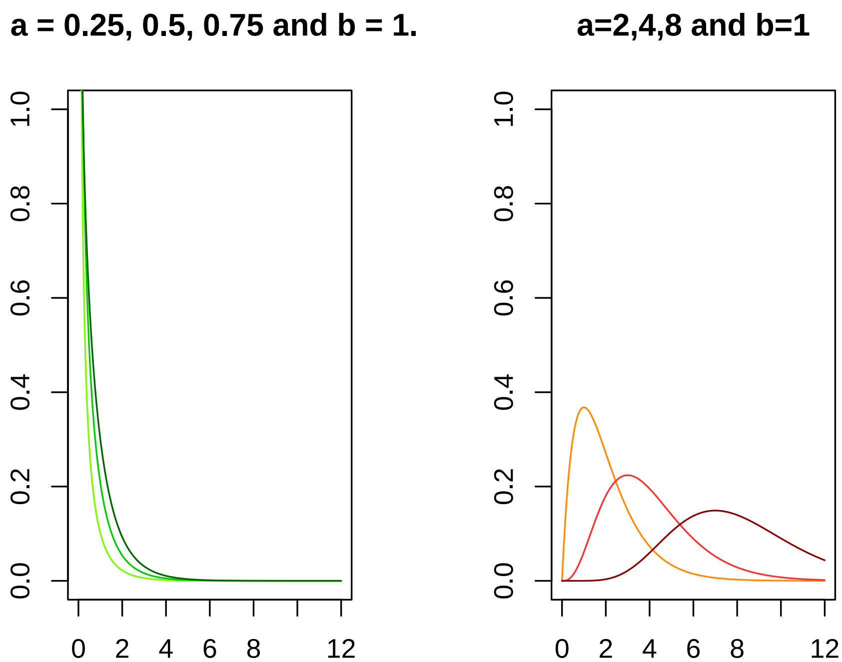
1. One intuitive way of understand continuous random variables is to plot their probability density function and see the changes in them as the parameters vary. Below we shall consider two densities namely:

   (a) The density of Gamma(a, b) is given by :

$$f(x; a, b) \equiv f(x) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

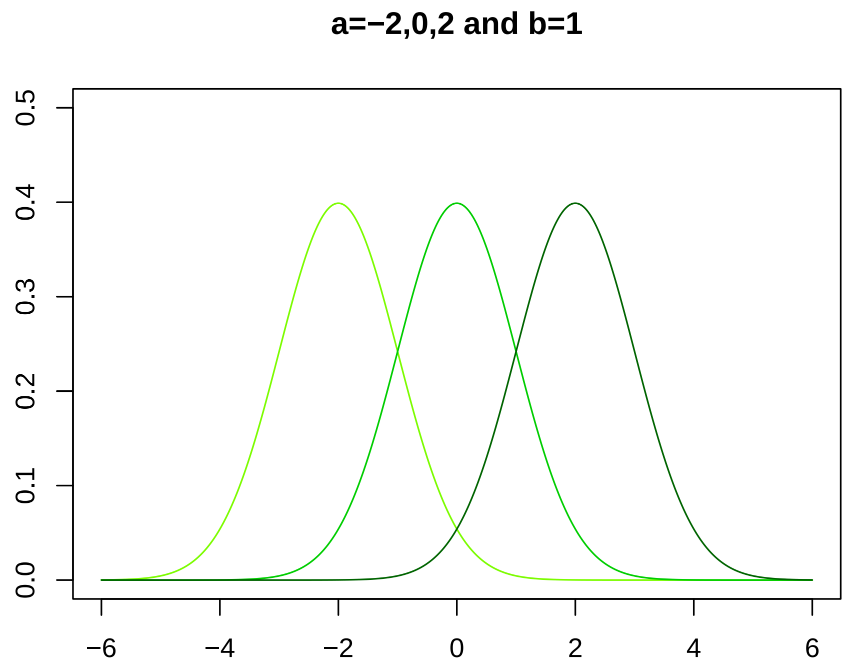


The above figure contains plots for the Gamma density for fixed $b = 1$ and varying parameters of a (in the range $a < 1$ and $a > 1$) respectively. Note that for $a < 1$ the Gamma density resembles an Exponential density but is unbounded. Further, for $a = 1$ the Gamma density is equal to the Exponential(b) density. Lastly when $a > 1$ the Gamma density has a finite maxima and it differs from the form of the Exponential Density.

(b) The density of Normal(a, b^2):

$$f(x) = \frac{1}{\sqrt{2\pi}b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}, \qquad x \in \mathbb{R}$$

where $a, b > 0$ be some given numbers.



The above plots the Normal density for fixed $b = 1$ and varying parameters of a. Note that as a moves across the real line the peak of the density moves accordingly.

(i) Using the given `R`-code below plot the Gamma densities for fixed a in the regions ($a > 1$ and $a < 1$) with varying b. Note down the changes in the density function.

(ii) Use the above `R`-code below plot the Normal densities for fixed a and with varying b. Note down the changes in the density function.

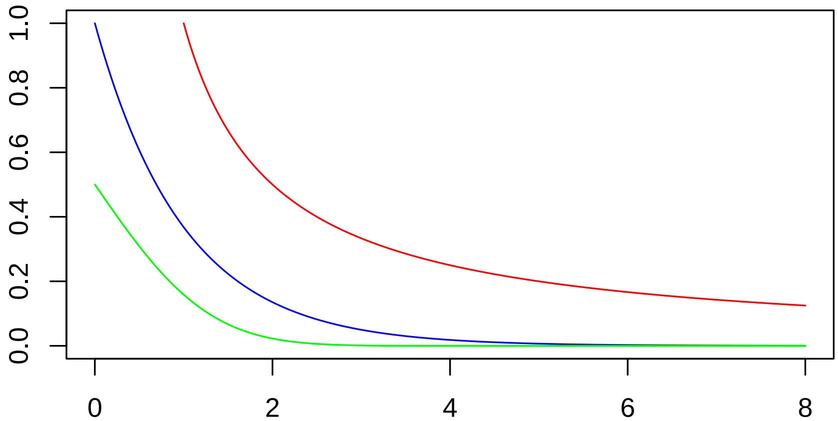
2. Suppose X is a random variable then we shall refer to $G : R \to [0, 1]$given by

$$G(x) = P(X > x),$$

as the tail probabilities. For example if $X \sim \text{Exp}(\lambda)$ then

$$G_\lambda(x) = P(X > x) = \begin{cases} \exp(-\lambda x) & \text{if } x \geq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Random variables whose tail probabilities decay slower than the expoential distribution are called Heavy-tailed distributions.



In the plot above, the graph in blue is the tail probability of Exponential(1), the graph in red is the tail probability of the Pareto$(1, 1)$ distribution and the graph in green is the tail probability of the Normal$(0, 1)$ distribution.

(a) Repeat the above plot for the distributions:

  i. Gamma$(4, 3)$
  ii. Cauchy$(4, 1)$
  iii. Pareto $(1, 3)$

(b) Repeat the above plots for the following:

  i. $X \sim$ Poisson (4)
  ii. X is a discrete random variable such that

$$P(X = k) = \frac{c}{k^4}, \qquad k = 1, 2 \ldots$$

Note that in the second part the tail probability function may not be available as an inbuilt function in `R`.

3. Repeat 1 and 2 for the following distributions:

(a) Cauchy Distribution
(b) Log-Normal Distribution
(c) Student's t Distribution

In each case decide if they are Heavy tail or not.

Below is a R-code for a plotting the tail probablities for Exponential(1), Pareto$(1,1)$ distribution and Normal$(0,1)$ distribution.

```
> library(EnvStats)
> axis=8
> loc=1
> shape=1
> plot(1, type="n", xlab="", ylab="", xlim=c(0, axis),
+      ylim=c(0, min(1,dpareto(loc,loc,shape))))
> lines(seq(loc,axis,length.out=axis*100),1- ppareto(seq(loc,axis,length.out
+                                                      =axis*100),loc,shape),col='red')
> lines(seq(0,axis,length.out=axis*100),1-pexp(seq(0,axis,
+                                                  length.out=axis*100),1),col='blue')
> lines(seq(0,axis,length.out=axis*100),1-pnorm(seq(0,axis,
+                                                   length.out=axis*100)),col='green')
```

Below is the R-code for plotting the Gamma Densities

```
> axis=12
> par(mfrow=c(1,2))
> plot(1, type="n", xlab="", ylab="", xlim=c(0, axis),
+   ylim=c(0, 1),main=" a = 0.25, 0.5, 0.75 and b = 1.")
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+   length.out=axis*100),0.25,1),col='lawngreen')
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+   length.out=axis*100),0.5,1),col='green3')
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+  length.out=axis*100),0.75,1),col='darkgreen')
> plot(1, type="n", xlab="", ylab="", xlim=c(0, axis),
+    ylim=c(0, 1), main="a=2,4,8 and b=1")
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+   length.out=axis*100),2,1),col='darkorange')
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+   length.out=axis*100),4,1),col='firebrick1')
> lines(seq(0,axis,length.out=axis*100),dgamma(seq(0,axis,
+    length.out=axis*100),8,1),col='red4')
```

Below is the R-code for plotting the Normal Densities

```
> axis=6
> plot(1, type="n", xlab="", ylab="", xlim=c(-axis, axis),
+      ylim=c(0, 1/2),main="a=-2,0,2 and b=1")
> lines(seq(-axis,axis,length.out=axis*200),dnorm(seq(-axis,
+       axis,length.out=axis*200),-2,1),col='lawngreen')
```

```
> lines(seq(-axis,axis,length.out=axis*200),dnorm(seq(-axis,
+                      axis,length.out=axis*200),0,1),col='green3')
> lines(seq(-axis,axis,length.out=axis*200),dnorm(seq(-axis,
+                      axis,length.out=axis*200),2,1),col='darkgreen')
```