Confidence interval for mean μ when σ is unknown

Suppose X is known to be Normally distributed mean μ and variance σ^2.

Sample $X_1, X_2, \ldots X_n$ from population X

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Recall: $Z = \dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim \text{Normal}(0,1)$

$\hookrightarrow$ Central limit Theorem.

was used to find Confidence interval.

statistic $\quad T := \dfrac{\sqrt{n}(\bar{X} - \mu)}{S}$

Q: What is the distribution of T ?

A:

step 1 :- $X_i \overset{d}{=} \text{Normal}(\mu, \sigma^2)$ and

$\text{Var}(X_i) = \sigma^2$, $\text{mean}(X_i) = \mu$, independent.

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1)$$

Step 2 :- $S^2 = \dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$

$$\frac{(n-1)}{\sigma^2} S^2 \sim \chi^2_{n-1} \equiv \text{Chi-Squared}$$

Step 2a :- $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ and $\dfrac{(n-1)}{\sigma^2} S^2$ are independent.

step 3 :-

$$T := \frac{\sqrt{n}(\bar{X} - \mu)}{S} = \frac{\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{1}{n-1} \dfrac{(n-1)}{\sigma^2} S^2}}$$

$$\sim \frac{\text{Normal}(0,1)}{\sqrt{\dfrac{\chi^2_{n-1}}{n-1}}} \overset{d}{=} t_{n-1}$$

Answer: $T := \frac{\sqrt{n}(\bar{X}-\mu)}{S}$ has the

t-distribution with $n-1$ degrees of freedom and has p.d.f given by

$$f_T(t) = \frac{\Gamma(\frac{n}{2})}{\sqrt{(n-1)\pi}\,\Gamma(\frac{n-1}{2})} \left(1+\frac{t^2}{n-1}\right)^{-\frac{n}{2}}$$

$t \in \mathbb{R}.$

Final step :-

Find a :- $\mathbb{P}(|\bar{X}-\mu| < a)$ – large

$$= \mathbb{P}\left(\frac{\sqrt{n}|\bar{X}-\mu|}{S} < \frac{a\sqrt{n}}{S}\right)$$

$$= \mathbb{P}\left(|T| < \frac{a\sqrt{n}}{S}\right)$$

$$= \mathbb{P}\left(-\frac{a\sqrt{n}}{S} < T < \frac{a\sqrt{n}}{S}\right)$$

where $T \sim t_{n-1}$

Suppose we want 95% C.I for μ when σ is unknown

ie Find a:

$$\mathbb{P}(|\overline{X} - \mu| < a) \simeq 0.95$$

ie Find a:

$$\mathbb{P}\left(-\frac{a\sqrt{n}}{S} < T < \frac{a\sqrt{n}}{S}\right) \simeq 0.95$$

n - known sample size

From t_{n-1} distribution we know

$$\mathbb{P}\left(|T| < t_{n-1,\,0.95}\right) = 0.95$$

Set $\frac{a\sqrt{n}}{S} = t_{n-1,0.95}$

$$\Rightarrow \quad a = \frac{S}{\sqrt{n}}\, t_{n-1,0.95}$$

95% C.I for μ when σ - is unknown is given by

$$\left(\overline{X} - \frac{S}{\sqrt{n}}\, t_{n-1,\,0.95}\ ,\ \overline{X} + \frac{S}{\sqrt{n}}\, t_{n-1,\,0.95}\right)$$

Review - Chi-Squared Distribution

$X_1 \stackrel{d}{=} \text{Normal}(0,1)$

$$f_{X_1}(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \qquad z \in \mathbb{R}$$

$\underline{Q}$:- $U = X_1^2$ Find distribution of U

$\underline{A}$:- (i) Find distribution function of U

$$F_U(u) = \mathbb{P}(U \le u)$$

$$= \mathbb{P}(X_1^2 \le u)$$

$$= \mathbb{P}(-\sqrt{u} \le X_1 \le \sqrt{u})$$

$$= \int_{-\sqrt{u}}^{\sqrt{u}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$= 2 \int_{0}^{\sqrt{u}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

(ii) Differentiate to find the p.d.f of U

$$f_{U}(u) = 2 \frac{e^{-u/2}}{\sqrt{2\pi}} \frac{1}{2} u^{1/2-1}$$

$$f_{U}(u) = \frac{e^{-u/2}}{\sqrt{2\pi}} u^{1/2-1} \sim \chi^2_1 \text{ distribution}$$

$$\overset{d}{=}$$

$$\Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

[Hard]
[OMIT PROOF]

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1} \sim \chi^2_{n-1} = \Gamma\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right)$$

has p.d.f.

$$f(s) = \begin{cases} \dfrac{2^{-n/2} s^{n/2-1} e^{-s/2}}{\Gamma(n/2)} & s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

χ^2_{n-1} = Chi-squared distribution with $n-1$ degrees of freedom.

from $$S = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$