Recall:

S - set sample space - all possible outcomes

$\mathcal{F}$ - space of all events - all subsets of S

ϕ, S

$\mathbb{P}$ - Probability on S $\quad \mathbb{P}: \mathcal{F} \to [0,1]$

① $\mathbb{P}(S) = 1$

② $E_1, E_2, \ldots$ countable sequence of disjoint events then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

Basic Properties:

Theorem 1.1.4: Let $\mathbb{P}$ be Probability on a sample space S. Then

(i) $\mathbb{P}(\phi) = 0$

(ii) If $E_1, E_2, \ldots E_n$ are a finite collection of disjoint events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$$

(iii) If E and F are events with $E \subseteq F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$

(iv) If E and F are event $E \subseteq F$

$$\mathbb{P}(F) - \mathbb{P}(E) = \mathbb{P}(F \setminus E)$$

(v) let E^c — Complement of E, then

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$$

(vi) If E and F are events then

$$\mathbb{P}(F \cup E) = \mathbb{P}(F) + \mathbb{P}(E) - \mathbb{P}(E \cap F)$$

Proof:- (i) $E_1 = \phi$, $E_2 = \phi$, ... is a countable sequence of disjoint events. Axiom (2) $\Rightarrow$

$$\mathbb{P}\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\phi}\Big) = \sum_{i=1}^{\infty} \mathbb{P}(\underbrace{E_i}_{\phi})$$

$$\Rightarrow \quad \underbrace{\mathbb{P}(\phi)}_{\text{Definition } 0 \leq P(\phi) \leq 1} = \underbrace{P(\phi) + P(\phi) + \cdots}_{\text{R.H.S converges by Axiom (2)}}$$

$$\Rightarrow \quad P(\phi) = 0$$

(ii) let $E_1, E_2, \ldots, E_n$ be given. Define $E_j = \phi$ $j \geq n+1$.
disjoint sets

$\Rightarrow$ $E_1, E_2, \ldots E_n, E_{n+1}, \ldots$ are a disjoint collection of sets

Axiom (2) : $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

(a) $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{n} E_i$ (b) By part (i) $\mathbb{P}(E_i) = 0 \ \forall i \geq n+1$

$$\Rightarrow \quad \mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$$

Observe: $E \subset F \Rightarrow F \setminus E$ & E are disjoint sets

AND $\underbrace{F \setminus E}_{E_1} \cup \underbrace{E}_{E_2} = F$

$\therefore$ apply (ii)

$$\mathbb{P}(F \setminus E \cup E) = \mathbb{P}(F \setminus E) + \mathbb{P}(E)$$

$$\Rightarrow \mathbb{P}(F) = \mathbb{P}(F \setminus E) + \mathbb{P}(E) \quad - (*)$$

(iii) From $(*)$ as $\mathbb{P}(F \setminus E) \geq 0 \Rightarrow$

$$\mathbb{P}(F) \geq \mathbb{P}(E)$$

(iv) $(*)$ readily implies (iv).

(v) In $(*)$ Take $F = S$,

$$\mathbb{P}(S) = \mathbb{P}(S \setminus E) + \mathbb{P}(E)$$

Axiom(I) $\quad 1 = \mathbb{P}(E^c) + \mathbb{P}(E)$

$$\Rightarrow \mathbb{P}(E) = 1 - \mathbb{P}(E^c)$$

(vi) $E \cup F = E \cup F \setminus E$ (disjoint union)

Apply (ii) $\quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F \setminus E) \quad - (\ddagger)$

$F \setminus E \subseteq F$ & $F \setminus (F \setminus E) = F \cap E$

Apply (iv) with F and $F \setminus E$ to get

$$\mathbb{P}(F \setminus E) = \mathbb{P}(F) - \mathbb{P}(F \setminus F \setminus E)$$

$$= \mathbb{P}(F) - \mathbb{P}(F \cap E) \quad (\ddagger\ddagger)$$

$(\ddagger)$ and $(\ddagger\ddagger)$ $\quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(F \cap E)$ $\square$

Example 1.1.5. A coin flip comes up Heads or Tails

$$S = \{\text{Heads}, \text{Tails}\}$$

- a coin is fair if every outcome in S is equally likely. $E = \{\text{Heads}\}$ & $F = \{\text{Tails}\}$

$$\Rightarrow \quad \mathbb{P}(E) = \mathbb{P}(F) = p \ (\text{say})$$

Axiom ①

$$1 = \mathbb{P}(S)$$

$$= \mathbb{P}(E \cup F)$$

Theorem 1.1.4 (ii)

$$= \mathbb{P}(E) + \mathbb{P}(F)$$

$$= p + p$$

$$= 2p$$

$$\Rightarrow \quad p = \frac{1}{2}.$$

Example 1.1.7 Suppose we know that there is a

- 60% chance that it will rain tomorrow and
- 70% chance that the high temperature will be above 30°C
- 40% chance that it will rain tomorrow and that the high temperature will be above 30°C.

How likely is it tomorrow will be a dry day & the high temperature does not go above 30°C ?

Ans: $E = \{$ it will rain tomorrow $\}$

$F = \{$ high temperature is above 30°C $\}$

$P(E) = 0.6, \quad P(F) = 0.7, \quad P(E \cap F) = 0.4$

$P(E^c \cap F^c) = ?$. As, $E^c \cap F^c = (E \cup F)^c$

$$P(E^c \cap F^c) = P((E \cup F)^c)$$

$$\underset{\text{(i) Theorem 1.1.4}}{=} 1 - P(E \cup F)$$

$$\underset{\text{(ii) Theorem 1.1.4}}{=} 1 - [P(E) + P(F) - P(E \cap F)]$$

$$= 1 - [0.6 + 0.7 - 0.4]$$

$$= 1 - 0.9 = 0.1 \qquad \square$$