Due Date: March 3rd, 2022
Problems due: None[1]

1. Let $n \geq 1$ and let $X_1, X_2, \ldots, X_n$ be a i.i.d random variabels. Let the X's be arranged in increasing order of magnitude denoted by
$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$
These ordered values are called the order statistics of the sample $X_1, X_2, \ldots, X_n$. For, $1 \leq r \leq n$, $X_{(r)}$ is called the r-th order statistic. Find the probability density function of, $X_{(r)}$, the r-th order statistic for $1 \leq r \leq n$ when
   (a) X_1 is distributed as Uniform $(0, 1)$
   (b) X_1 is distributed as Exponential (λ) for some $\lambda > 0$.

2. Suppose $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from a Normal mean 0 and variance 1 population and $Y_1, Y_2, \ldots, Y_m$ be an independent i.i.d. random sample from a Normal mean 0 and variance 1 population. Let
$$V = \sum_{i=1}^{m} Y_i^2 \quad \text{and} \quad U = \sum_{i=1}^{n} X_i^2$$
   (a) (*Chi-Square with n degrees of freedom*) Show that U has χ^2_n distribution, That is the probability density function of U is given by:
$$f(u) = \frac{2^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} = \begin{cases} \frac{2^{-\frac{n}{2}}}{(\frac{n}{2}-1)!} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} & \text{when } n \text{ is even,} \\ \frac{2^{n-\frac{n}{2}-1}(\frac{n-1}{2})!}{(n-1)!\sqrt{\pi}} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} & \text{when } n \text{ is odd,} \end{cases}$$
   for $u \in \mathbb{R}$.
   (b) (*F-distribution $F(n, m)$*) Let
$$Z = \frac{U/n}{V/m}.$$
   Show that Z has $F(n, m)$ distribution, that is its probability density function for $z > 0$ is given by
$$f(z) = \left(\frac{m}{n}\right)^{\frac{n}{2}} \frac{z^{\frac{n}{2}-1}}{(1 + \frac{n}{m}z)^{\frac{n+m}{2}}} \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}.$$
   (c) (*t_m-distribution*) Let $n = 1$,
$$W = \sqrt{Z} = \frac{X_1}{\sqrt{\frac{V}{m}}}.$$
   Show that W has the t_m distribution. That is, its probability density function for $w \in \mathbb{R}$ is given by
$$f_W(w) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\Gamma(\frac{m}{2})} \left(1 + \frac{w^2}{m}\right)^{-\frac{m+1}{2}}$$

[1] *Quiz 6 will be in assignment format and please solve all problems before hand for timely submission.*

3. Suppose Ruhi has a coin that Ruhi claim is "fair" (equally likely to come up heads or tails) and that her friend claims is weighted towards heads. Suppose Ruhi flip the coin twenty times and find that it comes up heads on sixteen of those twenty flips. While this seems to favor her friend's hypothesis, it is still possible that Ruhi is correct about the coin and that just by chance the coin happened to come up heads more often than tails on this series of flips. Let S be the sample space of all possible sequences of flips. The size of S is then 2^{20}, and if Ruhi is correct about the coin being "fair", each of these outcomes are equally likely.

   (a) Let E be the event that exactly sixteen of the flips come up heads. What is the size of E? What is the probability of E?

   (b) Let F be the event that at least sixteen of the flips come up heads. What is the size of F? What is the probability of F?

4. Two types of coin are produced at a factory: a fair coin and a biased one that comes up heads 55% of the time. We have one of these coins but do not know whether it is a fair or biased coin. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin comes up heads 525 or more times we shall conclude that it is a biased coin. Otherwise, we shall conclude that it is fair. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

5. *Extra Credit*[2]*.:* We will revisit the birthday problem. That is, we have N people in a room. We further assume that all years have 365 days and that birthrates are constant throughout[3]. Using `ggplot` plot the P(at least two people in the room who share a birthday) as a function N, with N varying from 0 to 100. From the plot deduce the following:

   (a) What is the value of N above which there is a 60% chance that two of the N people will have a common birthday ?

   (b) If $N = 20$ people in a room, from the plot, infer what is chance that two of the 20 people will have a common birthday ?

   In the same room of N people, what are the chances that 3 people share the same birthday ? Can you a write a `R`-code to implement the same for k people ?

[2]This is part of the bannana muffin challenge series. The question is not part of the course syllabus. If I get a complete solution to this question [From two students] along with H.W and it is written up properly, along with a clear report then I will bake bannana muffins for the class when you all return later this semester or early next semester.

[3]i.e. it is assumed that for each person all 365 possible birthdays are equally likely.