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Grading:
30 marks- Complete submission of Problem 2,4,6
70 marks- Problem 2

From Probability and Statistics with Examples Using R

1. Exercise 5.1.5
   Solution: 1

   (a).
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   i). $f(x) = \lambda e^{-\lambda x} \geq 0$ for all $x > 0$
   Because, $\lambda > 0$ (given) and $e^{-\lambda x} \geq 0$ for $x > 0$.
   Also, $f(x) = 0 \geq 0$ for all $x \leq 0$.
   Hence, $f(x) \geq 0 \quad ; x \in R$

   ii). $f(x) = \lambda e^{-\lambda x}$ is continuous for $x > 0$.
   And, $f(x) = 0$ being a constant function, is also continuous for $x \leq 0$
   Hence, $f(x)$ is piecewise-continuous.

   iii).
$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} 0dx + \int_{0}^{\infty} \lambda e^{-\lambda x}dx \\ &= 0 + \lambda \int_{0}^{\infty} e^{-\lambda x}dx \\ &= \lambda \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty \\ &= -e^{-\lambda x}|_0^\infty \\ &= 1\end{aligned}$$

   Therefore, f is a probability density function.

   (b). For $a > 0$
$$\begin{aligned}P((a,\infty)) &= \int_{a}^{\infty} f(x)dx \\ &= \int_{a}^{\infty} \lambda e^{-\lambda x}dx\end{aligned}$$

$$= \lambda \frac{e^{-\lambda x}}{-\lambda}\Big|_a^\infty$$
$$= -e^{-\lambda x}|_a^\infty$$
$$= 0 - (-e^{-\lambda a})$$
$$\therefore P((a,\infty)) = e^{-\lambda a}$$

2. Exercise 5.1.7

   Solution: 2

   (a).

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

   i). $f(x) = \frac{1}{x^2} \geq 0$ for all $x > 1$
   Because, $x^2 > 0$ for $x > 1$.
   Also, $f(x) = 0 \geq 0$ for all $x \leq 1$.
   Hence, $f(x) \geq 0 \quad ; x \in R$

   ii). $f(x) = \frac{1}{x^2}$ is continuous for $x > 1$.
   And, $f(x) = 0$ being a constant function, is also continuous for $x \leq 1$
   Hence, $f(x)$ is piecewise-continuous.

   iii).

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{1} 0dx + \int_{1}^{\infty} \frac{1}{x^2}dx$$
$$= 0 + \int_{1}^{\infty} \frac{1}{x^2}dx$$
$$= \frac{-1}{x}\Big|_1^\infty$$
$$= 0 - (-1)$$
$$= 1$$

   Therefore, f is a probability density function.

   (b). For $a > 1$

$$P((a,\infty)) = \int_a^\infty f(x)dx$$
$$= \int_a^\infty \frac{1}{x^2}dx$$
$$= \frac{-1}{x}\Big|_a^\infty$$
$$= 0 - \left(\frac{-1}{a}\right)$$
$$\therefore P((a,\infty)) = \frac{1}{a}$$

3. Exercise 5.1.8

Solution: 3

$$f(x) = \begin{cases} \frac{1}{6}x^2e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

i). $f(x) = \frac{1}{6}x^2e^{-x} \geq 0$ for all $x > 0$
Because, $\frac{1}{6} > 0$ (always), $x^2 > 0$ and $e^{-x} \geq 0$ for $x > 0$.
Also, $f(x) = 0 \geq 0$ for all $x \leq 0$.
Hence, $f(x) \geq 0 \quad ; x \in R$

ii). As, e^{-x} and x^2 are continuous for $x > 0 \implies e^{-x}x^2$ is also continuous for $x > 0$.
Because, the product of a finite number of continuous functions is also a continuous function.
Therefore, $f(x) = \frac{1}{6}x^2e^{-x}$ is continuous for $x > 0$.
And, $f(x) = 0$ being a constant function, is also continuous for $x \leq 0$
Hence, $f(x)$ is piecewise-continuous.

iii).

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} 0dx + \int_{0}^{\infty} \frac{1}{6}x^2e^{-x}dx \\ &= 0 + \int_{0}^{\infty} \frac{1}{6}x^2e^{-x}dx \\ &= \frac{1}{6}\int_{0}^{\infty} x^2e^{-x}dx \\ &= \frac{1}{6}\int_{0}^{\infty} x^2e^{-x}dx\end{aligned}$$

Using integration by-parts, we get:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \frac{1}{6}\left(x^2\frac{e^{-x}}{-1}\Bigg|_0^{\infty}\right) - \frac{1}{6}\int_{0}^{\infty} 2x\frac{e^{-x}}{-1}dx \\ &= 0 + \frac{2}{6}\int_{0}^{\infty} xe^{-x}dx\end{aligned}$$

Again, using integration by-parts, we get:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \frac{1}{3}\left(x\frac{e^{-x}}{-1}\Bigg|_0^{\infty}\right) - \frac{1}{3}\int_{0}^{\infty} \left(1 \times \frac{e^{-x}}{-1}\right)dx \\ &= 0 + \frac{1}{3}\int_{0}^{\infty} e^{-x}dx \\ &= \frac{1}{3} \neq 1\end{aligned}$$

Therefore, f is not a probability density function.

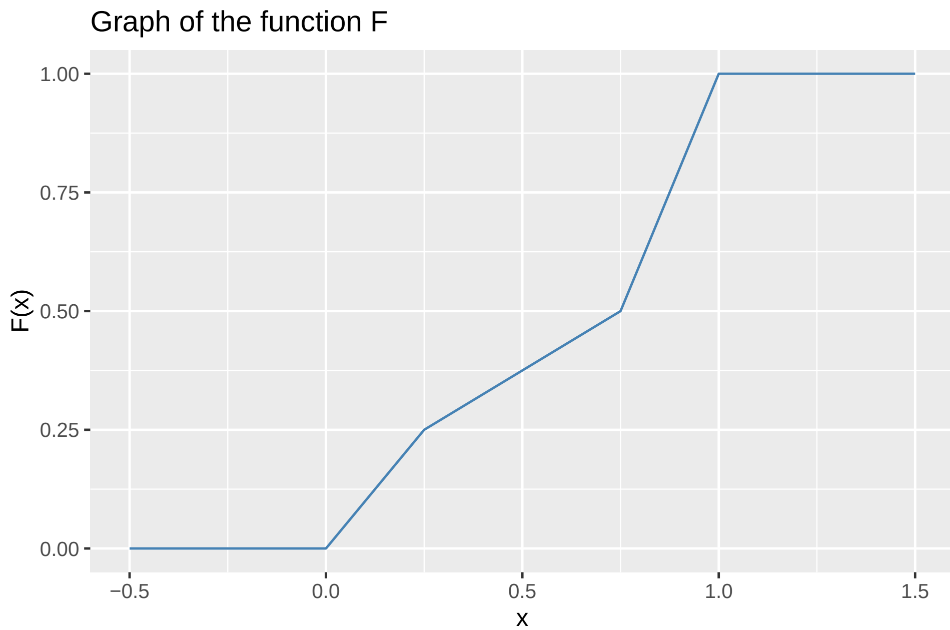
4. Exercise 5.2.3

Solution: 4

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 < x < \frac{1}{4} \\ \frac{x}{2} + \frac{1}{8} & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ 2x - 1 & \text{if } \frac{3}{4} \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

(a).

```
> F=function(x)
+ {
+  if (x<0)
+  return(0)
+  else if (0<=x & x<(1/4))
+  return(x)
+  else if ((1/4)<=x & x<(3/4))
+  return((x/2)+(1/8))
+  else if ((3/4)<=x & x<1)
+  return((2*x)-1)
+  else
+  return(1)
+ }
> y=seq(-0.5,1.5,0.0001)
> F_y=rep(NA,length(y))
> for(i in 1:length(y))
+ {
+  F_y[i]=F(y[i])
+ }
> df=data.frame(F_y,y)
> library(ggplot2)
> ggplot(df)+geom_line(aes(x=y,y=F_y),color="steelblue")+
+   labs(x='x',y='F(x)')+ggtitle("Graph of the function F")
```



(b).

$$
\begin{aligned}
P\left([0,\frac{1}{4})\right) &= P\left((-\infty,\frac{1}{4})\right) - P((-\infty,0)) \\
&= P\left(X \leq \frac{1}{4}\right) - P\left(X = \frac{1}{4}\right) - P(X \leq 0) \\
&= F\left(\frac{1}{4}\right) - P\left(X = \frac{1}{4}\right) - F(0) \\
&= \left(\left(\frac{1}{4}\times\frac{1}{2}\right) + \frac{1}{8}\right) - 0 - 0 \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
P\left(\left[\frac{1}{8},\frac{3}{2}\right]\right) &= P\left((-\infty,\frac{3}{2}]\right) - P\left((-\infty,\frac{1}{8})\right) \\
&= P\left(X \leq \frac{3}{2}\right) - P\left(X \leq \frac{1}{8}\right) + P\left(X = \frac{1}{8}\right) \\
&= F\left(\frac{3}{2}\right) - F\left(\frac{1}{8}\right) + P\left(X = \frac{1}{8}\right) \\
&= 1 - \frac{1}{8} + 0 \\
&= \frac{7}{8}
\end{aligned}
$$

$$
P\left(\left(\frac{3}{4},\frac{7}{8}\right]\right) = P\left((-\infty,\frac{7}{8}]\right) - P\left((-\infty,\frac{3}{4}]\right)
$$

$$
\begin{aligned}
&= P\left(X \leq \frac{7}{8}\right) - P\left(X \leq \frac{3}{4}\right) \\
&= F\left(\frac{7}{8}\right) - F\left(\frac{3}{4}\right) \\
&= \left(2 \times \frac{7}{8} - 1\right) - \left(2 \times \frac{3}{4} - 1\right) \\
&= \frac{1}{4}
\end{aligned}
$$

(c). We can obtain the probability density function by using $f(x) = F'(x)$.
Note: Since, densities are assumed to be piecewise continuous, their corresponding distribution functions are piecewise differentiable.

$$
f(x) = \begin{cases} \frac{d}{dx}(0) & \text{if } x < 0 \\ \frac{d}{dx}(x) & \text{if } 0 < x < \frac{1}{4} \\ \frac{d}{dx}\left(\frac{x}{2} + \frac{1}{8}\right) & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ \frac{d}{dx}(2x - 1) & \text{if } \frac{3}{4} \leq x < 1 \\ \frac{d}{dx}(1) & \text{if } x \geq 1 \end{cases}
$$

Therefore,

$$
f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{4} \\ \frac{1}{2} & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ 2 & \text{if } \frac{3}{4} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}
$$

5. Exercise 5.2.6

Solution: 5

$$
f(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & \text{if } -R < x < R \\ 0 & \text{otherwise} \end{cases}
$$

The Distribution function of X is defined as:
Case 1: $y \leq -R$:

$$
\begin{aligned}
F(y) &= \int_{-\infty}^{y} f(x)dx \\
&= \int_{-\infty}^{y} 0dx = 0
\end{aligned}
$$

Case 2: $-R < y < R$:

$$
\begin{aligned}
F(y) &= \int_{-\infty}^{y} f(x)dx \\
&= \int_{-\infty}^{-R} 0dx + \int_{-R}^{y} \frac{2}{\pi R^2}\sqrt{R^2 - x^2}dx \\
&= 0 + \frac{2}{\pi R^2}\int_{-R}^{y} \sqrt{R^2 - x^2}dx \\
&= \frac{2}{\pi R^2}\left(\frac{x\sqrt{R^2 - x^2}}{2} + \frac{R^2}{2} arcsin\left(\frac{x}{R}\right)\right)\Bigg|_{-R}^{y}
\end{aligned}
$$

$$= \frac{2}{\pi R^2}\left(\frac{y\sqrt{R^2-y^2}}{2} + \frac{R^2}{2}arcsin\left(\frac{y}{R}\right) - \frac{(-R)\sqrt{R^2-(-R)^2}}{2} - \frac{R^2}{2}arcsin\left(\frac{-R}{R}\right)\right)$$
$$= \frac{y\sqrt{R^2-y^2}}{\pi R^2} + \frac{1}{\pi}arcsin\left(\frac{y}{R}\right) + \frac{1}{2}$$

Case 3: $y \geq R$

$$\begin{aligned} F(y) &= \int_{-\infty}^{y} f(x)dx \\ &= \int_{-\infty}^{-R} 0dx + \int_{-R}^{y} \frac{2}{\pi R^2}\sqrt{R^2-x^2}dx + \int_{R}^{\infty} 0dx \\ &= 0+1+0 \\ &= 1 \end{aligned}$$

Hence, the distribution function of X:

$$F(x) = \begin{cases} 0 & \text{if } x \leq -R \\ \frac{x\sqrt{R^2-x^2}}{\pi R^2} + \frac{1}{\pi}arcsin\left(\frac{x}{R}\right) + \frac{1}{2} & \text{if } -R < x < R \\ 0 & \text{if } x \geq R \end{cases}$$

6. Exercise 5.2.7

Solution: 6

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{2}{\pi}arcsin(\sqrt{x}) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

We can obtain the probability density function by using $f(x) = F'(x)$.
Note: Since, densities are assumed to be piecewise continuous, their corresponding distribution functions are piecewise differentiable.
Now, if $x \leq 0$

$$\begin{aligned} F'(x) &= \frac{d}{dx}(0) \\ &= 0 \end{aligned}$$

If $0 < x < 1$

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{2}{\pi}arcsin(\sqrt{x})\right) \\ &= \frac{2}{\pi}\frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{\pi\sqrt{x(1-x)}} \end{aligned}$$

If $x \geq 1$

$$\begin{aligned} F'(x) &= \frac{d}{dx}(1) \\ &= 0 \end{aligned}$$

Hence, the probability density function of X:

$$f(x) = \begin{cases} \frac{1}{\pi\sqrt{x(1-x)}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

7. Exercise 5.2.10
   Solution: 7

(a). X $\sim$ Uniform(a, b)
The probability density function of X is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Let the median of X be 'm'.
i.e.

$$P(X < m) = \frac{1}{2} \qquad ------(1)$$

Also,

$$\begin{aligned} P(X < m) &= \int_{-\infty}^{m} f(x)dx \\ &= \int_{-\infty}^{a} 0dx + \int_{a}^{m} \frac{1}{b-a}dx \\ &= 0 + \frac{1}{b-a}\int_{a}^{m} 1dx \\ \therefore P(X < m) &= \frac{m-a}{b-a} \qquad ------(2) \end{aligned}$$

From (1) and (2):

$$\begin{aligned} \frac{1}{2} &= \frac{m-a}{b-a} \\ m &= a + \frac{b-a}{2} \\ \therefore m &= \frac{a+b}{2} \end{aligned}$$

Hence, median of X is $\frac{a+b}{2}$.

(b). Y $\sim$ Exp(λ)
The probability density function of Y is:

$$f(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let the median of Y be 'm'.
i.e.

$$P(Y < m) = \frac{1}{2} \qquad ------(1)$$

Also,

$$
\begin{aligned}
P(Y < m) &= \int_{-\infty}^{m} f(y)dy \\
&= \int_{-\infty}^{0} 0dy + \int_{0}^{m} \lambda e^{-\lambda y}dy \\
&= 0 + \lambda \frac{e^{-\lambda y}}{-\lambda}\Big|_0^m \\
\therefore P(X < m) &= 1 - e^{-\lambda m} \qquad - - - - - (2)
\end{aligned}
$$

From (1) and (2):

$$
\begin{aligned}
\frac{1}{2} &= 1 - e^{-\lambda m} \\
e^{-\lambda m} &= \frac{1}{2} \\
-\lambda m &= ln\left(\frac{1}{2}\right) \\
\therefore m &= \frac{ln2}{\lambda}
\end{aligned}
$$

Hence, median of Y is $\frac{ln2}{\lambda}$.

(c). Z $\sim$ Normal(μ, σ^2)
Let the median of Z be 'm'.
i.e.

$$
\begin{aligned}
P(Z < m) &= \frac{1}{2} \\
\implies P(\frac{Z-\mu}{\sigma} < \frac{m-\mu}{\sigma}) &= \frac{1}{2} \\
\implies P(X < \frac{m-\mu}{\sigma}) &= \frac{1}{2}
\end{aligned}
$$

where, $X \sim$ Normal(0,1)
Therefore,

$$\int_{-\infty}^{\frac{m-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \, \mathrm{d}x = \frac{1}{2} \qquad - - - - - (1)$$

Using R, we get:

```
> qnorm(1/2)
[1] 0
```

$$P(X < 0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \, \mathrm{d}x = \frac{1}{2} \qquad - - - - - (2)$$

From (1) and (2):

$$\frac{m-\mu}{\sigma} = 0 \implies m = \mu$$

Hence, median of Z is μ.