation

Let $X : S \to T$ be a discrete random variable (so T is countable). Then the expected value (or average) of X is written as $E[X]$ and is given by

$$E[X] = \sum_{t \in T} t \cdot P(X = t)$$

provided that the sum converges absolutely. In this case we say that X has "finite expectation".

- Refer to PSWEUR chapter 4 for Examples

- If the sum diverges to $\pm\infty$ we say the random variable has infinite expectation.
- If the sum diverges, but not to infinity, we say the expected value is undefined.

Example 2 : $X \sim \text{Binomial}(n, p)$

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0, 1, 2, \ldots, n$$

$$E[X] = \sum_{k=0}^{n} k \, \mathbb{P}(X=k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$[\text{Induction}] \overset{Ex}{=} np$$

Example :
- Toss a coin till the 1st Head appears
- p - Probability of Heads

let X = denote the trial at which the 1st head appears

$$E[X] = ?$$

Distribution of X : $\text{Range}(X) = \mathbb{N}$

$$\mathbb{P}(X=k) = \mathbb{P}(\underbrace{T\,T \ldots T}_{k-1} \underset{k}{H}) = (1-p)^{k-1} p$$

Expected value of X

$$E[X] = \sum_{k=1}^{\infty} k\, P(X=k) = \lim_{n\to\infty} T_n$$

where $\quad T_n = \sum_{k=1}^{n} k\, P(X=k) =$

$$= \sum_{k=1}^{n} k\,(1-p)^{k-1}\, p$$

$$= p \sum_{k=1}^{n} k\,(1-p)^{k-1} \xrightarrow{n\to\infty} 1/p$$

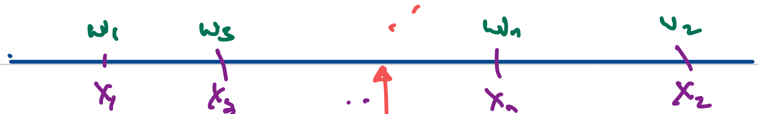
Ex: (Geometric Series)

$$E[X] = 1/p$$

Interpret $E[X]$:.

- $X \sim \text{Binomial}(10, \tfrac{1}{2})$ $\qquad E[X] = 5$ — Perform trial many times, on an average we will get 5
- $x_1, .., x_n$ — $\dfrac{\sum_{i=1}^{n} x_i}{n} = 3.5$
- $x_1, \dots, x_n$ — $\dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = 2$

"Point of Balance"



$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

other notions

- median
- mode

Properties of Expected Value

Suppose that X and Y are discrete random variables, both with finite expected value and both defined on the same sample space S. If a and b are real numbers then

$$E[aX] = aE[X];$$

$$E[X + Y] = E[X] + E[Y]; \quad \text{and}$$

$$E[aX + bY] = aE[X] + bE[Y].$$

If $X \geq 0$ then $E[X] \geq 0$.

From Definition of Expected value.

Variance

Let $X : S \rightarrow T$ be a discrete random variable with finite expected value. Then the variance of the random variable is written as $Var[X]$ and is defined as

$$Var[X] = E[(X - E[X])^2] = \sum_{t \in T} (t - E[X])^2) P(X = t)$$

- The standard deviation of X is written as $SD[X]$ and is defined as

$$SD[X] = \sqrt{Var[X]}$$

$x_1, \dots x_n$ = numbers $\quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \equiv Var(x)$$

$$\boxed{\text{statistics : } \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \equiv \text{Sample variance}}$$

$$Var[X] = \sum_{t \in T} \underbrace{(t - E[X])^2} \underbrace{\mathbb{P}(X=t)}$$

- If X has high probability of assuming t which is far from $E[X]$ then variance will be large.

- interpretation also available for small variance.

$$SD[X] = \sqrt{Var[X]}$$

Units $\begin{cases} X - \text{meters} \quad \dots \to \quad Var[X] = (\text{meters})^2 \\ SD[X] = \text{meters} \end{cases}$

Loosely speaking : $\quad \mu = E[X] \quad \sigma = SD[X]$

Effective Range of X = " $(\mu - \sigma, \mu + \sigma)$ "

Q:. $k \geq 1$; $\mu = E[X]$, $\sigma = SD[X]$; $\mathbb{P}(|X - \mu| \geq k\sigma) = ?$

Interpretation of Variance

- If X has a high probability of being far away from $E[X]$ the variance will tend to be large, while if X is very near $E[X]$ with high probability the variance will tend to be small.
- If we were to associate units with the random variable X (say *meters*), then the units of $Var[X]$ would be $meters^2$ and the units of $SD[X]$ would be *meters*.
- Informally we will view the standard deviation as a typical distance from average.
- It is possible that $Var[X]$ and $SD[X]$ could be infinite even if $E[X]$ is finite – meaning that the random variable has a clear average, but is so spread out that any finite number underestimates the typical distance of the random variable from its average.

Standardising Random Variables

- A standardized random variable X is one for which

$$E[X] = 0 \quad \text{and} \quad Var[X] = 1.$$

- Let X be a discrete random variable with finite expected value and finite, non-zero variance. Then $Z = \frac{X - E[X]}{SD[X]}$ is a standardized random variable.

$$\because \; E[Z] = E\left[\frac{X - E[X]}{SD[X]}\right] = \frac{E[X] - E[X]}{SD[X]} = 0$$

$$Var[Z] = E[Z - E[Z]]^2 = \frac{E(X - E[X])^2}{SD[X]^2} = 1$$

Example: $X \sim \text{Uniform}\{1,2,3,4,5,6\}$

$\mathbb{P}(X=k) = 1/6 \qquad k = 1,2,3,4,5,6$

$$E[X] = \frac{1+2+3+4+5+6}{6} = 3.5$$

(In my sample in R ~ 3.55)

$$\text{Var}[X] = \sum_{k=1}^{6} (k-3.5)^2 \cdot \frac{1}{6}$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + \cdots}{6}$$

$$= \frac{(2.5)^2 + (1.5)^2 + (0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2}{6}$$

$$= 2.91666\ldots$$

(In my sample in R ~ 2.82)

Sampling from a given distribution

- we can use the `sample` function.
- takes a sample of the specified size (specified by `size`) from the elements of x using either with or without replacement (specified by `replace`). The optional `prob` argument can be used to give a vector of weights for obtaining the elements of the vector being sampled.

```
> x = c(1,2,3,4,5,6)
> probx= c(1/6,1/6,1/6,1/6,1/6,1/6)
> Rolls=sample(x, size=1800,  replace=T, prob=probx)
```

Uniform(1,2,3,4,5,6)

```
> mean(Rolls)
[1] 3.501111
> var(Rolls)
[1] 3.001666
```

Sums of Rolls

Suppose we wish to simulate in R the experiment that we did in class of Rolling a die and noting down its sum. We can use the `sample` , `matrix` and `apply`.

```
> x = c(1,2,3,4,5,6)
> probx= c(1/6,1/6,1/6,1/6,1/6,1/6)
> Rolls=sample(x, size=1500,  replace=T, prob=probx)
> Rollm=matrix(Rolls, 5)
> # above creates a matrix 5 columns and 300 Rows
> Rollsums = apply(Rollm, 2, sum)
```

Sums of Rolls