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Grading:
30 marks- Complete submission of Problem 1,3,5,7
70 marks- Problem 3

From Probability and Statistics with Examples Using R

1. Exercise 3.2.4
   Solution: 1

   (a) Range(X)={0,1,2}
   Range(Y)={0,1,2}

Y/X	$X = 0$	$X = 1$	$X = 2$	$P(Y = y)$
Y=0	$1/12$	0	$3/12$	$4/12$
Y=1	$2/12$	$1/12$	0	$3/12$
Y=2	$3/12$	$1/12$	$1/12$	$5/12$
P(X=x)	$6/12$	$2/12$	$4/12$	

   Therefore, marginal distribution of X is:
   P(X=0)=$\frac{6}{12}$; P(X=1)=$\frac{2}{12}$ and, P(X=2)=$\frac{4}{12}$

   And, marginal distribution of Y is:
   P(Y=0)=$\frac{4}{12}$; P(Y=1)=$\frac{3}{12}$ and, P(Y=2)=$\frac{5}{12}$

   (b) Conditional distribution of X given Y=2 is:
   P(X=x|Y=2)=$\frac{P(X = x \cap Y = 2)}{P(Y = 2)}$; x=0,1,2
   Therefore,

$$\begin{aligned} P(X = 0|Y = 2) &= \frac{P(X = 0 \cap Y = 2)}{P(Y = 2)} \\ &= \frac{3/12}{5/12} \\ &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} P(X = 1|Y = 2) &= \frac{P(X = 1 \cap Y = 2)}{P(Y = 2)} \\ &= \frac{1/12}{5/12} \end{aligned}$$

$$= \frac{1}{5}$$

$$\begin{aligned} P(X = 2|Y = 2) &= \frac{P(X = 2 \cap Y = 2)}{P(Y = 2)} \\ &= \frac{1/12}{5/12} \\ &= \frac{1}{5} \end{aligned}$$

(c) Conditional distribution of Y given X=2 is:
P(Y=y|X=2)=$\dfrac{P(Y = y \cap X = 2)}{P(X = 2)}$; y=0,1,2
Therefore,

$$\begin{aligned} P(Y = 0|X = 2) &= \frac{P(Y = 0 \cap X = 2)}{P(X = 2)} \\ &= \frac{3/12}{4/12} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} P(Y = 1|X = 2) &= \frac{P(Y = 1 \cap X = 2)}{P(X = 2)} \\ &= \frac{0}{4/12} \\ &= 0 \end{aligned}$$

$$\begin{aligned} P(Y = 2|X = 2) &= \frac{P(Y = 2 \cap X = 2)}{P(X = 2)} \\ &= \frac{1/12}{4/12} \\ &= \frac{1}{4} \end{aligned}$$

(d) The random variables X and Y are independent if and only if
$P(X = x \cap Y = y) = P(X = x)P(Y = y)$; $x \in Range(X)$, $y \in Range(Y)$

$$\begin{aligned} P(X = 0, Y = 0) &= \frac{P(X = 0 \cap Y = 0)}{P(Y = 0)} \\ &= \frac{1/12}{4/12} \\ &= \frac{1}{4} \end{aligned}$$

And,

$$P(X = 0)P(Y = 0) = \left(\frac{6}{12}\right)\left(\frac{4}{12}\right)$$

$$= \frac{1}{6}$$
$$\neq P(X = 0 \cap Y = 0)$$

Hence, X and Y are not independent random variables.

2. Exercise 3.2.5

Solution: 2

Range(X)={0,1}

P(X=0)=$\frac{1}{3}$ and P(X=1)=$\frac{2}{3}$

Range(Y)={0,1,2}

P(Y=0)=$\frac{1}{5}$, P(Y=1)=$\frac{1}{5}$ and P(Y=2)=$\frac{3}{5}$

Since, X and Y are independent random variables.

$\therefore P(X = x \cap Y = y) = P(X = x)P(Y = y)$

The joint distribution of X and Y is:

Y/X	$X = 0$	$X = 1$	$P(Y = y)$
Y=0	$1/15$	$2/15$	$1/5$
Y=1	$1/15$	$2/15$	$1/5$
Y=2	$3/15$	$6/15$	$3/5$
P(X=x)	$1/3$	$2/3$	

3. Exercise 3.2.9

Solution: 3

N= Number of earthquakes in a year

M= Number of earthquakes in a year with magnitude at least 5

$N \sim Poisson(\lambda)$

P(N=n)=$\frac{e^{-\lambda}\lambda^n}{n!}$; n=0,1,2,...,∞

And,

$(M|N = n) \sim Binomial(n, p)$

P(M=m|N=n)= ${}^nC_m\, p^m\,(1-p)^{n-m}$; m=0,1,2,...,n

(a) The joint distribution of M and N is:

$$\begin{aligned} P(M = m \cap N = n) &= P(M = m|N = n)P(N = n) \\ &= {}^nC_m\, p^m\,(1-p)^{n-m} e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \frac{n!}{m!\,(n-m)!}\, p^m\,(1-p)^{n-m} e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \frac{1}{m!\,(n-m)!}\, p^m\,(1-p)^{n-m} e^{-\lambda}\lambda^n \end{aligned}$$

$$\therefore P(M = m \cap N = n) = \begin{cases} \frac{1}{m!\,(n-m)!}\, p^m\,(1-p)^{n-m} e^{-\lambda}\lambda^n & \text{; for m = 0,1,2,...,n and n=m,m+1,...,}\infty \\ 0 & \text{; otherwise} \end{cases}$$

(b) The marginal distribution of M is determined by:

$$P(M = m) = \sum_{n=m}^{\infty} P(M = m \cap N = n)$$

$$= \sum_{n=m}^{\infty} \frac{1}{m!\,(n-m)!}\, p^m\,(1-p)^{n-m} e^{-\lambda} \lambda^n$$

$$\therefore P(M=m) = \frac{1}{m!} e^{-\lambda} (\lambda p)^m \sum_{n=m}^{\infty} \frac{\lambda^{n-m}}{(n-m)!} (1-p)^{n-m} \ ; \ m > 0$$

(c) Put k=n-m
If $n = m \implies k = n - m = 0$
and, if $n = \infty \implies k = n - m = \infty$
Hence,
$P(M=m) = \frac{1}{m!} e^{-\lambda} (\lambda p)^m \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (1-p)^k \ ; \ m > 0$

(d) Using the infinite series equality $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we get:

$$P(M=m) = \frac{1}{m!} e^{-\lambda} (\lambda p)^m e^{\lambda(1-p)}$$
$$= \frac{1}{m!} e^{-\lambda(1-(1-p))} (\lambda p)^m$$
$$\therefore P(M=m) = \frac{1}{m!} e^{-(\lambda p)} (\lambda p)^m \ ; \ m > 0$$

Hence, $M \sim Poisson(\lambda p)$

4. Exercise 3.3.7
Solution: 4
Let X ~ Geometric(p) and Y ~ Geometric(p) be independent. Z=X+Y

(a) Range(X)={1,2,3,...}
Range(Y)={1,2,3,...}
$\implies$ Range(Z)={2,3,4,...}

(b)

$$P(Z=n) = P(X+Y=n)$$
$$= P(\cup_{i=1}^{n-1} (X=i, Y=n-i))$$
$$= \sum_{i=1}^{n-1} P(X=i, Y=n-i) \quad \text{(Mutually exclusive events)}$$
$$= \sum_{i=1}^{n-1} P(X=i) P(Y=n-i) \quad \text{(X and Y are independent random variables)}$$
$$= \sum_{i=1}^{n-1} p(1-p)^{i-1} \times p(1-p)^{n-i-1}$$
$$= \sum_{i=1}^{n-1} p^2 (1-p)^{n-2}$$
$$= p^2 (1-p)^{n-2} \sum_{i=1}^{n-1} 1$$

$\therefore P(Z=n) = (n-1)p^2(1-p)^{n-2}\ ;\ n = 2,3,4,...$

(c) For n=2,
$P(Z=2) = (2-1)p^2(1-p)^{2-2}$
$\therefore P(Z=2) = p^2$
For n=3,
$P(Z=3) = (3-1)p^2(1-p)^{3-2}$
$\therefore P(Z=3) = 2p^2(1-p)$
Now, we have to obtain the value of 'p' such that:

$$\begin{aligned} P(Z=3) &> P(Z=2) \\ 2p^2(1-p) &> p^2 \\ p^2(2-2p-1) &> 0 \\ \text{Since, } p^2 &> 0 \\ \implies 2-2p-1 &> 0 \\ \therefore p &< \frac{1}{2} \end{aligned}$$

Hence, for all $p < 0.5$, the P(Z=3) is larger than P(Z=2).

5. Exercise 3.3.11
Solution: 5
X_1, X_2, X_3 and X_4 are i.i.d Bernoulli(p)
$Y = X_1 + X_2 \sim Binomial(2,p)$
$Y = X_3 + X_4 \sim Binomial(2,p)$

(a)

$$\begin{aligned} P(Y=y \cap Z=z) &= P(X_1+X_2 = y \cap X_3+X_4 = z) \\ &= \sum_{x_1 \in Range(X_1)} \sum_{x_3 \in Range(X_3)} P(X_1 = x_1 \cap X_2 = y-x_1 \cap X_3 = x_3 \cap X_4 = z-x_3) \\ &= \sum_{x_1 \in Range(X_1)} \sum_{x_3 \in Range(X_3)} P(X_1 = x_1)P(X_2 = y-x_1)P(X_3 = x_3)P(X_4 = z-x_3) \end{aligned}$$

Because, X_1,X_2, X_3, X_4 are independent random variables

$$\begin{aligned} \therefore P(Y=y \cap Z=z) &= \sum_{x_1 \in Range(X_1)} P(X_1 = x_1)P(X_2 = y-x_1) \sum_{x_3 \in Range(X_3)} P(X_3 = x_3)P(X_4 = z-x_3) \\ &= P(Y = x_1+x_2)P(Z = x_3+x_4) \\ &= P(Y=y)P(Z=z) \end{aligned}$$

Hence, Y and Z are independent random variables. Now, the joint distribution of Y and Z is:

$$\begin{aligned} P(Y=y \cap Z=z) &= P(Y=y)P(Z=z) \\ &= {}^2C_y\, p^y(1-p)^{2-y} \times {}^2C_z\, p^z(1-p)^{2-z} \\ &= {}^2C_y\, {}^2C_z\, p^{y+z}(1-p)^{4-(y+z)} \end{aligned}$$

Y/Z	$Z = 0$	$Z = 1$	$Z = 2$
Y=0	$(1-p)^4$	$2p(1-p)^3$	$p^2(1-p)^2$
Y=1	$2p(1-p)^3$	$4p^2(1-p)^2$	$2p^3(1-p)$
Y=2	$p^2(1-p)^2$	$2p^3(1-p)$	p^4

(b) Y and Z are independent if and only if
$P(Y = y \cap Z = z) = P(Y = y)P(Z = z)$; $y \in Range(Y)$ and $z \in Range(Z)$
For y=0 and z=0:

$$\begin{aligned} P(Y = 0)P(Z = 0) &= {}^2C_0 p^0 (1-p)^2 \times {}^2C_0 p^0 (1-p)^2 \\ &= (1-p)^4 \\ \therefore P(Y = 0)P(Z = 0) &= P(Y = 0 \cap Z = 0) \end{aligned}$$

Similarly, for other values of Y and Z, it can be shown that:
$P(Y = y \cap Z = z) = P(Y = y)P(Z = z)$.

(c) The theorem 3.3.6 states that:
Let $n > 1$ be a positive integer. For each $j \in \{1, 2, ..., n\}$ define a positive integer m_j and suppose $X_{i,j}$ is an array of mutually independent random variables for $j \in \{1, 2, ..., n\}$ and $i \in \{1, 2, ..., m_j\}$. Let f_j be functions such that the quantity

$$Y_j = f_j(X_{1,j}, X_{2,j}, ..., X_{m,j})$$

is defined for the outputs of the $X_{i,j}$ variables. Then the resulting variables $Y_1, Y_2, ...Y_n$ are mutually independent.
Here,
$Y_1 = Y = X_1 + X_2$
$Y_2 = Z = X_3 + X_4$
$\therefore$ Y and Z are independent random variables.

6. Exercise 3.3.15
Solution: 6
X $\sim$ Geometric(p)
Y $\sim$ Geometric(q)
X and Y are independent random variables. (given)
Z=min(X,Y)
METHOD I:

(a) Let us define the events:
A={X=n and Y=n}
B={X=n and Y>n}
C={X>n and Y=n}
Event {Z=n} occurs if either A or B or C occurs.
Also, the events A, B and C disjoint events as:
$\{A \cap B\} = \phi$; $\{A \cap C\} = \phi$; $\{B \cap C\} = \phi$ and $\{A \cap B \cap C\} = \phi$
Therefore, the event {Z=n} can be written as disjoint union:
$\{Z = n\} = \{X = n, Y = n\} \cup \{X = n, Y > n\} \cup \{X > n, Y = n\}$

(b) $P(Z = n) = P(X = n, Y = n) + P(X = n, Y > n) + P(X > n, Y = n)$
Because, disjoint union

Since, X and Y are independent random variables.

$$\begin{aligned} P(Z=n) &= P(X=n)P(Y=n) + P(X=n)P(Y>n) + P(X>n)P(Y=n) \\ &= (p(1-p)^{n-1}) \times (q(1-q)^{n-1}) + (p(1-p)^{n-1}) \times (1-q)^n + (1-p)^n \times (q(1-q)^{n-1}) \\ \therefore P(Z=n) &= [(1-p)(1-q)]^{n-1}(pq + p(1-q) + (1-p)q) \end{aligned}$$

(c)

$$\begin{aligned} P(Z=n) &= [(1-p)(1-q)]^{n-1}(pq + p(1-q) + (1-p)q) \\ &= [(1-p)(1-q)]^{n-1}[1-(1-p)(1-q)] \\ &= (1-r)^{n-1}r \end{aligned}$$

Therefore, Z $\sim$ Geometric(r) where, r=1-(1-p)(1-q)

METHOD II:

$A_j \sim Bernoulli(p)$; j=1,2,...
$B_k \sim Bernoulli(q)$; k=1,2,...
Also, A_j and B_k variables collectively are mutually independent.
X=Number of first A_j that produces a result of 1
Y=Number of first B_k that produces a result of 1

(a)

$$C_j = \begin{cases} 1 & ;\text{ if either } A_j = 1 \text{ or } B_j = 1 \text{ or both} \\ 0 & ;\ A_j\text{=0=}B_j \end{cases}$$

Now,

$$\begin{aligned} P(C_j=1) =& P(A_j=1, B_j=0) + P(A_j=0, B_j=1) + P(A_j=1, B_j=1) \\ =& P(A_j=1)P(B_j=0) + P(A_j=0)P(B_j=1) + P(A_j=1)P(B_j=1) \\ =& p(1-q) + (1-p)q + pq \\ \therefore P(C_j=1) =& 1-(1-p)(1-q) \end{aligned}$$

And,

$$\begin{aligned} P(C_j=0) =& P(A_j=0, B_j=0) \\ =& P(A_j=0)P(B_j=0) \\ \therefore P(C_j=0) =& (1-p)(1-q) \end{aligned}$$

Hence,

$$P(C_j=c) = \begin{cases} 1-(1-p)(1-q) & ;\text{ for } C_j = 1 \\ (1-p)(1-q) & ;\text{ for } C_j = 0 \end{cases}$$

Therefore, $C_j \sim Bernoulli(r)$ where, r=1-(1-p)(1-q)

(b) Since, A_j and B_k variables collectively are mutually independent and C_j depends on A_j and B_j only.
Hence, the sequence C_1, C_2,... are mutually independent random variables.

(c) The random variable Z represents the number of first C_j that results in a 1.
The random variable $C_j = 1$ if either $A_j = 1$ or $B_j = 1$ or both.
Also, X=Number of first A_j that produces a result of 1, and Y=Number of first B_k that produces a result of 1.
Hence, $\{Z = n\} = \{X = n, Y = n\} \cup \{X = n, Y > n\} \cup \{X > n, Y = n\}$
$\implies$ Z=min{X,Y}

(d) $P(Z = n) = P(X = n, Y = n) + P(X = n, Y > n) + P(X > n, Y = n)$
Because, disjoint union
Since, X and Y are independent random variables.

$$\begin{aligned}P(Z = n) &= P(X = n)P(Y = n) + P(X = n)P(Y > n) + P(X > n)P(Y = n)\\ &= (p(1-p)^{n-1}) \times (q(1-q)^{n-1}) + (p(1-p)^{n-1}) \times (1-q)^n + (1-p)^n \times (q(1-q)^{n-1})\\ &= [(1-p)(1-q)]^{n-1}(pq + p(1-q) + (1-p)q)\\ &= [(1-p)(1-q)]^{n-1}(pq + p(1-q) + (1-p)q)\\ &= [(1-p)(1-q)]^{n-1}[1-(1-p)(1-q)]\end{aligned}$$

$\therefore P(Z = n) = (1-r)^{n-1}r$

Therefore, Z $\sim$ Geometric(r) where, r=1-(1-p)(1-q)

7. Exercise 4.4.3
Solution: 7
Let X be the random variable representing the return on investment and let A,B, and C represent the events that the economy will be stronger, the same, and weaker in the next quarter, respectively.
We are given that:
E(X|A)=3 ; E(X|B)=1 ; E(X|C)=-1
P(A)=0.1 ; P(B)=0.4 ; P(C)=5
S.D(X|A)=3 ; S.D(X|B)=2 ; S.D(X|C)=3
Therefore,
V(X|A)=9 ; V(X|B)=4 ; V(X|C)=9
Now,

$$\begin{aligned}E(X) &= E(X|A)P(A) + E(X|B)P(B) + E(X|C)P(C)\\ &= (3 \times 0.1) + (1 \times 0.4) + ((-1) \times 0.5)\\ \therefore E(X) &= 0.2\end{aligned}$$

$$\begin{aligned}V(X) =&(V(X|A) + (E(X|A))^2)P(A) + (V(X|B) + (E(X|B))^2)P(B) + (V(X|C)+\\ &(E(X|C))^2)P(C) - (E(X))^2\\ =&(9 + 3^2) \times 0.1 + (4 + 1^2) \times 0.4 + (9 + (-1)^2) \times 0.5 - (0.2)^2\\ =&1.8 + 2 + 5 - 0.04\\ \therefore V(X) =&8.76\end{aligned}$$

8. Exercise 4.4.4
Solution: 8

Let A be the event that a standard light bulb is selected from the box and B be the event that a Super D-Lux light bulb is selected from the box.
P(A)=0.9 ; P(B)=0.1
Let, X be the random variable that represent the lifetime of a bulb.
E(X|A)=4 ; S.D(X|A)=1 $\implies$ Var(X|A)=1
E(X|B)=8 ; S.D(X|B)=3 $\implies$ Var(X|B)=9
Now,

$$\begin{aligned} E(X) &= E(X|A)P(A) + E(X|B)P(B) \\ &= (4 \times 0.9) + (8 \times 0.1) \\ \therefore E(X) &= 4.4 \end{aligned}$$

$$\begin{aligned} V(X) &= (V(X|A) + (E(X|A))^2)P(A) + (V(X|B) + (E(X|B))^2)P(B) - (E(X))^2 \\ &= (1 + 4^2) \times 0.9 + (9 + 8^2) \times 0.1 - (4.4)^2 \\ &= 15.3 + 7.3 - 19.36 \\ \therefore V(X) &= 3.24 \end{aligned}$$

Hence,

$$\begin{aligned} S.D(X) &= \sqrt{Var(X)} \\ &= \sqrt{3.24} \\ \therefore S.D(X) &= 1.8 \end{aligned}$$