Ishaan Taneja

Grading:
30 marks- Complete submission of Problem 2,4,6
70 marks- Problem 6

From Probability and Statistics with Examples Using R

1. Exercise 1.4.6

   Solution: 1

   A={The ball is either black or green}

$$\begin{aligned}P(A) &= P(\text{The ball is either black or green})\\ &= P(\text{Black ball} \cup \text{Green ball})\\ &= P(\text{Black ball}) + P(\text{Green ball}) \quad \text{(because, mutually exclusive events)}\\ &= \frac{1}{27} + \frac{8}{27}\\ &= \frac{9}{27}\\ &= \frac{1}{3}\end{aligned}$$

   B={The ball is either black or red}

$$\begin{aligned}P(B) &= P(\text{The ball is either black or red})\\ &= P(\text{Black ball} \cup \text{red ball})\\ &= P(\text{Black ball}) + P(\text{red ball}) \quad \text{(because, mutually exclusive events)}\\ &= \frac{1}{27} + \frac{8}{27}\\ &= \frac{9}{27}\\ &= \frac{1}{3}\end{aligned}$$

   C={The ball is either black or blue}

$$\begin{aligned}P(C) &= P(\text{The ball is either black or blue})\\ &= P(\text{Black ball} \cup \text{Green ball})\\ &= P(\text{Black ball}) + P(\text{blue ball}) \quad \text{(because, mutually exclusive events)}\\ &= \frac{1}{27} + \frac{8}{27}\\ &= \frac{9}{27}\\ &= \frac{1}{3}\end{aligned}$$

(a). Now,

$$
\begin{aligned}
P(A \cap B \cap C) &= P(\text{Black ball}) \\
&= \frac{1}{27}
\end{aligned}
$$

(b).

$$
\begin{aligned}
P(A)P(B)P(C) &= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{27}
\end{aligned}
$$

(c). From (a) and (b), we know that:
$P(A \cap B \cap C) = P(A)P(B)P(C)$
But,

$$
\begin{aligned}
P(A \cap B) &= P(\text{Black ball}) \\
&= \frac{1}{27} \\
&\neq P(A)(B) = \frac{1}{9}
\end{aligned}
$$

Similarly,
$P(A \cap C) \neq P(A)P(C)$
And, $P(B \cap C) \neq P(B)P(C)$
Hence, A, B and C are not mutually independent events

2. Exercise 1.4.7
Solution: 2
A_1={The student is female}

$$
\begin{aligned}
P(A_1) &= \frac{90}{150} \\
&= \frac{3}{5}
\end{aligned}
$$

A_2={The student uses a pencil}

$$
\begin{aligned}
P(A_2) &= \frac{60}{150} \\
&= \frac{2}{5}
\end{aligned}
$$

A_3={The student is wearing eye glasses}

$$
\begin{aligned}
P(A_3) &= \frac{30}{150} \\
&= \frac{1}{5}
\end{aligned}
$$

(a). If $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ then,

$$
P(A_1 \cap A_2 \cap A_3) = \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5}
$$

$$= \frac{6}{125}$$

Hence, cardinality of set $(A_1 \cap A_2 \cap A_3) = \frac{6}{125} \times 150 = 7.2$.
which is not possible, as cardinality of a set can only be non-negative integers.
Therefore, it is impossible for these events to be mutually independent.

(b). For pairwise independence of events A_1, A_2 and A_3:

i).

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)P(A_2) \\ &= \frac{3}{5} \times \frac{2}{5} \\ &= \frac{6}{25} \end{aligned}$$

Hence, cardinality of set $(A_1 \cap A_2) = \frac{6}{25} \times 150 = 36$.
which is possible, as cardinality of a set can only be non-negative integers.

ii).

$$\begin{aligned} P(A_1 \cap A_3) &= P(A_1)P(A_3) \\ &= \frac{3}{5} \times \frac{1}{5} \\ &= \frac{3}{25} \end{aligned}$$

Hence, cardinality of set $(A_1 \cap A_3) = \frac{3}{25} \times 150 = 24$.
which is possible, as cardinality of a set can only be non-negative integers.

iii).

$$\begin{aligned} P(A_2 \cap A_3) &= P(A_2)P(A_3) \\ &= \frac{2}{5} \times \frac{1}{5} \\ &= \frac{2}{25} \end{aligned}$$

Hence, cardinality of set $(A_2 \cap A_3) = \frac{2}{25} \times 150 = 16$.
which is possible, as cardinality of a set can only be non-negative integers.

Hence, it is possible for these events to be pairwise independent, if we take the values obtained above as an example.

3. Exercise 3.3.3
Solution: 3
Let X $\sim$ Bernoulli(p) and Y $\sim$ Bernoulli(q) be independent.
Range(X)={0,1}
Range(Y)={0,1}

(a). Let, $Z = XY$
Range(Z)={0,1}
Now, for Z=0,

$$\begin{aligned} P(Z=0) &= P(XY=0) \\ &= P(X=0,Y=0) + P(X=1,Y=0) + P(X=0,Y=1) \end{aligned}$$

Since, X and Y are independent Bernoulli random variables

$$\begin{aligned} &= P(X=0)P(Y=0) + P(X=1)P(Y=0) + P(X=0)P(Y=1) \\ &= (1-p)(1-q) + p(1-q) + (1-p)q \\ \therefore P(Z=0) &= 1 - pq \end{aligned}$$

For Z=1,

$$\begin{aligned} P(Z=1) &= P(XY=1) \\ &= P(X=1,Y=1) \end{aligned}$$

Since, X and Y are independent Bernoulli random variables

$$\begin{aligned} &= P(X=1)P(Y=1) \\ &= pq \\ \therefore P(Z=1) &= pq \end{aligned}$$

In general, $P(Z=z) = (pq)^z(1-pq)^{1-z}$; z=0,1
Hence, Z=XY is a also a Bernoulli random variable with parameter 'pq'.

(b). Let, $Z = 1 - X$
Range(Z)={0,1}
Now, for Z=0,

$$\begin{aligned} P(Z=0) &= P(1-X=0) \\ &= P(X=1) \\ &= p \\ \therefore P(Z=0) &= p \end{aligned}$$

For Z=1,

$$\begin{aligned} P(Z=1) &= P(1-X=1) \\ &= P(X=0) \\ &= (1-p) \\ \therefore P(Z=1) &= (1-p) \end{aligned}$$

In general, $P(Z=z) = (1-p)^z(p)^{1-z}$; z=0,1
Hence, $Z = 1 - X$ is a also a Bernoulli random variable with parameter '(1-p)'.

(c). Let, $Z = X + Y - XY$
Range(Z)={0,1}
Now, for Z=0,

$$\begin{aligned} P(Z=0) &= P(X+Y-XY=0) \\ &= P(X=0,Y=0) \end{aligned}$$

Since, X and Y are independent Bernoulli random variables

$$= P(X = 0)P(Y = 0)$$

$$\therefore P(Z = 0) = (1 - p)(1 - q)$$

For Z=1,

$$P(Z = 1) = P(X + Y - XY = 1)$$
$$= P(X = 0, Y = 1) + P(X = 1, Y = 0) + P(X = 1, Y = 1)$$

Since, X and Y are independent Bernoulli random variables

$$= P(X = 0)P(Y = 1) + P(X = 1)P(Y = 0) + P(X = 1)P(Y = 1)$$
$$= (1 - p)q + p(1 - q) + pq$$
$$\therefore P(Z = 1) = 1 - (1 - p)(1 - q)$$

In general, $P(Z = z) = [1 - (1 - p)(1 - q)]^z[(1 - p)(1 - q)]^{1-z}$; z=0,1
Hence, $Z = X + Y - XY$ is a also a Bernoulli random variable with parameter '[1-(1-p)(1-q)]'.

4. Exercise 3.3.6
   Solution: 4
   Let, X and Y be the random variable representing the number of heads and number of tails in one flip of a single fair coin respectively.

   (a) Range(X)={0,1}
   Range(Y)={0,1}
   Since, it is a fair coin
   $\implies P(Head) = P(X = 1) = \frac{1}{2} = P(Y = 0)$
   And,
   $P(Tail) = P(X = 0) = \frac{1}{2} = P(Y = 1)$
   $\therefore$ X,Y $\sim$ Bernoulli $\left(\frac{1}{2}\right)$

   (b) Z=X+Y
   Range(Z)={1}
   Because, If we flip a fair coin, we will get either head or a tail but not both at the same time.
   So, there won't be any case when both the random variables X and Y will simultaneously takes the value 0 or 1.
   Only possible cases are:
   1) X=0, Y=1 $\implies$ Z=1
   2) X=1, Y=0 $\implies$ Z=1
   Therefore, Z=1 is a sure event.
   Hence, P(Z=1)=1.

   (c) This result doesn't conflict with the conclusion of Example 3.3.3 because, in this case, even though the X,Y $\sim$ Bernoulli(p) but they are not independent random variables. While in Example 3.3.3, the X,Y $\sim$ Bernoulli(p) and are independent random variables.

5. Exercise 3.3.7
   Solution: 5
   Let X $\sim$ Geometric(p) and Y $\sim$ Geometric(p) be independent.
   Z=X+Y

(a) Range(X)={1,2,3,...}
Range(Y)={1,2,3,...}
$\implies$ Range(Z)={2,3,4,...}

(b)

$$
\begin{aligned}
P(Z=n) &= P(X+Y=n) \\
&= P(\cup_{i=1}^{n-1}(X=i, Y=n-i)) \\
&= \sum_{i=1}^{n-1} P(X=i, Y=n-i) \quad \text{(Mutually exclusive events)} \\
&= \sum_{i=1}^{n-1} P(X=i)P(Y=n-i) \quad \text{(X and Y are independent random variables)} \\
&= \sum_{i=1}^{n-1} p(1-p)^{i-1} \times p(1-p)^{n-i-1} \\
&= \sum_{i=1}^{n-1} p^2(1-p)^{n-2} \\
&= p^2(1-p)^{n-2} \sum_{i=1}^{n-1} 1
\end{aligned}
$$

$\therefore P(Z=n) = (n-1)p^2(1-p)^{n-2}$; $n = 2, 3, 4, ...$

(c) For n=2,
$P(Z=2) = (2-1)p^2(1-p)^{2-2}$
$\therefore P(Z=2) = p^2$
For n=3,
$P(Z=3) = (3-1)p^2(1-p)^{3-2}$
$\therefore P(Z=3) = 2p^2(1-p)$
Now, we have to obtain the value of 'p' such that:

$$
\begin{aligned}
P(Z=3) &> P(Z=2) \\
2p^2(1-p) &> p^2 \\
p^2(2-2p-1) &> 0 \\
\text{Since, } p^2 &> 0 \\
\implies 2-2p-1 &> 0 \\
\therefore p &< \frac{1}{2}
\end{aligned}
$$

Hence, for all $p < 0.5$, the P(Z=3) is larger than P(Z=2).

6. Exercise 4.4.1
Solution: 6
X $\sim$ Geometric(p)
Event A = $\{X \leq 3\}$
Range(X)={1,2,3,...}

$P(X = x|A) = \dfrac{P(X = x \cap A)}{P(A)}$
If $x > 3$ then,
$P(X = x \cap A) = 0$
Therefore, $P(X = x|A) = 0$; x=4,5,6,...

If $x \leq 3$ then,

$$\begin{aligned} P(X = x|A) &= \frac{P(X = x)}{P(X \leq 3)} \\ &= \frac{p^(1-p)^{x-1}}{\sum_{x=1}^{3} p^(1-p)^{x-1}} \\ &= \frac{p^(1-p)^{x-1}}{p(p^2 - 3p + 3)} \\ \therefore P(X = x|A) &= \frac{(1-p)^{x-1}}{p^2 - 3p + 3} \ ; \ x = 1, 2, 3 \end{aligned}$$

Hence,

$$P(X = x|A) = \begin{cases} \dfrac{(1-p)^{x-1}}{p^2 - 3p + 3} & \text{; for x = 1,2,3} \\ 0 & \text{; for x=4,5,6,...} \end{cases}$$

$$\begin{aligned} E(X|A) &= \sum_{x=1}^{\infty} xP(X = x|A) \\ &= 1 \times \left(\frac{1}{p^2 - 3p + 3}\right) + 2 \times \left(\frac{1-p}{p^2 - 3p + 3}\right) + 3 \times \left(\frac{(1-p)^2}{p^2 - 3p + 3}\right) + 0 \\ &= \frac{1 + (2 - 2p) + (3 - 6p + 3p^2)}{p^2 - 3p + 3} \\ \therefore E(X|A) &= \frac{3p^2 - 8p + 6}{p^2 - 3p + 3} \end{aligned}$$

Now,

$$\begin{aligned} E(X^2|A) &= \sum_{x=1}^{\infty} x^2 P(X = x|A) \\ &= 1^2 \times \left(\frac{1}{p^2 - 3p + 3}\right) + 2^2 \times \left(\frac{1-p}{p^2 - 3p + 3}\right) + 3^2 \times \left(\frac{(1-p)^2}{p^2 - 3p + 3}\right) \\ &= \frac{1 + (4 - 4p) + 9 - 18p + 9p^2)}{p^2 - 3p + 3} \\ \therefore E(X^2|A) &= \frac{9p^2 - 22p + 14}{p^2 - 3p + 3} \end{aligned}$$

We know that,

$$\begin{aligned} Var(X|A) &= E(X^2|A) - (E(X|A))^2 \\ \therefore Var(X|A) &= \frac{9p^2 - 22p + 14}{p^2 - 3p + 3} - \left(\frac{3p^2 - 8p + 6}{p^2 - 3p + 3}\right)^2 \end{aligned}$$

7. Exercise 4.5.2

Solution: 7

X ∼ Uniform({0,1,2})

Y= Number of heads in X flips of a coin.

(a) We should expect X and Y to be positively correlated because if we increase the number of flips, the chances of getting more heads also increases.
In other words, If we increase the value of X, the value of Y should also increase i.e. positively correlated.

(b) Range(X)={0,1,2}
Range(Y)={0,1,2}

$$\begin{aligned} P(X=0,Y=0) &= P(Y=0|X=0)P(X=0) \\ &= 1 \times \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

Since, it is impossible to get either 1 or 2 heads in 0 flips of a coin.
Therefore, $P(X=0,Y=1)=0=P(X=0,Y=2)$

$$\begin{aligned} P(X=1,Y=0) &= P(Y=0|X=1)P(X=1) \\ &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} P(X=1,Y=1) &= P(Y=1|X=1)P(X=1) \\ &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

Again, it is impossible to get 2 heads in 1 flip of a coin.
Therefore, $P(X=1,Y=2)=0$

$$\begin{aligned} P(X=2,Y=0) &= P(Y=0|X=2)P(X=2) \\ &= \frac{1}{4} \times \frac{1}{3} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} P(X=2,Y=1) &= P(Y=1|X=2)P(X=2) \\ &= \frac{2}{4} \times \frac{1}{3} \\ &= \frac{2}{12} \end{aligned}$$

$$\begin{aligned}P(X=2,Y=2) &= P(Y=2|X=2)P(X=2)\\ &= \frac{1}{4}\times\frac{1}{3}\\ &= \frac{1}{12}\end{aligned}$$

Therefore,

Y/X	$X=0$	$X=1$	$X=2$	$P(Y=y)$
Y=0	$1/3$	$1/6$	$1/12$	$7/12$
Y=1	0	$1/6$	$2/12$	$4/12$
Y=2	0	0	$1/12$	$1/12$
P(X=x)	$1/3$	$1/3$	$1/3$	

$$\begin{aligned}E(X) &= \sum_{x=0}^{2} xP(X=x)\\ &= 0\times\frac{1}{3}+1\times\frac{1}{3}+2\times\frac{1}{3}\\ \therefore E(X) &= 1\end{aligned}$$

$$\begin{aligned}E(Y) &= \sum_{y=0}^{2} yP(Y=y)\\ &= 0\times\frac{7}{12}+1\times\frac{4}{12}+2\times\frac{1}{12}\\ \therefore E(Y) &= \frac{1}{2}\end{aligned}$$

Let, Z=XY
Range(Z)={0,1,2,4}
Now,

$$\begin{aligned}P(Z=0) &= P(X=0,Y=0)+P(X=0,Y=1)+P(X=0,Y=2)\\ &+P(X=1,Y=0)+P(X=2,Y=0)\\ &= \frac{1}{3}+0+0+\frac{1}{6}+\frac{1}{12}\\ &= \frac{7}{12}\end{aligned}$$

Also,

$$\begin{aligned}P(Z=1) &= P(X=1,Y=1)\\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}P(Z=2) &= P(X=1,Y=2)+P(X=2,Y=1)\\ &= 0+\frac{2}{12}\end{aligned}$$

$$= \frac{1}{6}$$

$$\begin{aligned} P(Z = 4) &= P(X = 2, Y = 2) \\ &= \frac{1}{12} \end{aligned}$$

Therefore,

$$\begin{aligned} E(Z) &= E(XY) \\ &= \sum_{z \in Range(Z)} zP(Z = z) \\ &= 0 \times \frac{7}{12} + 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 4 \times \frac{1}{12} \\ &= \frac{5}{6} \end{aligned}$$

We know that,

$$\begin{aligned} Cov(X, Y) &= E(XY) - E(X)E(Y) \\ &= \frac{5}{6} - 1 \times \left(\frac{1}{2}\right) \\ &= \frac{1}{3} \end{aligned}$$

Since, $Cov(X, Y) > 0$
Therefore, it confirms our answer to (a).