1.2 Equally likely Outcomes

S – sample space – (Countable)

It is enough to describe Probability of each outcome in S to describe the Probability of all events.

$P(\{\omega\})$ – known $\forall\ \omega \in S$

$A \subseteq S$, A – event

$$\mathbb{P}(A) = \mathbb{P}(\underbrace{\bigcup_{\omega \in A} \{\omega\}}_{\text{Countable disjoint union}}) \overset{\text{Axiom 2}}{=} \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

The assignment of Probabilities to each outcome is called a "distribution" on S.

– Interesting & understable case is when S is finite.

Example :– – Toss a fair Coin $S = \{\text{Heads}, \text{Tails}\}$

$$\mathbb{P}(\{\text{Heads}\}) = \mathbb{P}(\{\text{tails}\}) = 1/2$$

S – outcomes were all "equally likely"

– Roll a fair dice $S = \{1, 2, 3, 4, 5, 6\}$

One can verify that $\mathbb{P}(\{k\}) = 1/6$ for $k \in S$.

Theorem 1.2.1 Uniform $(\{\omega_1, \omega_2, \dots, \omega_n\})$:-

Let $S = \{\omega_1, \omega_2, \dots, \omega_n\}$ be a non-empty finite set.

If $E \subseteq S$, let

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|E|}{n},$$

where $|E|$ = # of elements in the set E.

Then $\mathbb{P}$ - defines a Probability on S &

$\mathbb{P}$ - assigns equal probability to every individual outcome in S.

Proof :- step 1 - $\mathbb{P}: \mathcal{F} \longrightarrow [0,1]$ is a well defined function

this indeed true $E \subseteq S \Rightarrow 0 \le |E| \le |S| (= n)$

$$\Rightarrow \mathbb{P}(E) = \frac{|E|}{n} \in [0,1]$$

step 2 - Verify Axiom ①

$$P(S) = \frac{|S|}{n} = \frac{n}{n} = 1$$

$\therefore$ Axiom ① holds

Step3:- Verify Axiom ②

let $E_1, E_2, \ldots E_n, \ldots$ be a countable sequence of disjoint events.

S - is finite, $|S| = n$

(*') - we may assume without loss of generality

ie $E_j = \phi$ $j > n$ [Rearranging $\{E_j\}_{j \geq 1}$]

(*'') $|E_1 \cup E_2 \ldots \cup E_n| = \sum_{j=1}^{n} |E_j|$ [disjoint sets]

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) \overset{(*')}{=} \mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) \overset{\text{definition}}{=} \frac{\left|\bigcup_{j=1}^{n} E_j\right|}{n} \overset{(*'')}{=} \sum_{j=1}^{n} \frac{|E_j|}{n}$$

$$= \sum_{j=1}^{n} P(E_j) \underset{P(\phi)=0}{\overset{(*')}{=}} \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

$\Rightarrow$ Axiom ② is verified.

Step4:- "Equally likely" outcome distribution

$\omega \in S$, $\mathbb{P}(\{\omega\}) = \frac{|\{\omega\}|}{n} = \frac{1}{n}$ [Same for any $\omega \in S$]

So each outcome is equally likely.

Example 1.2.3 :- Two fair dice are rolled. How likely is it that their sum will equal eight?

Method 1: $S = \{2, 3, \ldots, 12\}$ - sample space

possibility - turns out it is a bit harder to assign probabilities. - It is not the case that all outcomes are equally likely.

Method 2 :- Rolling one dice & Rolling another dice

$$S = \left\{ \begin{array}{cccccc} (1,1), & (1,2) & ,(1,3), & (1,4) & ,(1,5) & ,(1,6) \\ (2,1) & ,(2,2) \ldots & & \ldots & & (2,6) \\ \vdots & \ldots & & & & \vdots \\ (6,1) & ,(6,2) & \ldots & & & ,(6,6) \end{array} \right\}$$

- any outcome in S is equally likely.

$|S| = 36$, Theorem 1.2.1 $\mathbb{P}(E) = \frac{|E|}{36}$

$E = \{$ sum of the rolls equals 8 $\}$

$= \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

& $|E| = 5$

$$\mathbb{P}(\text{sum equals eight}) = P(E) = \frac{|E|}{36} = \frac{5}{36} \quad \square$$