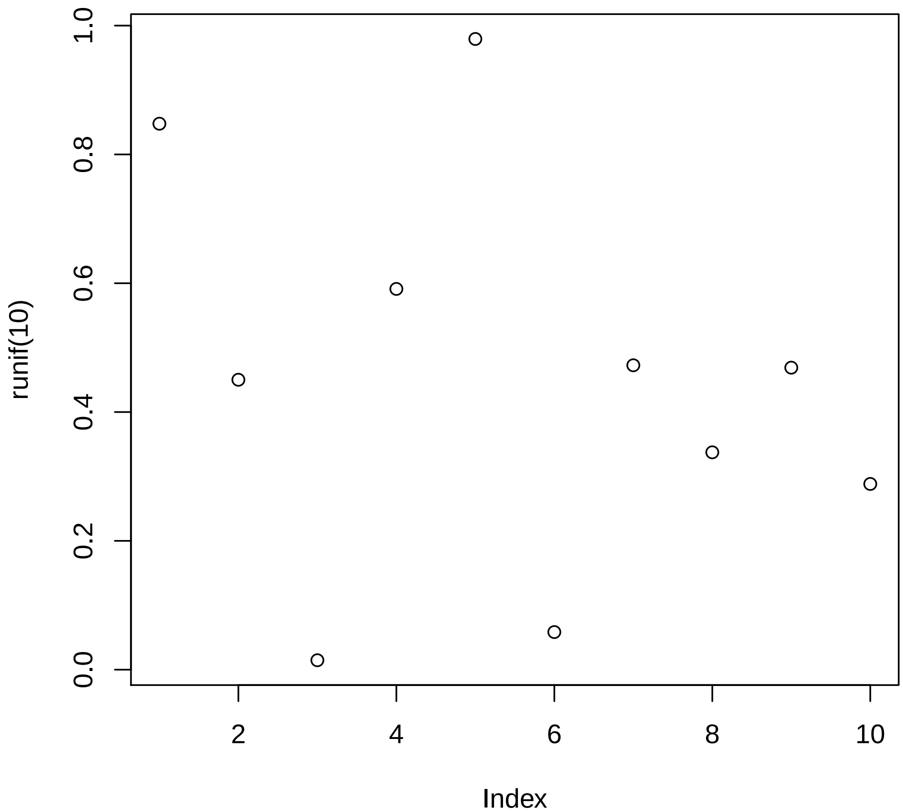
1. (a) Please write the down the R command that will provide the below plot. Describe in detail what the points in the plot represent.



Solution: `runif(n, min=0, max=1)` is an imbuilt function in R that generates n random samples from the uniform distribution on the interval from 'min' to 'max'. If 'min' or 'max' are not specified they assume the default values of '0' and '1' respectively.

The above plot is executed in R by:

```
> plot(runif(10))
```

function. It is a simple scatter plot of 10 samples generated by `runif(10)`. The x-axis are the index set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ of the samples generated and the y-axis are the corresponding sample generated.

(b) The following `R` code simulates a random variable `X`

```
> L = 10
> i = 0
> U = runif(1, min=0, max =1)
> Y = -log(U)/L
> Sum = Y
> while (Sum<1) {
+     U = runif(1, min=0, max =1)
+     Y = -log(U)/L
+     Sum = Sum +Y
+     i = i + 1
+ }
> X = i
```

(i) (10 points) Find $P(X = 0)$ and $P(X \geq 1)$.

(ii) (10 points) Suppose for $\lambda > 0$ and $T_1, T_2, \ldots T_n$ being i.i.d. Exp(λ) random variables it is known that for all $a > 0$,

$$P(\sum_{i=1}^{n} T_i \leq a) = \int_0^a \frac{\lambda^n}{n-1!} e^{-\lambda z} z^{n-1} dz$$

then find $P(X = n)$ for $n \geq 1$.

Solution: Let $\{U_n\}_{n\geq 1}$ be Uniform $(0, 1)$ random variables. Then the output X in the above algorithm is given by

$$X = \begin{cases} 0 & \text{if } \frac{-1}{10}\ln(U_1) > 1 \\ \max\{j \geq 1 : \frac{-1}{10}\ln(U_1 U_2 \ldots U_j) \leq 1\} & \text{otherwise} \end{cases}$$

First note that,

$$\text{Range}\{X\} = \{0\} \cup \mathbb{N}. \tag{1}$$

Observe that

$$\mathbb{P}(X = 0) = \mathbb{P}(\frac{-1}{10}\ln(U_1) > 1) = \mathbb{P}(U_1 \leq e^{-10}) = e^{-10} \tag{2}$$

For $i \geq 1$, let $T_i = -\frac{1}{10}\ln(U_i)$. Then $T_i \sim \text{Exp}(10)$ and consequently

$$\frac{-1}{10}\ln(U_1 U_2 \ldots U_j) = \frac{-1}{10}\sum_{i=1}^{j}\ln(U_i) = \sum_{i=1}^{j} T_i \sim \text{Gamma}(j, 10).$$

For $n \geq 1$, observe that

$$\{X \geq n\} = \{\frac{-1}{10}\ln(U_1 U_2 \ldots U_n) \leq 1\} = \{\sum_{i=1}^{n} T_i \leq 1\}$$

Using this, for $n \geq 1$,

$$
\begin{aligned}
\mathbb{P}(X = n) &= \mathbb{P}(X \geq n) - \mathbb{P}(X \geq n+1) \\
&= \mathbb{P}(\sum_{i=1}^{n} T_i \leq 1) - \mathbb{P}(\sum_{i=1}^{n+1} T_i \leq 1) \\
&= \frac{10^n}{n-1!} \int_0^1 z^{n-1} e^{-10z} dz - \frac{10^{n+1}}{n!} \int_0^1 z^n e^{-10z} dz \\
&\quad \text{(Using the p.d.f of Gamma distribution)} \\
&= \frac{10^n}{n-1!} \int_0^1 z^{n-1} e^{-10z} dz - \frac{10^{n+1}}{n!} \left[-\frac{e^{-10z} z^n}{10} \Big|_0^1 + \frac{n}{10} \int_0^1 z^{n-1} e^{-10z} dz \right] \\
&\quad \text{(Integration by parts)} \\
&= e^{-10} \frac{10^n}{n!}. \qquad \text{(As the above of limit of partial sums)} \qquad (3)
\end{aligned}
$$

From (1),(2), and (3) we conclude that $X \sim$ Poisson(10)

2. Abhiti and Daughters Pvt. Ltd. produces resistors and markets them as 10–ohm resistors. However the actual ohms of resistance produced by the resistors may vary. Research has established that 10% of the values are below 9.5 ohms and 20% are above 10.5 ohms. Two resistors, randomly selected, are used in a system.

   Solution: a) Probability that both resistors have actual values between 9.5 and 10.5 ohms. Let the two resistances be R_1 and R_2. Then

$$
\begin{aligned}
&P(9.5 \leq R_1 \leq 10.5, 9.5 \leq R_2 \leq 10.5) \\
&= P(9.5 \leq R_1 \leq 10.5) P(9.5 \leq R_2 \leq 10.5) \\
&= (1 - 0.1 - 0.2)^2 = 0.49
\end{aligned}
$$

   and b) Probability that at least one resistor has an actual value in excess of 10.5

$$
\begin{aligned}
&P(R_1 > 10.5 \text{ or } R_2 > 10.5) \\
&= P(R_1 > 10.5) P(R_2 > 10.5) + P(R_1 > 10.5) P(R_2 \leq 10.5) \\
&\qquad + P(R_1 \leq 10.5) P(R_2 > 10.5) \\
&= (0.2)^2 + (0.2)(0.8) + (0.8)(0.2) = 0.36
\end{aligned}
$$

3. Two assembly lines (I and II) have the same rate of defectives in their production of voltage regulators. Five regulators are sampled from each line and tested. Among the total of 10 tested regulators, 4 are defective. Find the probability that exactly 3 of the defectives came from line I.

   Solution: Let p be the probability that any given voltage regulator is defective. The probability that 4 of 10 given regulators are defective is $\binom{10}{4} p^4 (1-p)^6$. The probability that 3 of the 5 regulators from I and 1 of the 5 regulators from II are defective is $\binom{5}{3}p^3(1-p)^2\binom{5}{1}p^1(1-p)^4$. Hence the probability that 3 of the 5 regulators from I and 1 of the 5 regulators from II are defective, **given** that 4 of the 10 from I and II are defective, is

$$\frac{\binom{5}{3}p^3(1-p)^2\binom{5}{1}p^1(1-p)^4}{\binom{10}{4}p^4(1-p)^6} = \frac{\binom{5}{3}\binom{5}{1}}{\binom{10}{4}} = \frac{\frac{5\times4}{2}5}{\frac{10\times9\times8\times7}{4\times3\times2}} = \frac{5}{\frac{9\times7}{3}} = \frac{5}{21}$$

4. The duration of a certain computer game is a random variable with an exponential distribution with mean 10 minutes. Suppose that when you enter a video arcade two players are playing the game (independently). What is the probability that at least one of them is still playing 20 minutes later?

   Solution: Let D be the duration of the computer game. Suppose that the player has been playing for time b before you entered the arcade. By the *memoryless property* of the exponential distribution

   $$P(D \geq b+20|D \geq b) = P(D \geq 20) =< \int_{20}^{\infty} \frac{1}{10}e^{-t/10}\,dt = \int_{2}^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_2^{\infty} = e^{-2}$$

   So, the probability that they are both playing is $\left(e^{-2}\right)^2$, the probability that exactly one of the two is playing is $2\left(e^{-2}\right)\left(1-e^{-2}\right)$ and the probability that at least one is playing is

   $$\left(e^{-2}\right)^2 + 2\left(e^{-2}\right)\left(1-e^{-2}\right) = \left(e^{-2}\right)\left(2-e^{-2}\right) = 0.2524$$