Hypothesis Testing :-

Example :- Coin ; Probability of heads $= p$

Question: Estimate p ?

Generate :- $\underbrace{X_1, X_2, \ldots, X_{100}}_{\text{i.i.d Bernoulli } (p)}$ coin tosses

Compute :- $\sum_{i=1}^{100} X_i \underset{(say)}{=} 67$

Estimate : $\hat{p} = \frac{67}{100} = 0.67$ — Confidence interval

Question :- Test if the coin is fair or not

i.e Is $p = 0.5$?

Based on the observed data how does one make a judgement on this question?

Ex :- Suppose Coin was fair ; $X_i \sim \text{Bernoulli}(1/2)$

Then

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i \geq 67\right) \sim 0.04$$

i.e. the observed event has very small Probability $\Rightarrow$ Coin perhaps is not fair ie the experimental evidence does not support $p = 0.5$.

Basic framework X – Population has a p.m.f or p.d.f $f(\cdot \mid p)$ where $p \in \mathcal{P} \subseteq \mathbb{R}^d$.

Conjecture :: X – Is $p \in \mathcal{P}_0 \subsetneq \mathcal{P}$?

Example: $\mathcal{P} = [0,1]$, $\mathcal{P}_0 = \{0.5\}$

Refered to as Null hypothesis

Formal Theory of Hypothesis Testing ::

- Sample $X_1, \ldots, X_n$ i.i.d X
  - Develop a "test statistic" = function of $X_1 \ldots, X_n$

    Distribution will depend on parameter p

- To "verify" or test the conjecture if $p \in \mathcal{P}_0$ – one asks if the observed statistic could arise from a $p \in \mathcal{P}_0$

- quantifies the degree of this possibility
through a probability $\equiv$ p-value.

- Requirement :- the distribution of the test-statistic is fully known when $p \in \mathbb{P}_0$.

- Necessity :- To find the "best" test-statistic
defined through some optimality condition
- hard to establish.

Intuitive Approach :-

- unknown parameter of interest $- p \in \mathbb{P}$
- null hypothesis of interest about the parameter, ie $p \in \mathbb{P}_0$
- test-statistic - whose distribution is known under null.

Example : $X \sim \text{Normal}(\mu, \sigma^2)$ $\quad \sigma \equiv$ known

$p \equiv (\mu, \sigma^2) \subseteq \mathbb{R} \times \mathbb{R}_+$ $\quad \mu \in \mathbb{R}$ - unknown.

- let $X_1, X_2, \ldots, X_n$ be i.i.d. X
- Conjecture :- Is $\mu = c$? $\quad c$ - known value.
(Null Hypothesis)

Is $p \in \mathbb{P}_0 = \{c\} \times \{\sigma^2\}$

Test statistic as T : if $\mu = c$

$$T(X_1, \ldots X_n) \equiv \sqrt{n}\left(\frac{\overline{X} - c}{\sigma}\right) \sim \text{Normal}(0,1)$$

Formalise it :- Given $X_1, \ldots, X_n$ from population.

let $Y_1, \ldots, Y_n$ be i.i.d X

(mimic samples)

Calculate :-

$$\mathbb{P}(\overline{Y} \geq \overline{X}) = \mathbb{P}\left(\underbrace{\sqrt{n}\frac{(\overline{Y} - c)}{\sigma}} \geq \sqrt{n}\left(\frac{\overline{X} - c}{\sigma}\right)\right)$$

Normal $(0,1)$ if $X \sim \text{Normal}(c, \sigma^2)$

$$= \mathbb{P}\left(Z \geq \sqrt{n}\left(\frac{\overline{X} - c}{\sigma}\right)\right)$$

Procedure :- Fix a threshold say α

(Significance level $\alpha \in (0,1)$)

If $\mathbb{P}(\overline{Y} \geq \overline{X}) < \alpha$ then we would conclude $\mu = c$ is incorrect if $\mathbb{P}(\bar{Y} \geq \bar{x}) \geq \alpha$ then we would conclude that $\mu = c$ cannot be rejected.

<u>Z-test</u> :- <u>Null</u> : $\mu = c$.

- Fix $\alpha \in (0,1) \equiv$ threshold
- Compute $\sqrt{n}\left(\frac{\bar{X} - c}{\sigma}\right)$
- Check $\mathbb{P}\left(Z \geq \sqrt{n}\left(\frac{\bar{X}-c}{\sigma}\right)\right) \equiv \begin{cases} < \alpha \\ \geq \alpha \end{cases}$

Reject null hypothesis

Cannot reject the null hypothesis

<u>Example</u> :- - $X \sim \text{Normal}(\mu, 9)$

(a) - Sample $X_1, X_2, \ldots, X_{16}$ of i.i.d random variable X, $\bar{X} = 10.2$

- <u>Null-hypothesis</u> : $\mu = 9.5$.

against

<u>Alternative</u> <u>hypothesis</u> : $\mu > 9.5$

- Fix $\alpha = 0.05$

Compute :- $\sqrt{n}\left(\frac{\bar{X} - 9.5}{\sigma}\right) = \frac{4(10.2 - 9.5)}{3}$

$$\mathbb{P}\left(Z \geq \frac{4(10.2 - 9.5)}{3} \right) = 0.175$$

Ex

$Z \sim \text{Normal}(0,1)$

- As $0.175 \gg 0.05 = \alpha$, we would not reject the null hypothesis.

- We observed that $\bar{X} = 10.2$ has a 0.175 i.e. 17.5% chance of occuring when $\mu = 9.5$, hence we will not reject the null hypothesis.

Ex:- Replace (a) with (a)'

(a)' :- $\bar{X} = 10.2$ but

Null hypothesis : $\mu = 8.5$

Alternative hypothesis : $\mu > 8.5$

- Decide with $\alpha = 0.05$?