Linear Statistical Models

Week-4: Graded Assignment

Subjective Assignment: (Manual-grading) **Max. Marks: 25**

Note: *R is not required for this assignment.*

1. Consider a Linear Model:

$$y_{ij} = \mu + \mu_i + \epsilon_{ij} \quad ; \quad 1 \le i \le 2\,,\ 1 \le j \le 2$$

Decide if the parameters μ and μ_i are identifiable from the model. If not then specify a condition for the same. [2 Marks]

Solution:
Identifiability means parameters can be uniquely estimated from the data, i.e., there should be enough information to estimate it.
As per the given model, the parameters μ and μ_i are not identifiable from the model. Because, we have total 4 independent observations, i.e., $y_{11}, y_{12}, y_{21}, y_{22}$ and number of parameters are 3 with rank of X as 2.
To identify the parameters, we need to impose a constraint on the model as $\mu_1 + \mu_2 = 0$.

2. Consider a model:

$$y_1 = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \epsilon_1 \quad ;$$
$$y_2 = \beta_1 + \beta_3 - \beta_2 - \beta_4 + \epsilon_2 \quad ;$$
$$y_3 = \beta_1 + \beta_2 - \beta_3 - \beta_4 + \epsilon_3 \quad ;$$
$$y_4 = \beta_1 + \beta_4 - \beta_2 - \beta_3 + \epsilon_4$$

where, $\beta_i \in R$; $i = 1, 2, 3, 4$ and ϵ_i's $(i = 1, 2, 3, 4)$ are uncorrelated random variables with variance σ^2.

(a) If we want to rewrite the model as $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, I_{4\times 4})$, then find $\underset{\sim}{Y}, X, \underset{\sim}{\beta}$ and $\underset{\sim}{\epsilon}$. [2 Marks]

Solution:
To rewrite the model in matrix form, i.e., $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, I_{4\times 4})$, we can write the $\underset{\sim}{Y}, X, \underset{\sim}{\beta}$ and $\underset{\sim}{\epsilon}$ as follows:

$$\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},\ X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},\ \underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \text{ and } \underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}$$

(b) Write down the Normal equation for the above model. [3 Marks]

Solution:

As we know that, for a linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ any least square estimate $\underset{\sim}{\hat{\beta}}$ of $\underset{\sim}{\beta}$ satisfies the following equation

$$(X^T X)\underset{\sim}{\hat{\beta}} = X^T \underset{\sim}{Y} \qquad \ldots (*)$$

Equation (*) is called Normal equation. Now,

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \text{ and}$$

$$X^T \underset{\sim}{Y} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

On substituting the values in equation (*), we get

$$\implies \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\begin{pmatrix} 4\hat{\beta}_1 \\ 4\hat{\beta}_2 \\ 4\hat{\beta}_3 \\ 4\hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 - y_3 + y_4 \end{pmatrix}$$

(c) Using the normal equation obtained in part (c), find the least square estimates of $\beta_1, \beta_2, \beta_3$ and β_4. [4 Marks]

Solution:

Approach(I): As we know the normal equation from part(b) which is as follows:

$$\begin{pmatrix} 4\hat{\beta}_1 \\ 4\hat{\beta}_2 \\ 4\hat{\beta}_3 \\ 4\hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 - y_3 + y_4 \end{pmatrix}$$

$$\implies 4 \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 - y_3 + y_4 \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 - y_3 + y_4 \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(y_1 + y_2 + y_3 + y_4) \\ \frac{1}{4}(y_1 - y_2 + y_3 - y_4) \\ \frac{1}{4}(y_1 + y_2 - y_3 - y_4) \\ \frac{1}{4}(y_1 - y_2 - y_3 + y_4) \end{pmatrix}$$

Least square estimates of $\beta_1, \beta_2, \beta_3$ and β_4 are $\underset{\sim}{\hat{\beta}_1} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4), \underset{\sim}{\hat{\beta}_2} = \frac{1}{4}(y_1 - y_2 + y_3 - y_4), \underset{\sim}{\hat{\beta}_3} = \frac{1}{4}(y_1 + y_2 - y_3 - y_4)$ and $\underset{\sim}{\hat{\beta}_4} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4)$ respectively.

Approach(II): As we know for a linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ normal equation is

$$(X^T X)\underset{\sim}{\hat{\beta}} = X^T \underset{\sim}{Y}$$

So,

$$\underset{\sim}{\hat{\beta}} = (X^T X)^{-1} X^T \underset{\sim}{Y} \qquad \ldots (**)$$

From part(b), we have $X^T X = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$. So, $(X^T X)^{-1} = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}$

Now, substituting the values of $(X^T X)^{-1}$ and $X^T Y$ in the equation (**), we get

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 - y_3 + y_4 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\end{pmatrix} = \begin{pmatrix}0.25(y_1+y_2+y_3+y_4)\\ 0.25(y_1-y_2+y_3-y_4)\\ 0.25(y_1+y_2-y_3-y_4)\\ 0.25(y_1-y_2-y_3+y_4)\end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\end{pmatrix} = \begin{pmatrix}\frac{1}{4}(y_1+y_2+y_3+y_4)\\ \frac{1}{4}(y_1-y_2+y_3-y_4)\\ \frac{1}{4}(y_1+y_2-y_3-y_4)\\ \frac{1}{4}(y_1-y_2-y_3+y_4)\end{pmatrix}$$

Least square estimates of $\beta_1, \beta_2, \beta_3$ and β_4 are $\underset{\sim}{\hat{\beta}_1} = \frac{1}{4}(y_1+y_2+y_3+y_4), \underset{\sim}{\hat{\beta}_2} = \frac{1}{4}(y_1-y_2+y_3-y_4), \underset{\sim}{\hat{\beta}_3} = \frac{1}{4}(y_1+y_2-y_3-y_4)$ and $\underset{\sim}{\hat{\beta}_4} = \frac{1}{4}(y_1-y_2-y_3+y_4)$ respectively.

We can observe that least square estimates by both approaches are same.

3. Consider a model:

$$y_1 = 12\beta_1 + 6\beta_2 + \epsilon_1 \quad ;$$
$$y_2 = 10\beta_1 - 2\beta_2 + \epsilon_2.$$

where, ϵ_i's $(i = 1, 2)$ are uncorrelated random variables with variance 1.

(a) Compute the expression for $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$. [2 Marks]

Solution:

From the above model, we can write $\underset{\sim}{Y}, X$ and $\underset{\sim}{\beta}$ as follows:

$\underset{\sim}{Y} = \begin{pmatrix}y_1\\ y_2\end{pmatrix}$, $X = \begin{pmatrix}12 & 6\\ 10 & -2\end{pmatrix}$ and $\underset{\sim}{\beta} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$.

Now,

$$\begin{aligned}\underset{\sim}{Y} - X\underset{\sim}{\beta} &= \begin{pmatrix}y_1\\ y_2\end{pmatrix} - \begin{pmatrix}12 & 6\\ 10 & -2\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\\ &= \begin{pmatrix}y_1\\ y_2\end{pmatrix} - \begin{pmatrix}12\beta_1 + 6\beta_2\\ 10\beta_1 - 2\beta_2\end{pmatrix}\\ &= \begin{pmatrix}y_1 - 12\beta_1 - 6\beta_2\\ y_2 - 10\beta_1 + 2\beta_2\end{pmatrix}\end{aligned}$$

As we know that if $z \in \mathbb{R}^n$, then $\|z\|_2 = \sqrt{z^T z} = \sqrt{\sum_{i=1}^n z_i^2}$. Thus,

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(\underset{\sim}{Y} - X\underset{\sim}{\beta})^T(\underset{\sim}{Y} - X\underset{\sim}{\beta})}$$

$$= \sqrt{(y_1 - 12\beta_1 - 6\beta_2, y_2 - 10\beta_1 + 2\beta_2)\begin{pmatrix} y_1 - 12\beta_1 - 6\beta_2 \\ y_2 - 10\beta_1 + 2\beta_2 \end{pmatrix}}$$

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$$

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(b) Let $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix}$, then for which of the following value of $\underset{\sim}{\beta}$ will the value of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$ is minimum? [2 Marks]

(i). $\underset{\sim}{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(ii). $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(iii). $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(iv). $\underset{\sim}{\beta} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Solution: We have $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$ from the above part.

Given, $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Now, we will substitute the each values of $\underset{\sim}{beta}$ and $\underset{\sim}{Y}$ in the expression of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$ and observe the values of $\underset{\sim}{\beta}$ for which $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$ will be minimum.

(I): Substitute the value of $\underset{\sim}{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in the expression of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$, we get

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$$

$$= \sqrt{(48 - 6)^2 + (12 + 2)^2}$$
$$= \sqrt{(42)^2 + (14)^2}$$
$$= \sqrt{1764 + 196}$$
$$= \sqrt{1960}$$
$$= 44.27$$

(II): Substitute the value of $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in the expression of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$, we get

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$$

$$= \sqrt{(48-12)^2 + (12-10)^2}$$
$$= \sqrt{(36)^2 + (2)^2}$$
$$= \sqrt{1296 + 4}$$
$$= \sqrt{1300}$$
$$= 36.06$$

<u>(III):</u> Substitute the value of $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in the expression of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$, we get

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$$

$$= \sqrt{(48-12-6)^2 + (12-10+2)^2}$$
$$= \sqrt{(30)^2 + (4)^2}$$
$$= \sqrt{900 + 16}$$
$$= \sqrt{916}$$
$$= 30.27$$

<u>(IV):</u> Substitute the value of $\underset{\sim}{\beta} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in the expression of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$, we get

$$\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2 = \sqrt{(y_1 - 12\beta_1 - 6\beta_2)^2 + (y_2 - 10\beta_1 + 2\beta_2)^2}$$

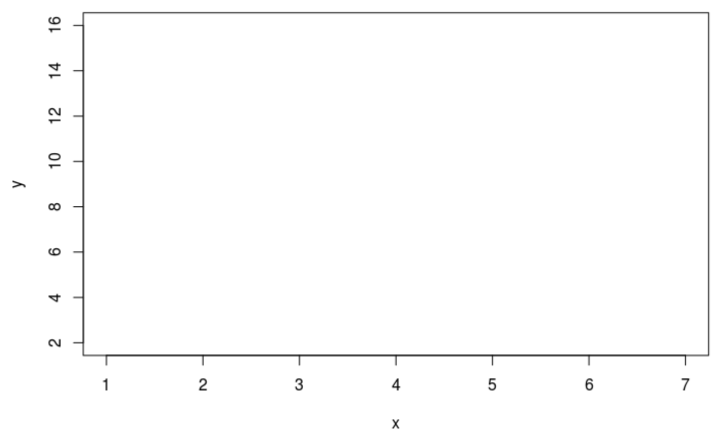
$$= \sqrt{(48-6-3)^2 + (12-5+1)^2}$$
$$= \sqrt{(39)^2 + (8)^2}$$
$$= \sqrt{1521 + 64}$$
$$= \sqrt{1585}$$
$$= 39.81$$

It can be clearly observe that for the value of $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\underset{\sim}{Y} = \begin{pmatrix} 48 \\ 12 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ the value of $\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|_2$ is minimum. Hence, $\underset{\sim}{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the least square estimate for the given model.
Thus, option(c) is correct.

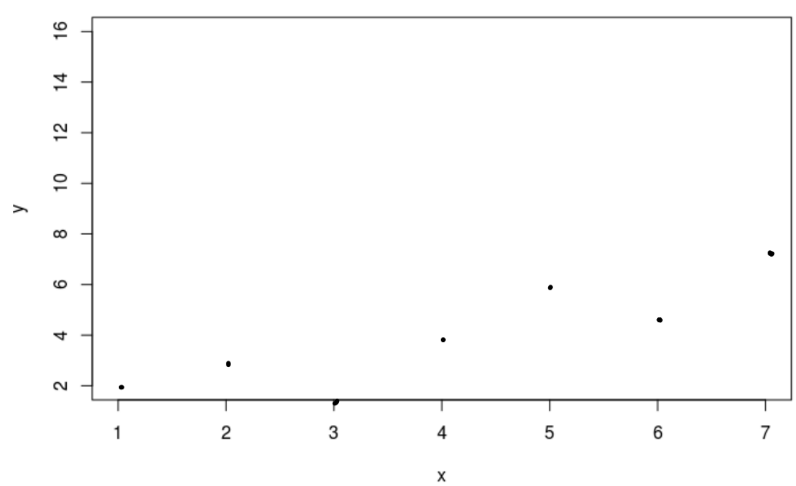
4. Consider the following data.

$\underset{\sim}{x}$	$\underset{\sim}{y}$
1	2
2	3
3	1
4	4
5	6
6	5
7	7

(a) Make a scatter plot of $(\underset{\sim}{x}, \underset{\sim}{y})$ in the graph below: [2 Marks]



Solution: Scatter plot for the given data is:



(b) Suppose we assume the linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, I_{7\times 7})$ for the data obtained from the scatter plot. Find the least square estimate for $\underset{\sim}{\beta}$. [4 Marks]

Solution:

From the data obtained from scatter plot of above part, we have

$$\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 6 \\ 5 \\ 7 \end{pmatrix}, \underset{\sim}{X} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \\ 1 & x_6 \\ 1 & x_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix} \text{ and } \underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

As we know for a linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ least square estimates satisfies the normal equation which is

$$(X^T X)\underset{\sim}{\hat{\beta}} = X^T \underset{\sim}{Y}$$

So,

$$\underset{\sim}{\hat{\beta}} = (X^T X)^{-1} X^T \underset{\sim}{Y} \quad \ldots (***)$$

Now, $$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 7 & 28 \\ 28 & 140 \end{pmatrix}.$$

So, $$(X^T X)^{-1} = \frac{1}{196} \begin{pmatrix} 140 & -28 \\ -28 & 7 \end{pmatrix}$$

$$X^T \underset{\sim}{Y} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 6 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 28 \\ 136 \end{pmatrix}$$

Now, substituting the values of $(X^T X)^{-1}$ and $X^T Y$ in the equation (***), we get

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{196} \begin{pmatrix} 140 & -28 \\ -28 & 7 \end{pmatrix} \begin{pmatrix} 28 \\ 136 \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{196} \begin{pmatrix} 112 \\ 168 \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \frac{112}{196} \\ \frac{168}{196} \end{pmatrix}$$

$$\implies \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} 0.571 \\ 0.857 \end{pmatrix}$$

Which is the least square estimate for $\underset{\sim}{\beta}$.

5. Consider a 2×2 matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find Column space (A), Row space (A), Null space (A) and Column space (A^T). [4 Marks]

Solution:

We are given a 2×2 matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

I: Row space (A) can be computed as follows:
Given matrix is:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

$R_2 \rightarrow \frac{-1}{3} R_2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

As both rows are independent, therefore

$$R(A) = span \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

II: Column space (A) is defined as:

$$C(A) = \{ b_{2 \times 1} \mid \exists x \in \mathbb{R}^2 : Ax = b \}$$

Now,

$$C(A) = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\}$$

Now, $b \in C(A) \Leftrightarrow$ the system $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ has a solution. Let's perform elementary row operations on matrix A:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 - 2b_2 \end{pmatrix}$$

$R_2 \rightarrow \frac{-1}{3} R_2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ \frac{-1}{3}(b_1 - 2b_2) \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1 + 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 - 2b_2 \end{pmatrix}$$

Now, we get $x_2 = \frac{-1}{3}(b_1 - 2b_2), x_1 = b_2 - 2x_2$. Thus, this system of equation has a solution.

Also, the pivot columns are non-zero and thus columns are linearly independent. Therefore,

$$C(A) = span \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

<u>III:</u> Since, $A^T = A$. Therefore, Column space(A^T) = Column space (A).

<u>IV:</u> Null space (A) is defined as:

$$N(A) = \{x \in \mathbb{R}^2 : Ax = 0\}$$

Let's perform elementary row operations on matrix A:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1 + 2x_2 \\ -3x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, we get $x_1 + 2x_2 = 0$ and $-3x_2 = 0$. On solving, we get $x_1 = 0, x_2 = 0$. Hence, Null space (A) consists of the zero vector alone.