Z-test :- Test for sample mean when variance is known.

$X \sim N(\mu, \sigma^2)$ → known

- Sample $x_1, x_2, \dots, x_n$ from X

(I) $H_0 : \mu = c$ $\quad H_A \; \mu > c$

- Fix $\alpha \in (0,1)$
- Compute. $\mathbb{P}\left(Z \geq \frac{\sqrt{n}\,(\bar{x} - c)}{\sigma} \right) \equiv$ p-value

- If p-value $< \alpha$ then we reject the null hypothesis. otherwise we conclude that there is no evidence to reject the null hypothesis.

(II) $H_0 : \mu = c$ $\quad H_A \; \mu < c$

- Fix $\alpha \in (0,1)$
- Compute. $\mathbb{P}\left(Z \leq \frac{\sqrt{n}\,(\bar{x} - c)}{\sigma} \right) \equiv$ p-value

- If p-value $< \alpha$ then we reject the null hypothesis. otherwise we conclude that there is no evidence to reject the null hypothesis

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(III) $H_0 : \mu = c$ $\quad H_A \; \mu \neq c$

- Fix $\alpha \in (0,1)$
- Compute. $\mathbb{P}\left(|Z| \geq \left| \frac{\sqrt{n}\,(\bar{x} - c)}{\sigma} \right| \right) \equiv$ p-value

- If p-value $< \alpha$ then we reject the null hypothesis. otherwise we conclude that there is no evidence to reject the null hypothesis.

t-test : Test for sample mean when variance is unknown.

Assume $X \sim \text{Normal}(\mu, \sigma^2)$ & both μ and σ are unknown.

let $X_1, X_2, \dots, X_n$ be i.i.d. $\text{Normal}(\mu, \sigma^2)$

$H_0 : \mu = c$ $\qquad$ $H_A : \mu < c$

let $Y_1, Y_2, \dots Y_n$ be random variables that "mimic" the sampling procedure. $\boxed{Y \sim \text{Normal}(c, s^2)}$

Under H_0: i.e. assume $\mu = c$

$$\sqrt{n}\left(\frac{\bar{Y} - c}{S}\right) \sim t_{n-1} \quad \text{———} \circledast$$

$$\left[S^2 := \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \right]$$

Q:- $\mathbb{P}(\bar{Y} < \bar{X}) = ?$

$$= \mathbb{P}\left(\frac{\sqrt{n}(\bar{Y} - c)}{S} < \frac{\sqrt{n}(\bar{X} - c)}{S} \right)$$

$$= \mathbb{P}\left(T < \frac{\sqrt{n}(\bar{X} - c)}{S} \right)$$

where $T \sim t_{n-1}$

Fix $\alpha \in (0,1)$. If $\mathbb{P}\left(T < \frac{\sqrt{n}(\bar{X} - c)}{S}\right) < \alpha$ then reject H_0

Ex:- Prescribe the t-test when

- $H_0: \mu = c$ $\quad$ $H_A: \mu > c$
- $H_0: \mu = c$ $\quad$ $H_A: \mu \neq c$

General Approach :-

Assumption :-

X – has p.d.f / p.m.f. $f(\cdot \mid p)$

$$p \in \mathcal{P} \subseteq \mathbb{R}^d.$$

Sample: $X_1, X_2, \ldots, X_n$ i.i.d. X

Likelihood given sample $X_1, X_2, \ldots, X_n$ is

$$L(p\,;\, X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} f(X_i \mid p)$$

Recall: MLE $\hat{p} = \underset{p \in \mathcal{P}}{\operatorname{argmax}}\, L(p\,;\, X_1, X_2, \ldots, X_n)$

View Hypothesis Test :- as restriction of $\mathcal{P}$ to a smaller subset $\mathcal{P}_0$.

For example : $\mathcal{P}_0 = \{c\}$ in the "intuitive approach" above.

$H_0: p \in \mathcal{P}_0$ $\quad$ $H_A: p \notin \mathcal{P}_0$

$(\mu = c)$ $\quad$ $(\mu \neq c)$

MLE approach under null hypothesis $p \in \mathcal{P}_0 \subsetneq \mathcal{P}$

$$\hat{p}_0 = \underset{p \in \mathcal{P}_0}{\operatorname{argmax}} \; L(p ; X_1, X_2, \ldots, X_n)$$

Likelihood Ratio : Given a sample $X_1, X_2, \ldots, X_n$

$$\lambda(X_1, X_2, \ldots X_n) = \frac{L(\hat{p}_0, X_1, X_2, \ldots, X_n)}{L(\hat{p}, X_1, X_2, \ldots, X_n)}$$

as the likelihood ratio and

$$\Lambda(X_1, X_2, \ldots, X_n) = -\log \lambda(X_1, X_2, \ldots, X_n)$$

$$= -\log \frac{L(\hat{p}_0, X_1, X_2, \ldots, X_n)}{L(\hat{p}, X_1, X_2, \ldots, X_n)}$$

Intuition :-

$$\mathcal{P}_0 \subseteq \mathcal{P} \overset{\text{Ex.}}{\Longrightarrow} 0 \leq \frac{L(\hat{p}_0, X_1, X_2, \ldots, X_n)}{L(\hat{p}, X_1, X_2, \ldots, X_n)} \leq 1$$

$$\Rightarrow \quad 0 \leq \lambda(X_1, X_2, \ldots X_n) \leq 1$$

$$\Rightarrow \quad 0 \leq \Lambda(x_1, x_2, \dots, x_n)$$

$$= -\log \frac{L(\hat{p}_0, x_1, x_2, \dots, x_n)}{L(\hat{p}, x_1, x_2, \dots, x_n)}$$

$$= \log \frac{L(\hat{p}, x_1, x_2, \dots, x_n)}{L(\hat{p}_0, x_1, x_2, \dots, x_n)}$$

$\hat{p}$ is further away from P_0 in terms of L then less likely is P_0 is true as the null hypothesis. [I.e for larger values of Λ]

<u>Z-test</u> :- $X \sim \text{Normal}(\mu, \underbrace{\sigma^2}_{\text{known.}})$

$$\mu \in P = \mathbb{R}$$

$H_0: \mu = c \qquad H_A: \mu \neq c$

i.e. $P_0 = \{c\}$. Given sample $x_1, \dots, x_n$:

$$L(\mu; x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

<u>Ex</u>:- $$\hat{\mu} = \underset{\mu \in \mathbb{R}}{\arg\max}\, L(\mu; x_1, \dots, x_n)$$

$$= \overline{X}$$

$$\hat{\mu}_0 = \underset{\mu \in P_0}{\arg\max}\, L(\mu; X_1, \dots, X_n) = c$$

$$\Lambda(X_1, X_2, \dots, X_n) = \log \frac{L(\hat{\mu}, X_1, X_2, \dots, X_n)}{L(\hat{\mu}_0, X_1, X_2, \dots, X_n)}$$

$$= \log \frac{L(\bar{X}, X_1, X_2, \dots, X_n)}{L(c, X_1, X_2, \dots, X_n)} =$$

$$= \log \left[\frac{\prod_{i=1}^{n} \frac{e^{-\left(\frac{X_i - \bar{X}}{2\sigma^2}\right)^2}}{\sqrt{2\pi}\,\sigma}}{\prod_{i=1}^{n} \frac{e^{-\left(\frac{X_i - c}{2\sigma^2}\right)^2}}{\sqrt{2\pi}\,\sigma}} \right]$$

(Ex.) $\frac{1}{2} \frac{n}{\sigma^2} (\bar{X} - c)^2 = \frac{1}{2} \left(\frac{\sqrt{n}(\bar{X} - c)}{\sigma} \right)^2$

let $Y_1, Y_2, \dots, Y_n$ be i.i.d random variables "imitate" sample under H_0. We have to check

$$\underbrace{\mathbb{P}\left(\Lambda(Y_1, Y_2, \dots, Y_d) \geq \underline{\Lambda(X_1, X_2, \dots, X_n)} \right)}_{\text{p-value of the test}}$$

We know: $\Lambda(Y_1, Y_2, \ldots, Y_d) = \frac{1}{2}\left(\frac{\sqrt{n}(\bar{Y}-c)}{\sigma}\right)^2$

$Z \sim N(0,1) \quad := \frac{Z^2}{2}$

$\therefore$ one can compute the p-value

$$= \mathbb{P}\left(Z^2 \geq \left(\frac{\sqrt{n}(\bar{X}-c)}{\sigma}\right)^2 \right)$$

$X \sim \text{Normal}(\mu, \sigma^2)$ $\quad \sigma \equiv$ known

$H_0: \mu \leq c$ $\boxed{\text{vs}}$ $H_A \; \mu > c$ [not interested in a single value of μ] but seeing if mean is larger than c or lower]

Sample $X_1, X_2, \ldots, X_n$ from population:-

Compute: $\Lambda(X_1, X_2, \ldots, X_n) = \log \dfrac{L(\hat{\mu}, X_1, X_2, \ldots, X_n)}{L(\hat{\mu}_0, X_1, X_2, \ldots, X_n)}$

$$\hat{\mu}_0 = \underset{\mu \in P_0}{\arg\max}\, L(\mu; X_1, \ldots, X_n) \qquad P_0 = (-\infty, c]$$

$$\hat{\mu} = \underset{\mu \in P}{\arg\max}\, L(\mu; X_1, \ldots, X_n) \qquad P = \mathbb{R}$$

Ex:-

- $\hat{\mu} = \overline{X}$
- $\hat{\mu}_0 = \underset{\mu \in (-\infty, c]}{\operatorname{argmax}} \text{ "} \prod_{i=1}^{n} \frac{e^{-\frac{(X_i - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \text{"}$

$$= \min(\overline{X}, c)$$

Ex 2:-

$$\Lambda(X_1, X_2, \ldots, X_n) = \log \frac{L(\hat{\mu}, X_1, X_2, \ldots, X_n)}{L(\hat{\mu}_0, X_1, X_2, \ldots, X_n)}$$

$$= \begin{cases} 0 & \text{if } \overline{X} \leq c \\ \dfrac{n(\overline{X} - c)^2}{2\sigma^2} & \text{if } \overline{X} - c \end{cases}$$

Compute:

$$\mathbb{P}\left(\sqrt{n}\left(\frac{\overline{Y} - c}{\sigma}\right) \geq \frac{\sqrt{n}(\overline{X} - c)}{\sigma}\right)$$