(Sample Space)

Recall: Toss a coin $n=5$ times
$S = \{H, T\}^n$

- A sample space S is a set.
- The elements of the set S will be called "outcomes" and should be viewed as a listing of all possibilities that might occur.
- We will call the process of actually selecting one of these outcomes an "experiment".

(Temporary Definition of Event) Given a sample space S, an "event" is any subset $E \subset S$.

Example: $A = \{$ 3 Heads in 5 Tosses $\}$

(Probability Space Axioms)

Probability $\longleftrightarrow$ Relative frequency

$\hookrightarrow$ we will make this connection precise later in this course.

Let S be a sample space and let $\mathcal{F}$ be the collection of all events. A "**probability**" is a function $P : \mathcal{F} \to [0, 1]$ such that

$P(S) = 1$; and

If $E_1, E_2, \ldots$ are a <u>countable</u> collection of disjoint events (that is, $E_i \cap E_j = \varnothing$ if $i \neq j$), then

$$P(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} P(E_j). \tag{1}$$

$\rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + P(B)$ if $A \cap B = \phi$.

Random Variable

X = # of Heads in $n=5$ tosses.

- A "discrete random variable" is a function $X : S \to T$ where S is a sample space equipped with a probability P, and T is a countable (or finite) subset of the real numbers.
- P generates a probability on T and since it is a discrete space, the distribution may be determined by knowing the likelihood of each possible value of X.
- $f_X : T \to [0,1]$ given by

$$f_X(t) = P(X = t)$$

referred to as a "probability mass function".

Binomial Distribution

$X \sim$ **Binomial**(n, p): Let $0 \leq p \leq 1$ and let $n \geq 1$ be an integer. If X is a random variable taking values in $\{0, 1, \ldots, n\}$ having a probability mass function

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for all $0 \leq k \leq n$, then X is a binomial random variable with parameters n and p. We have seen that such a quantity describes the number of successes in n Bernoulli trials.

Heads – success (p)

• Each toss as a trial (Bernoulli Trial)

Tails – failure

Plotting Binomial Probabilities

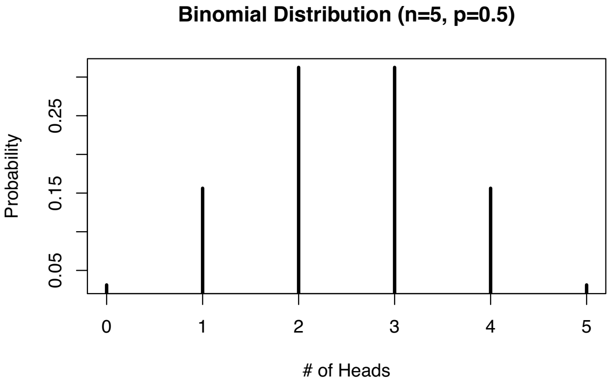
Please try this on R-studio or R-console or ..

x 0, 1, 2, 3, 4, 5

computing ${}^{5}C_{x} (0.5)^{x} (0.5)^{5-x}$

```
> x <- 0:5
> plot(x,dbinom(x,size=5,prob=.5),
+     type='h',
+     main='Binomial Distribution (n=5, p=0.5)',
+     ylab='Probability',
+     xlab ='# of Heads',
+     lwd=3 )
```

Plotting Binomial Probabilities



X-outcome of fair die roll
Y be the number of heads in X coin flips.

- What is the distribution of X?

$T := \text{Range}(X) = \{1,2,3,4,5,6\}$

$$f_X : T \to [0,1]$$
$$f_X(t) = \mathbb{P}(X=t) = \frac{1}{6} \quad \forall t \in T$$

- ____________________________

Equally likely outcome experiment

S - sample space ; $|S| < \infty$

$A \subseteq S \qquad \mathbb{P}(A) = \frac{|A|}{|S|} \leftarrow$ Equally likely outcome

X-outcome of fair die roll

Y be the number of heads in X coin flips.

- What is the distribution of $(Y|X = n)$ for $n = 1, \ldots \cancel{5?}$, 6 ?

Given $X=n$

→ what values Y can assume?

– what probabilities does Y assign to these values?

– Conditional Probability

$X = 4$ i.e. - Roll of the die $\equiv 4$
- Toss a fair coin 4 times
$Y = \#$ of heads in 4 tosses.

$$\text{Range}(Y \mid X=4) = \{0, 1, 2, 3, 4\}$$

$$\mathbb{P}(Y=k \mid X=4) \qquad k \in \{0,1,2,3,4\}$$

$$= {}^{4}C_{k} \left(\tfrac{1}{2}\right)^{k} \left(\tfrac{1}{2}\right)^{4-k}$$

In general: $X = n$ then $Y \mid X=n$ has:

$$\text{Range}(Y \mid X=n) = \{0, 1, \ldots, n\}$$

$$\mathbb{P}(Y=k \mid X=n) = {}^{n}C_{k} \left(\tfrac{1}{2}\right)^{k} \left(\tfrac{1}{2}\right)^{n-k}$$

$$k \in \{0, 1, \ldots, n\}$$

Conditional distribution of

$$Y \mid X=n \sim \text{Binomial}\left(n, \tfrac{1}{2}\right)$$

Conditional Probability

Let S be a sample space with probability P. Let A and B be two events with $P(B) > 0$. Then the conditional probability of A given B written as $P(A|B)$ and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Exercise

Derive This in The Equally likely out come Experiment Sets up

X-outcome of fair die roll

Y be the number of heads in X coin flips.

- What is the conditional probability ~~$P(X|Y=0)$?~~

$$\mathbb{P}(X=1 \mid Y=0) = ?$$

From definition:

$$\mathbb{P}(X=1 \mid Y=0) = \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0)} = \frac{\mathbb{P}(Y=0, X=1)}{\mathbb{P}(Y=0)}$$

$$\mathbb{P}(Y=0, X=1) = \mathbb{P}(Y=0 \mid X=1)\, \mathbb{P}(X=1)$$

$$= \frac{1}{2} \cdot \frac{1}{6}$$

$$= \frac{1}{12}$$

Set up the experiment "Conditionally"

- Roll a die once - outcome is X
  - Toss a fair coin X times
    $Y = \#$ of Heads.

$$\{Y=0\} = \bigsqcup_{i=1}^{6} \{Y=0, X=i\}$$

↳ Disjoint union.

$$\mathbb{P}(Y=0) = \mathbb{P}\left(\bigsqcup_{i=1}^{6} \{Y=0, X=i\}\right)$$

Axiom ② of Probability

$$= \sum_{i=1}^{6} \mathbb{P}(Y=0, X=i)$$

$$= \sum_{i=1}^{6} \mathbb{P}(Y=0 \mid X=i)\ \mathbb{P}(X=i)$$

Binomial $(i, \frac{1}{2})$

Equally likely outcome experiment

$$= \sum_{i=1}^{6} \left(\frac{1}{2}\right)^i \cdot \frac{1}{6}$$

$$\therefore\ \mathbb{P}(X=1 \mid Y=0) = \frac{1/12}{1/6 \sum_{i=1}^{6} (1/2)^i} \ldots$$

Bayes Theorem

<u>Example</u> :- $A = \{Y=0\}$ $B_k = \{X=k\}$ $\boxed{i=1}$

Theorem

Suppose A is an event, $\{B_i : 1 \leq i \leq n\}$ are a collection of disjoint events whose union contains all of A. Further assume that $P(A) > 0$ and $P(B_i) > 0$ for all $1 \leq i \leq n$. Then for any $1 \leq i \leq n$,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}.$$