Recall :

Definition : (BLUE ; Best linear Unbiased Estimate)

A linear estimate $\underline{a}^T \underline{y}$ is called BLUE for estimating an estimable linear parametric function $\underline{p}^T \underline{\beta}$ if

(i) $E[\underline{a}^T \underline{y}] = \underline{p}^T \underline{\beta} \quad \forall \ \underline{\beta} \in \mathbb{R}^n$

(ii) $Var(\underline{a}^T \underline{y}) \le Var(\underline{l}^T \underline{y}) \quad \forall \ \underline{l} \in \mathbb{R}^n$

s.t $\underline{l}^T \underline{y}$ is unbiased estimator of $\underline{p}^T \underline{\beta}$.

- last class

Theorem (Gauss- Markov Theorem): Suppose $(\underline{y}, X\underline{\beta}, \sigma^2 I_{n\times n})$ be a linear model and $\underline{p}^T \underline{\beta}$ be an estimable linear parametric function.

Then the least square estimate of $\underline{p}^T \underline{\beta}$, written as $\underline{p}^T \hat{\underline{\beta}}$ for some solution of the normal equations $(X^T X)\hat{\underline{\beta}} = X^T \underline{y}$ is the BLUE of $\underline{p}^T \underline{\beta}$.

Comments :-

- $\underline{p}^T \underline{\beta}$ is estimable $\iff$ least square estimator is unique

- $\underline{\beta}$ - all components are estimable $\iff$ $X^T X$ is non-singular

$$\hat{\underline{\beta}} = (X^T X)^{-1} X^T \underline{y}$$

Remarks:

[Gauss - Markov Theorem] -

✓ L.S.E is best among all linear unbiased estimators.

ⓧ It does not say: L.S.E is best estimator.

(there could be other estimators that are biased or non-linear that are better)

✓ No assumption on the distribution of the model

only mean is specified and uncorrelatedness.

(there is No: - Confidence interval for $\hat{\beta}$)

- hypothesis testing

If we assume the errors $\underline{\varepsilon} \overset{d}{=} N(0, \sigma^2 I_{n \times n})$

then we saw M.L.E. & L.S.E for β

(likelihood) $$L(\underline{\beta}, \sigma^2 \mid \underline{y}) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp\left(-\frac{1}{2\sigma^2} (\underline{Y} - X\underline{\beta})^T (\underline{Y} - X\underline{\beta})\right)$$

$$= \frac{1}{(2\pi)^{n/2} \sigma^n} \exp\left(-\frac{1}{2\sigma^2} \|\underline{Y} - X\underline{\beta}\|^2\right)$$

then if $\sigma^2 > 0$ is given

$$\max_{\beta \in \mathbb{R}^m} L(\underline{\beta}, \sigma^2 \mid \underline{y}) = \min_{\beta \in \mathbb{R}^m} \|\underline{Y} - X\underline{\beta}\|^2$$

M.L.E. of $\underline{\beta}$

L.S.E of $\underline{\beta}$

Theorem (The First fundamental Theorem of least Square)

Suppose $(\underline{Y}, X_{n\times m}\underline{\beta}, \sigma^2 I_{n\times n})$ be a linear model

where $\underline{\varepsilon} = \underline{Y} - X\underline{\beta} \sim N_n(0, \sigma^2 I_{n\times n})$ distribution.

If $R_0^2 = \min_{\beta \in \mathbb{R}^m} \| \underline{Y} - X\underline{\beta} \|_2^2$

then $\dfrac{R_0^2}{\sigma^2} \sim \chi^2_{n-r}$ where $r = \text{rank}(X)$.

Proof :-

- let the S.V.D. of $X_{n\times m}$ be given by

$$X = P \begin{pmatrix} D_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} \Phi^T$$

$$[\, PP^T = P^TP = I_n \; ; \quad \Phi\Phi^T = \Phi^T\Phi = I_m$$

$$D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

- $$P^T X = \begin{pmatrix} D_{r\times r} (\Phi^1)^T \\ 0 \end{pmatrix}$$ where $\Phi = [\Phi^1_{m\times r} : \Phi^2]$

- $$R_0^2 = \min_{\beta \in \mathbb{R}^m} \| \underline{Y} - X\underline{\beta} \|_2^2$$

$$\overset{(Ex)}{=} \min_{\beta \in \mathbb{R}^m} \| P^T\underline{Y} - P^TX\underline{\beta} \|_2^2$$

$$\underset{\sim}{z} = P^T \underline{y} \qquad \tilde{v} = P^T X \underline{\beta} = \begin{pmatrix} D_{r \times r} (Q')' \underline{\beta} \\ 0 \end{pmatrix}$$

$$\therefore \quad R_0^2 = \min_{\beta \in \mathbb{R}^m} \left\| z - \begin{pmatrix} D_{r\times r} (Q')' \underline{\beta} \\ 0 \end{pmatrix} \right\|_2^2$$

$$= \min_{\tilde{v} \in \mathbb{R}^r} \| z - \tilde{v} \|_2^2 \qquad \tilde{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$= \sum_{j=r+1}^{n} z_j^2$$

$$\underset{\sim}{z} = P^T \underset{\sim}{y} \underset{Ex}{\sim} N(P^T X \underline{\beta}, \sigma^2 I_n)$$

$$\Rightarrow (z_{r+1}, \dots z_n) \sim N(\underset{\sim}{0}, \sigma^2 I_{n-r})$$

$$\Rightarrow \frac{z_j}{\sigma} \sim N(0,1) \text{ and independent } \quad r+1 \leq j \leq n$$

$$\Rightarrow \frac{R_0^2}{\sigma^2} = \sum_{j=r+1}^{n} z_j^2 \sim \chi^2_{n-r}$$

$\square$