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Grading:
30 marks- Complete submission of Problem 2,4,6
70 marks- Problem 6

From Probability and Statistics with Examples Using R

1. Ex. 4.1.2
   Solution: 1

   (a) X $\sim$ Binomial (n=5,p=0.30)
   Because, X denotes the number of successes i.e. Number of times lottery will be won, in next 5 days (with each day independent of every other), with probability of success equal 0.30 (30%)
   P(X=x)= ${}^5C_x(0.30)^x(0.70)^{5-x}$; $x = 0, 1, 2, 3, 4, 5$

   (b) For binomial distribution;
   E(X)=np
   E(X)=$5 \times 0.30 = 1.5$
   Hence, on an average, in next five days lottery will be won 1.5(≈ 2) times.

   (c) Mode of binomial distribution is given by:

   $$Mode(X) = \begin{cases} [(n+1)p] & \text{if (n+1)p is 0 or non-integer} \\ (n+1)p \quad and \quad (n+1)p-1 & \text{if (n+1)p} \in \{1,2,...,n\} \\ n & \text{if (n+1)p =n+1} \end{cases}$$

   where, [ ]= Greatest Integer function
   Here, n=5 and p=0.30
   (n+1)p = $6 \times 0.30 = 1.8$ i.e. non-integer.
   Hence, Mode(X)=1 i.e. integral part of (n+1)p

   (d) Probability that the lottery will be won in either one or two of the next five days is given by:
   P(X=1 or X=2)= P(X=1) + P(X=2) = ${}^5C_1(0.30)^1(0.70)^4 + {}^5C_2(0.30)^2(0.70)^3$
   Therefore, P(X=1 or X=2)= 0.6689

2. Ex 4.1.3
   Solution: 2

   (a) Y $\sim$ Geometric(p=0.12)
   Because, Y denotes the number of questions asked until the first success where success is that she does not know the correct answer. And, chances that she knows the answer of any given question is independent of every other.

Probability of success = 0.12
P(X=x)=$(0.12)(0.88)^{x-1}$; $x = 1, 2, 3, ...$

(b) For geometric distribution;
E(X)=$\frac{1}{p}$
E(X)=$\frac{1}{0.12} = 8.33$
Hence, on an average, 8.33($\approx$ 8) questions are asked until the first question for which contestant does not know the answer.

(c) Mode of geometric distribution is 1 if 0<p<1 , as it will have the highest probability.
Hence, Mode(Y)=1

(d) Probability that she wins the grand prize i.e. She knows the answer of 12 successive questions, is given by:
P[12 failures]= $(0.88)^{12} = 0.2157$
As, chances that she knows the answer of any given question is independent of every other.

3. Ex 4.1.6
Solution: 3
Let, X be a random variable representing the amount of money (in $) one wins in the game.
Range(X)={2,4,8,16,...}
The distribution of X is given by:
P(X=2^n)= $\frac{1}{2^n}$; n=1,2,3,...
The amount of money one wins, on an average, in the game is:

$$E(X) = \sum_{i=1}^{\infty} 2^i P(X = 2^i)$$

$$E(X) = \sum_{i=1}^{\infty} 2^i \frac{1}{2^i} = \sum_{i=1}^{\infty} 1$$

which diverges to infinity.
Hence, there is no amount of money the player could pay to make this a fair game.

4. Ex 4.2.1
Solution: 4
The probability mass function of random variable X is:

$$P(X = x) = \begin{cases} 0.2 & \text{if x=0} \\ 0.5 & \text{if x=1} \\ 0.2 & \text{if x=2} \\ 0.1 & \text{if x=3} \end{cases}$$

The expected value of X:

$$E(X) = \sum_{x=0}^{3} xP(X = x)$$

$$E(X) = (0 \times 0.2) + (1 \times 0.5) + (2 \times 0.2) + (3 \times 0.1) = 1.2$$

Also,

$$E(X^2) = \sum_{x=0}^{3} x^2 P(X = x)$$

$$E(X^2) = (0^2 \times 0.2) + (1^2 \times 0.5) + (2^2 \times 0.2) + (3^2 \times 0.1) = 2.2$$

Hence, Variance of X is given by:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = 2.2 - (1.2)^2 = 0.76$$

Therefore, Standard deviation of X is:

$$S.D(X) = \sqrt{Var(X)} = 0.8718$$

Now, Probability that X will produce a result more than one standard deviation from its expected value is:

$$\begin{aligned} P(X > E(X) + S.D(X)) &= P(X > 1.2 + 0.8718) \\ &= P(X > 2.0718) \\ &= P(X = 3) = 0.1 \end{aligned}$$

5. Ex 4.2.2

   Solution: 5

   (a) Let X_1 be a random variable representing the number of heads.
   $X_1 \sim$ Binomial (n=100, p=0.5)
   Because, X_1 denotes the number of successes i.e. Number of heads, in 100 flips of a fair coin.

   $$\begin{aligned} S.D(X_1) &= \sqrt{np(1-p)} \\ S.D(X_1) &= \sqrt{100 \times 0.5 \times (1-0.5)} = 5 \end{aligned}$$

   (b) If the number of coins quadrupled then,
   $X_2 \sim$ Binomial(n=400, p=0.5)
   where, X_2= Number of heads

   $$\begin{aligned} S.D(X_2) &= \sqrt{np(1-p)} \\ S.D(X_2) &= \sqrt{400 \times 0.5 \times (1-0.5)} = 10 \end{aligned}$$

   Hence, S.D(X_2)= $2 \times S.D(X_1)$

6. Ex 4.2.3

   Solution: 6

   Let, X be the random variable representing the number of rolls needed before we see the first 3.
   Hence, X $\sim$ Geometric(p=$\frac{1}{6}$)
   P(X=x)=$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1}$; $x = 1, 2, 3, ...$

(a) For geometric distribution;

$$E(X) = \frac{1}{p}$$
$$E(X) = \frac{1}{\frac{1}{6}} = 6$$

(b)

$$S.D(X) = \sqrt{\frac{1-p}{p^2}}$$
$$S.D(X) = \sqrt{\frac{1-\frac{1}{6}}{\frac{1}{36}}} = 5.4772$$

(c) Effective range for X is given by:

$$((E(X) - S.D(X)), (E(X) + S.D(X)) = ((6 - 5.4772), (6 + 5.4772))$$

Hence, effective range is {1,2,3,4,5,6,7,8,9,10,11}
Since, 9 do lies in the effective range of X.
Therefore, it would not seem unusual to roll the die more than 9 times before seeing 3.

(d)

$$\begin{aligned}
P(X > 9) &= 1 - P(X \leq 9) \\
&= 1 - (P(X = 1) + P(X = 2) + ... + P(X = 9)) \\
&= 1 - [p + (1-p)p + ... + (1-p)^8 p] \\
&= 1 - p[1 + (1-p) + ... + (1-p)^8] \\
&= 1 - p\left(\frac{1-(1-p)^9}{1-(1-p)}\right) \\
&= 1 - [1 - (1-p)^9] \\
&= (1-p)^9 \\
\therefore P(X > 9) &= \left(1 - \frac{1}{6}\right)^9 = 0.1938
\end{aligned}$$

(e) The probability that X produces a result within one standard deviation of its expected value is given by:

$$\begin{aligned}
P[E(X) - S.D(X) \leq X \leq E(X) + S.D.(X)] &= P[0.5228 \leq X \leq 11.4772] \\
\implies P[1 \leq X \leq 11] &= P(X = 1) + P(X = 2) + ... + P(X = 11) \\
&= p + (1-p)p + (1-p)^2 p + ... + (1-p)^{10} p \\
&= p[1 + (1-p) + (1-p)^2 + ... + (1-p)^{10}] \\
&= p\left(\frac{1-(1-p)^{11}}{1-(1-p)}\right) \\
&= 1 - (1-p)^{11} \\
&= 1 - \left(1 - \frac{1}{6}\right)^{11} \\
\therefore P[1 \leq X \leq 11] &= 1 - 0.1346 = 0.8654
\end{aligned}$$