Linear Statistical Models

Week-5: Graded Assignment

1. Consider a linear model as :

$$y_{ij} = \mu + \mu_i + \epsilon_{ij} \quad ; \quad 1 \leq i \leq k \ , \ 1 \leq j \leq n_i$$

Where, $\mu, \mu_i \in \mathbb{R} \ ; \ 1 \leq i \leq k$ and $\epsilon_{ij} \sim$Normal$(0, \sigma^2)$ and independent for $1 \leq i \leq k \ , \ 1 \leq j \leq n_i$. If population is divided in i group with size of n_i in each group such that $n = n_1 + n_2 + \ldots + n_k$ is total sample size, then

(a) Define sum of squares within group (SSW) and sum of squares between group (SSB) and prove that SSW is independent of SSB.

(b) If the given model is represented as $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$, then prove that rank$(X) = k$.

2. Suppose $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$ be a linear model, where

$$\underset{\sim}{Y} = (y_1, y_2, y_3, \ldots, y_n)^T,$$

$$\underset{\sim}{\beta} = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T,$$

$$X = ((x_{ij})) \ ; 1 \leq i \leq n, 1 \leq j \leq m$$

and

$$\underset{\sim}{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n)^T$$

where, $\underset{\sim}{\epsilon}$ has mean $\underset{\sim}{0}$ and variance covariance matrix $\sigma^2 I_{n\times n}$.

Suppose for a $\underset{\sim}{l} \in \mathbb{R}^n, \underset{\sim}{l}^T \underset{\sim}{y}$ is an unbiased estimate of $\underset{\sim}{p}^T \underset{\sim}{\beta}$ show that:

$$X^T \underset{\sim}{l} = \underset{\sim}{p}$$

. [5 Marks]

3. Consider the following models

(i) One way classification model as

$$y_{ij} = \mu + \mu_i + \epsilon_{ij} \quad ; \quad 1 \leq i \leq k \ , \ 1 \leq j \leq n_i$$

(ii) Nested classification model as

$$y_{ijk} = \mu + \mu_i + \delta_{ij} + \epsilon_{ijk} \quad ; \quad 1 \leq k \leq n_{ij}, \ 1 \leq j \leq m_i, 1 \leq i \leq k$$

(iii) Two way classification model as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \quad ; \quad 1 \le k \le n_{ij},\ 1 \le j \le J, 1 \le i \le I$$

Express the models (i), (ii) and (iii) as $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I_{n\times n})$.

4. (a) Consider a simple linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ given by

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i; \quad 1 \le i \le n.$$

For the given model, find $X^T X$ and $X^T \underset{\sim}{Y}$.

(b) If the values of least square estimates $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ and $\hat{\beta}_1 = \dfrac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$, then find $Var(\hat{\beta}_0)$

5. Show that $\underset{\sim}{p}^T \underset{\sim}{\beta}$ is estimable, (where, $\underset{\sim}{p}^T \underset{\sim}{\beta} = p_1\beta_1 + p_2\beta_2 + \ldots + p_m\beta_m\ \forall\ p \in \mathbb{R}^m$) under the linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ if and only if p is orthogonal to the Null space of X.

6. Consider the linear model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ in which

$$X = \begin{pmatrix} 1 & 1 & k \\ 1 & 2 & 2k \\ \vdots & \vdots & \vdots \\ 1 & k & k^2 \\ 1 & 1 & l \\ 1 & 2 & 2l \\ \vdots & \vdots & \vdots \\ 1 & l & l^2 \end{pmatrix}, \qquad \underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

where k, l are integers bigger than or equal to 2.

(a) Determine which of the following linear functions of $\underset{\sim}{\beta}$ are always estimable, with no further assumptions on k, l.

i. β_1

ii. $\beta_1 + \beta_2 k + \beta_3 s$

iii. $\beta_2 + \beta_3 \frac{(l+k)}{2}$

(b) Determine necessary and sufficient conditions on k and l for all linear functions of the elements of $\underset{\sim}{\beta}$ to be estimable.