ELEMENTARY RIEMANNIAN GEOMETRY

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Abstract. In these lectures, we cover some basic material on Riemannian Geometry.

1. Advanced Calculus

1.1. **Derivatives.** In single variable calculus, one says that a real-valued function $f : (c, d) \to \mathbb{R}$ is differentiable at $a \in (c, d)$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it does, one calls this limit L the derivative of f at a, and denotes it by $L = \frac{df}{dx}(a)$ or $f'(a)$.

In order to generalise this to a real-valued function of several variables, i.e. a function (or map) $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is an open set, one cannot blindly carry over the above one-variable definition, because one would have to divide the scalar $f(a + h) - f(a)$ by the vector $h \in \mathbb{R}^n$. But one may recast the one-variable definition above in the following form:

$$\lim_{h \to 0} \left\| \frac{f(a+h) - f(a)}{h} - L \right\| = 0$$

which is the same as saying:

$$\lim_{\|h\| \to 0} \frac{\|f(a+h) - f(a) - Lh\|}{\|h\|} = 0$$

This last fomulation easily generalises to functions of several variables. One merely has to note that the scalar L must now be replaced by some map that can be applied to the vector $h \in \mathbb{R}^n$ to yield a scalar. Also, since multiplication by a scalar is a linear map from $\mathbb{R}$ to $\mathbb{R}$, it is reasonable to require $L : \mathbb{R}^n \to \mathbb{R}$ to be a linear map. This motivates the following:

Definition 1.1.1 (Differentiability)**.** Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^m$ be a map. One says that f is *differentiable at* $a \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$$\lim_{\|h\| \to 0} \frac{\|f(a+h) - f(a) - L.h\|}{\|h\|} = 0$$

The linear map L is called the *derivative of* f at a and denoted by $L = Df(a)$. If f is differentiable at all points $a \in U$, we say f is *differentiable on* U.

Remark 1.1.2. Another reformulation of the definition above is: there exists a linear map L such that for h small enough,:

$$f(a + h) = f(a) + L.h + g(h)$$

where $g(h) = o(\|h\|)$, viz.

$$\lim_{\|h\| \to 0} \frac{g(h)}{\|h\|} = 0$$

This formulation says that there is a linear approximation L to the map f, in the sense that the discrepancy between $f(a + h) - f(a)$ and $L.h$ is of "order strictly higher than 1" in $\|h\|$, for small h.

Exercise 1.1.3.

(i): (Derivative is well-defined) Show that if a linear map L exists, in accordance with the Definition 1.1.1 above, then it is unique.

(ii): Show that f is differentiable at a implies that f is continuous at a.

(iii): (The Chain Rule) If $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^l$ where $U \subset \mathbb{R}^n$ is open, $V \subset \mathbb{R}^m$ is open and contains $f(U)$, f is differentiable at $a \in U$, and g is differentiable at $f(a) \in V$, then the composite $g \circ f : U \to \mathbb{R}^l$ is differentiable at a, and has the derivative $Dg(f(a)) \circ Df(a)$. (Hint: Use the Remark 3.3.12).

(iv): Show that $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$ iff each of the component functions f_i $(1 \leq i \leq m)$ of f is differentiable at a, and that

$$Df(a) = \begin{pmatrix} Df_1(a) \\ Df_2(a) \\ . \\ . \\ . \\ Df_m(a) \end{pmatrix}$$

(v): If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then its derivative is L at all points of $\mathbb{R}^n$ (the best linear approximation of a linear map is itself). More generally, for an *affine map* $f : \mathbb{R}^n \to \mathbb{R}^m$, (that is, $f(x) = Lx + b$, where $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, and $b \in \mathbb{R}^m$ is some fixed vector) the derivative $Df(a) = L$ for all $a \in \mathbb{R}^n$.

(vi): Show that if $f : U \to \mathbb{R}^m$ and $g : U \to \mathbb{R}^m$ are differentiable at $a \in U$, $U \subset \mathbb{R}^n$ an open set, then so is $\alpha f + \beta g$ for fixed $\alpha, \beta \in \mathbb{R}$. When $m = 1$, then show that the map fg (defined by $fg(x) = f(x)g(x)$) is also differentiable at a. If $f(x) \neq 0$ for all $x \in U$, show that $x \mapsto (f(x))^{-1}$ is also differentiable at a. Thus any polynomial function on $\mathbb{R}^n$ is differentiable at all points in $\mathbb{R}^n$, and a rational function (i.e. $f = P/Q$, where P, Q are two polynomial functions) is differentiable at all points of the open set $\mathbb{R}^n \setminus Z(Q)$, where $Z(Q)$ is the closed zero-set $Q^{-1}(0)$ of Q.

(vii): One can identify the vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices with the euclidean space $\mathbb{R}^{n^2}$. Let $GL(n, \mathbb{R}) \subset M(n, \mathbb{R})$ be the open set of all real *nonsingular* matrices. Show that the bijective map of $GL(n, \mathbb{R})$ to itself defined by $A \mapsto A^{-1}$ is differentiable at all points in $GL(n, \mathbb{R})$. (Use Cramer's formula for the inverse of a matrix).

1.2. **Directional and partial derivatives.**

Definition 1.2.1 (Directional and partial derivatives)**.** Let $U \subset \mathbb{R}^n$, and $f : U \to \mathbb{R}^m$ be differentiable at a point $a \in U$. Let $v \in \mathbb{R}^n$ be some vector. The *directional derivative* of f along v is the quantity:

$$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}$$

This quantity exists, and is easily shown to be $Df(a)v$ (Exercise).

Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. The directional derivatives along these directions e_i are called the *partial derivatives* of f. If we write $f = (f_1, .., f_m)$ in terms of its components, and denote:

$$\frac{\partial f_i}{\partial x_j}(a) := Df_i(a)(e_j)$$

then with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the linear map $Df(a)$ can be represented as the matrix:

$$Df(a) = \begin{pmatrix} Df_1(a) \\ Df_2(a) \\ . \\ . \\ . \\ Df_m(a) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & ... & \frac{\partial f_1}{\partial x_n}(a) \\ . & . & ... & . \\ . & . & ... & . \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & ... & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

The $m \times n$ matrix on the right is called the *Jacobian matrix* of f. Note that a matrix representation for $Df(a)$ can be done with respect to *any* bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, but the Jacobian matrix is the representation of $Df(a)$ with respect to the standard bases.

Remark 1.2.2 (Caution)**.** Note that for a function f as above, differentiable or not, it is possible to define the partial derivative:

$$\frac{\partial f}{\partial x_j}(a) := \lim_{t \to 0} \frac{f(a + te_j) - f(a)}{t}$$

if it exists. Further, it is possible for all the partial derivatives $\frac{\partial f}{\partial x_j}(a)$ of a function f (indeed, even all directional derivatives) to exist at a point a without the function f being differentiable at a. (See Exercise 1.3.3 below).

1.3. **Higher derivatives, smooth functions and maps.** We note that the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is itself a linear vector space, of dimension mn. If we denote this vector space by $\hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, then by choice of bases $\{e_i\}$ and $\{f_j\}$ of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can identify this vector space with the vector space of $m \times n$ real matrices, which is isomorphic to $\mathbb{R}^{mn}$.

Definition 1.3.1 (C^r-maps)**.** Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be a map. If f is differentiable on U and the map:

$$\begin{aligned} Df : U &\to \hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \\ a &\mapsto Df(a) \end{aligned}$$

is continuous, then we say f is C^1. More generally, we inductively define f to be C^r if the map Df above is C^{r-1}. If a function is C^r for all $r \geq 1$, then we say f is C^∞ or *smooth*. By convention, a C^0 map means a continuous map.

Exercise 1.3.2. Show that f is C^r implies that all its mixed partial derivatives $D^\alpha f_i := \frac{\partial^{i_1} \partial^{i_2} .. \partial^{i_n}}{\partial x_1^{i_1} \partial x_2^{i_2} .. \partial x_n^{i_n}} f_i$ exist and are continuous for all $1 \leq i \leq m$ and all multi-indices $\alpha := (i_1, .., i_n)$ with $|\alpha| := \sum_j i_j \leq r$. Conversely, show that if all these mixed partials exist and are continuous everywhere, then f is C^r.

Exercise 1.3.3. Consider the function

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \frac{x \mid y \mid}{(x^2 + y^2)^{1/2}} \qquad (x, y) \neq (0, 0) \\ (0, 0) &\mapsto 0 \end{aligned}$$

Show that the function above is not differentiable at $(0, 0)$, even though both partial derivatives exist at the origin. In fact, show that the restriction of f to every line passing through the origin is smooth, and hence all directional derivatives of f exist at $(0, 0)$.

1.4. **Diffeomorphisms.**

Definition 1.4.1 (Diffeomorphism)**.** Let $1 \leq r \leq \infty$. A C^r map $f : U \to V$, where U and V are open subsets of some euclidean spaces, is called a *C^r-diffeomorphism* if there exists a C^r map $g : V \to U$ which satisfies $g \circ f = \mathrm{Id}_U, \quad f \circ g = \mathrm{Id}_V$. A C^∞-diffeomorphism is called a *smooth diffeomorphism*. For $r = 0$, the definition also makes sense, though it is more customary to call a C^0-diffeomorphism a *homeomorphism*.

Remark 1.4.2. The Jacobian of a C^r-diffeomorphism $f : U \to V$ is pointwise invertible, as a linear map, for $r \geq 1$. For, by the chain rule of Exercise 1.1.3 (iii), it follows that the linear maps $Df(a)$ and $Dg(f(a))$ are inverses of each other for each $a \in U$. Thus U and V have to be subsets of euclidean spaces of the *same dimension* if they are C^r-diffeomorphic, for $r \geq 1$. (For $r = 0$, the result is still true and is much harder to prove. It is called the *Brouwer Domain Invariance Theorem*, and requires the use of homology theory. See e.g. E. H. Spanier's book *Algebraic Topology* for a proof).

Exercise 1.4.3. Let f be a C^r-map which is a C^1- diffeomorphism, for $r \geq 1$. Then f is a C^r-diffeomorphism (Use (vii) of the Exercise 1.1.3). Show by an example that this conclusion is false if C^1 above is replaced by C^0. That is, give an example of a C^1-map which is a homeomorphism but not a C^1-diffeomorphism.

Definition 1.4.4 (Local diffeomorphism)**.** Let $f : U \to \mathbb{R}^n$ be a C^r map, where U is an open set in $\mathbb{R}^n$. We say that f is a *C^r-local diffeomorphism at $a \in U$* if there is a neighbourhood W of a such that $f_{|W} : W \to f(W)$ is a C^r- diffeomorphism. Note that a map which is a C^r-local diffeomorphism at every point of U will also have invertible Jacobian at every point of U, for $r \geq 1$.

Example 1.4.5 (A local diffeomorphism which is not a diffeomorphism)**.** Consider the map:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (e^x \cos y, e^x \sin y) \end{aligned}$$

This is nothing but the map $z \mapsto e^z$ from $\mathbb{C}$ to $\mathbb{C}$, written out in long-hand. It is clearly not a diffeomorphism because the points $(0, 2n\pi)$, $n \in \mathbb{Z}$ all map to $(1, 0)$. However, it is not difficult to show that f restricted to any open strip $\mathbb{R} \times (b, b + 2\pi)$ is a diffeomorphism. f maps this open strip diffeomorphically to the half-slit plane, i.e. $\mathbb{R}^2$ minus the half-ray $\{(\lambda \cos b, \lambda \sin b) : \lambda \geq 0\}$.

Example 1.4.6 (Polar Coordinates)**.** There is also the (related) smooth map which is a local diffeomorphism:

$$\begin{aligned} f : (0, \infty) \times \mathbb{R} &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r \cos \theta, r \sin \theta) \end{aligned}$$

Over any open strip $(0, \infty) \times (\alpha, \alpha + 2\pi)$, it is a diffeomorphism on to a half slit plane. Note that the smooth diffeomorphism $t \to e^t$ taking $\mathbb{R}$ to $(0, \infty)$ converts this example to the previous one.

Exercise 1.4.7. Prove that a smooth bijection $f : U \to V$ of open subsets of $\mathbb{R}^n$ is a smooth diffeomorphism iff it is a C^1-local diffeomorphism at each point of U.

1.5. **The Inverse Function Theorem.** As we noted above, if a map $f : U \to \mathbb{R}^n$ is a local diffeomorphism at the point $a \in U$, then its derivative $Df(a)$ at a is an invertible linear map. There is a fundamental result which states that the converse is also true. That is:

Theorem 1.5.1 (Inverse Function Theorem)**.** Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ be a C^r map with $r \geq 1$. If the derivative $Df(a)$ is invertible for $a \in U$, then f is a C^r-local diffeomorphism at a.

In order to prove this theorem, we need a lemma:

Lemma 1.5.2 (Banach's Contraction Mapping Principle)**.** Let B be a complete metric space with a metric d. Let $T : B \to B$ be a *contraction mapping*. That is, there is a constant $0 \leq C < 1$ such that:

$$d(Tx, Ty) \leq Cd(x, y) \quad \text{for all } \ x, y \in B$$

Then there is a unique fixed point $b \in B$ for T, viz. $T(b) = b$.

Proof of the Lemma: Just take an arbitrary point $a \in B$, and consider the sequence $x_n := T^n(a)$. Note that by the hypothesis on T we have

$$d(x_{n+2}, x_{n+1}) \leq Cd(x_{n+1}, x_n) \leq ...C^{n+1}d(x_1, x_0)$$

which easily implies that $\{x_n\}$ is a Cauchy sequence, since $C < 1$. Because B is complete, $\{x_n\}$ converges to some limit $b \in B$. Also, T being a contraction map, is continuous, so:

$$Tb = T(\lim_n x_n) = \lim_n Tx_n = \lim_n x_{n+1} = b$$

proving that b is a fixed point. Its uniqueness is clear from the contraction property of T. □

Proof of the Inverse Function Theorem:

Before we get into the proof, we make a simple remark. If f is an *affine* map, viz., a map of the form:

$$f(z) = Az + b$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear, and $b \in \mathbb{R}^n$ is some fixed vector, then we would have the formula:

$$f(w) - f(z) = Aw - Az = Df(z)(w - z)$$

so that if $A = Df(z)$ were invertible, and we wanted the inverse image w of some $y = f(w)$, we would have the solution as:

$$w = z - Df(z)^{-1}(f(z) - y) \qquad\qquad (*)$$

Now let us get back to a general map f as in the hypothesis. We can simplify the setting somewhat. By translating in the domain and image, and composing with the linear map $Df(a)^{-1}$, we can assume, without loss of generality that $a = 0$, $f(a) = 0$, and $Df(a) = I$, the identity map of $\mathbb{R}^n$. Let $B = \overline{B(0, \delta)}$ be the closed ball of radius δ around the origin so that $B \subset U$. We will suitably choose δ in the sequel.

For $y \in \mathbb{R}^n$, define the map:

$$\begin{aligned} T_y : B &\to \mathbb{R}^n \\ z &\mapsto z - (f(z) - y) \end{aligned}$$

This map T_y is motivated by the solution in $(*)$, in the sense that it would yield the inverse image of y in case the map f were affine. Note that a fixed point z for this map T_y would determine z as the inverse image of y. We proceed to analyse T_y so as to be able to apply the Banach contraction mapping principle of Lemma 1.5.2 above.

Now, since f is C^r with $r \geq 1$, the map $z \mapsto \|I - Df(z)\|_{op}$ (the operator norm[1] of $(I - Df(z))$) is a continuous function of z. Thus by choosing δ suitably small, one can guarantee that:

$$\|I - Df(z)\|_{op} < \frac{1}{8} \quad \text{for} \quad z \in B \tag{1}$$

[1]The operator norm of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ is defined as $\|A\|_{op} := \sup\{\|Ax\| : \|x\| = 1\}$

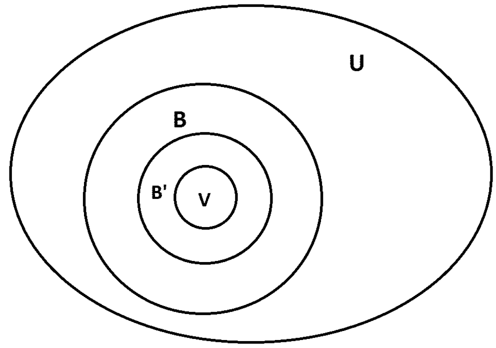

FIGURE 1. Inverse Function Theorem

We also choose δ small enough so that $Df(z)$ is invertible all over $B = \overline{B(0,\delta)}$. This is possible since $Df(0) = I$ is invertible and $z \mapsto \det Df(z)$ is a continuous function.

Now, we would like to make T_y a contraction map. Note that for $z, w \in U$, $z - w \in U_1 = \bigcup_{w\in U}(U - w)$ where $U - w$ is the $(-w)$ translate of U. Clearly U_1 is open.

Define the map:

$$\begin{aligned} g : U \times U_1 &\rightarrow \mathbb{R}^n \\ (w,h) &\mapsto g(w,h) := f(w+h) - f(w) - Df(w)h \end{aligned}$$

Since f is C^1, Df is continuous, and hence g is continuous (jointly in the variables w and h). The definition of differentiability of f also implies that the function ϕ, defined on $U \times U_1$ by $\phi(w,h) := \frac{g(w,h)}{\|h\|}$ for $h \neq 0$ and $\phi(w,0) \equiv 0$, is continuous. In particular, the subset $U_2 \subset U \times U_1$ defined by:

$$U_2 := \left\{(w,h) \in U \times U_1 : \ \|\phi(w,h)\| < \frac{1}{8}\right\}$$

is an open set. Since $(0,0) \in U_2$, and since U_2 is open, we can choose δ small enough so that $\overline{B(0,\delta/2)} \times \overline{B(0,\delta)} \subset U_2$. If $z, w \in B' = \overline{B(0,\delta/2)}$, then $z - w \in B = \overline{B(0,\delta)}$, so that we have, for δ as above:

$$\|g(w, z-w)\| \leq \frac{1}{8}\|z - w\| \quad \text{for all } \ z, w \in B' = \overline{B(0,\delta/2)} \tag{2}$$

where we have put the equality sign to cover the situation $z = w$.

Hence, for δ as above,

$$\begin{aligned} \|T_y(z) - T_y(w)\| &= \|z - w - (f(z) - f(w))\| \\ &\leq \|(I - Df(w))(z-w) - g(w, z-w)\| \\ &\leq \|I - Df(w)\|_{op}\|z - w\| + \|g(w, z-w)\| \\ &\leq \frac{1}{8}\|z-w\| + \frac{1}{8}\|z - w\| \\ &\leq \frac{1}{4}\|z - w\| \quad \text{for all } \ z, w \in B' = \overline{B(0,\delta/2)} \end{aligned} \tag{3}$$

by using the equations (1) and (2).

Note that setting $w = 0$ in the above inequality and letting $z \in B'$, implies that

$$\|T_y(z) - T_y(0)\| = \|T_y z - y\| \leq \frac{1}{4}\|z\| \leq \frac{\delta}{8}$$

Thus if we let $y \in B(0,\delta/4)$, we have $\|T_y(z)\| \leq \delta/4 + \delta/8 < \delta/2$. Thus for this choice of δ, and $y \in B(0,\delta/4)$, the map T_y maps $B' = \overline{B(0,\delta/2)}$ to itself, and is a contraction map on B'.

So there exists a point $b \in B'$ such that $b = T_y(b) = b - (f(b) - y)$. That is $f(b) = y$. Since this fixed point is unique in B', b is the unique inverse image of y in B'. We let $V = B(0,\delta/4)$ and $W = f^{-1}(V) \cap B(0,\delta/2)$. By the foregoing, $f : W \to V$ is a bijection. We need to show that the inverse map $f^{-1} : V \to W$ is smooth. Let y and $y' = y + k \in V$, and $z = f^{-1}(y), w = f^{-1}(y') = z + h \in W$. The equation:

$$f(z) - f(w) = z - w - (T_y(z) - T_y(w))$$

implies that

$$\|z - w\| - \|T_y(z) - T_y(w)\| \leq \|f(z) - f(w)\| \leq \|z - w\| + \|T_y(z) - T_y(w)\|$$

which, in view of (3) above, implies that for $z, w \in B'$

$$\frac{3}{4}\|z - w\| \leq \|f(z) - f(w)\| \leq \frac{5}{4}\|z - w\| \tag{4}$$

for $z, w \in B'$. From (4) it follows that we can assume that h is very small if k is very small. (Note, as an aside, that the injectivity of f over B', as also the continuity of f^{-1} over V, also follow from (4)). The differentiability of f at z implies that:

$$f(z + h) - f(z) = k = Df(z)h + g(h)$$

with $g(h) = o(h)$. This equation can be rewritten, upon applying $(Df(z))^{-1}$ all over, as:

$$Df(z)^{-1}(y + k - y) = Df(z)^{-1}(k) = h - \widetilde{g}(k) = f^{-1}(y + k) - f^{-1}(y) - \widetilde{g}(k)$$

where $\widetilde{g}(k) := -Df(z)^{-1}g(h)$. This implies

$$f^{-1}(y + k) - f^{-1}(y) = Df(z)^{-1}(k) + \widetilde{g}(k)$$

It is clear that $\widetilde{g}(k)$ is $o(\|k\|)$ in the light of the the inequality (4) above, and the fact that $Df(z)^{-1}$ exists and has operator norm bounded above for all $z \in B'$. This implies that f^{-1} is differentiable at $y \in V$, with derivative $Df(z)^{-1}$ where $z = f^{-1}(y)$. The continuity of $D(f^{-1})$ follows from the continuity of Df, the continuity of f^{-1} and Cramer's formula. Thus f is a C^1-diffeomorphism. From the Exercise 1.4.3 it is a local C^r-diffeomorphism, and the theorem is proved. $\square$

Remark 1.5.3.

(i): Note that the inverse function theorem concludes the *local* behaviour of a function (i.e. all over a neighbourhood of a point a) from information at a *point* about its derivative $Df(a)$.

(ii): As the Examples 1.4.5 and 1.4.6 show, it is possible for a map f to be a local diffeomorphism at each point, without being a global diffeomorphism. On the other hand, the reader can easily check that a map $f : \mathbb{R} \to \mathbb{R}$ which is a C^r-local diffeomorphism at each point (for $r \geq 1$) is a global C^r-diffeomorphism.

Exercise 1.5.4. For each $r \geq 1$, give an example of a C^r-diffeomorphism which is not a C^{r+1}-diffeomorphism.

Exercise 1.5.5. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ has non-vanishing derivative $Df(0) = \frac{1}{2}$ at $x = 0$. However, f is not a diffeomorphism in any neighbourhood of $x = 0$. (In fact, f is not even injective in any neighbourhood of 0!). Hence the hypothesis of f being at least C^1 in the inverse function theorem cannot be relaxed. (*Hint:* Examine sign changes of Df)

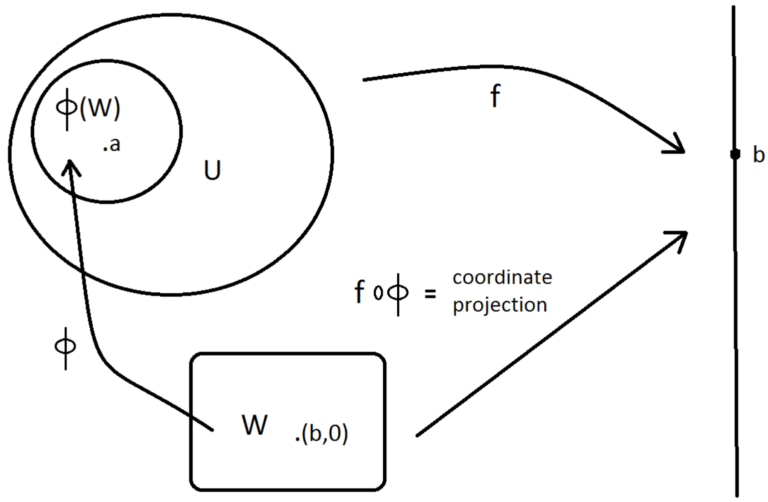

FIGURE 2. The Implicit Function Theorem (Submersive form)

1.6. **The Implicit Function Theorems.** There are two more important theorems that deduce the local behaviour of a map from information about its derivative at a point. Both follow from the inverse function theorem. First some definitions. As always, U is an open subset of $\mathbb{R}^n$.

Definition 1.6.1 (Submersion)**.** Let $r \geq 1$. A C^r-map $f : U \to \mathbb{R}^m$ is called a C^r*-submersion at* $a \in U$ if its derivative $Df(a)$ is surjective. This means that $n \geq m$, and the Jacobian matrix of f at a is of rank m.

Definition 1.6.2 (Immersion)**.** Let $r \geq 1$. A C^r-map $f : U \to \mathbb{R}^m$ is called a C^r*-immersion at* $a \in U$ if its derivative $Df(a)$ is injective. This means that $n \leq m$, and the Jacobian matrix of f at a is of rank n.

Example 1.6.3. An obvious example of a submersion is a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ which is surjective. By a change of basis in the *domain* $\mathbb{R}^n$, one can view this as the projection map onto the first m coordinates. Similarly, for $n \leq m$, the inclusion of a linear m-dimensional subspace $V \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is an immersion. Again, by a change of basis in the *range* $\mathbb{R}^n$, this map can be viewed as the map keeping the first n coordinates as they are, and inserting zeros in the last $m - n$ coordinates. Note that the change of basis is always happening in the *larger* dimensional space.

The two implicit function theorems in the sequel say that, *locally*, a submersion (resp. immersion) is equivalent to the two prototype models of submersions (resp. immersions) of the above example. More precisely:

Theorem 1.6.4 (Implicit Function Theorem, submersive form)**.** Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^m$ be a C^r-map with $r \geq 1$. Assume that f is a submersion at a, and $f(a) = b \in \mathbb{R}^m$, say. Then there exists a C^r-local diffeomorphism $\phi : W \to \phi(W)$, with W a neighbourhood of $(b_1, .., b_m, 0, .., 0) = (b, 0)$ in $\mathbb{R}^n$, $\phi(W)$ a neighbourhood of a contained in U, and $\phi(b, 0) = a$, satisfying:

$$f \circ \phi(x_1, ..., x_n) = (x_1, ..., x_m) \quad \text{for} \quad (x_1, .., x_n) \in W$$

Proof: By rearranging the coordinates in the domain $\mathbb{R}^n$ (noting that this can be absorbed into ϕ without changing the statement), one can assume that the $m \times m$-submatrix:

$$A = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{1\leq i,j\leq m}$$

is of rank m. That is, it is invertible. Now, consider the map:

$$\begin{array}{rcl} F: U & \to & \mathbb{R}^n \\ (x_1,..,x_n) & \mapsto & (f_1(x_1,..,x_n),...,f_m(x_1,..,x_n),x_{m+1}-a_{m+1},..,x_n-a_n) \end{array}$$

Clearly, $c := F(a) = (b, 0)$, and the Jacobian of F at a is:

$$DF(a) = \begin{pmatrix} A & * \\ 0 & I_{n-m} \end{pmatrix}$$

where I_{n-m} is the identity matrix of size $n-m$. Since A is invertible, $DF(a)$ is also invertible. Thus, by the Inverse Function Theorem 1.5.1, there is a neighbourhood $\phi(W)$ of a and a C^r-diffeomorphism $\phi : W \to \phi(W)$ where W is a neighbourhood of c, $\phi(c) = \phi(b, 0) = a$, and such that the composite:

$$F \circ \phi(x_1,....,x_n) = (x_1,...,x_n) \quad \text{for} \quad (x_1,..,x_n) \in W$$

By reading the first m entries of the left hand side, we obtain:

$$f \circ \phi(x_1,...,x_n) = (x_1,...,x_m) \quad \text{for} \quad (x_1,..,x_n) \in W$$

This proves our assertion. □

It isn't very transparent why the above theorem is called the Implicit Function Theorem. The following corollary will clarify this.

Corollary 1.6.5. Let U and f be as in the statement of the Theorem 1.6.4 above. Let $a \in U$, and $b = f(a)$ as above. Assume that the $m \times m$ submatrix of $Df(a)$ given by:

$$\left[\frac{\partial f_i}{\partial x_j}(a)\right]_{1\leq i,j\leq m}$$

is nonsingular, as above. Then, there exists a neighbourhood V of $(a_{m+1},..,a_n)$ and a C^r-map $\psi : V \to \mathbb{R}^m$ such that:

(i): $\psi(a_{m+1},..,a_n) = (a_1,..,a_m)$.
(ii): $f(\psi(x_{m+1},..,x_n), x_{m+1},..,x_n) = b$ for all $(x_{m+1},..,x_n) \in V$.
(iii): In a small enough neighbourhood N of a, $f^{-1}(b) \cap N$ consists precisely of points of the kind $(\psi(x_{m+1},..,x_n), x_{m+1},..,x_n)$ (viz. the level set $f^{-1}(b)$ is the *graph* of ψ in the neighbourhood N of a).

To summarise, under the jacobian hypothesis above, the *implicit equation* $f(x_1,..,x_n) = b$ can be solved *locally* for the first m variables x_i as C^r functions of the last $n-m$ free variables in the *explicit* form $x_i = \psi_i(x_{m+1},..,x_m)$.)

Proof: From the equation

$$F \circ \phi(x_1,....,x_n) = (x_1,...,x_n) \quad \text{for} \quad (x_1,..,x_n) \in W$$

in the proof of the Theorem 1.6.4 above it follows, upon reading the *last* $n-m$ coordinates and the definition of F, that:

$$\phi_j(x_1,..,x_n) - a_j = x_j \quad \text{for} \quad m+1 \leq j \leq n$$

By plugging in the particular values b_i for the first m variables and $x_j - a_j$ for the last $n-m$ variables, we get:

$$\phi_j(b_1,...b_m, x_{m+1}-a_{m+1},..,x_n-a_n) = x_j \quad \text{for} \quad m+1 \leq j \leq n$$

so that from the conclusion of the theorem above we have:

$$f(\phi_1(b, x_{m+1} - a_{m+1}, .., x_n - a_n), ..\phi_m(b, x_{m+1} - a_{m+1}, .., x_n - a_n), x_{m+1}, .., x_n) = b$$

for $(b, x_{m+1} - a_{m+1}, ..., x_n - a_n) \in W$. If we let V be the neighbourhood of $(a_{m+1}, .., a_n) \in \mathbb{R}^{n-m}$ which is the inverse image of W under the map

$$\theta : (x_{m+1}, .., x_n) \mapsto (b_1, .., b_m, x_{m+1} - a_{m+1}, .., x_n - a_n)$$

and define:

$$\psi(x_{m+1}, .., x_n) := (\phi_1(\theta(x_{m+1}, .., x_n)), .., \phi_m(\theta(x_{m+1}, .., x_n))$$

the conclusions (i) and (ii) follow. The last conclusion (iii) is left as an exercise. □

Example 1.6.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map defined by $f(x, y) = x^2 + y^2 - 1$. Then the Jacobian of f at $(1, 0)$ is the matrix:

$$Df(1, 0) = (2, 0)$$

which is clearly surjective. In fact $\frac{\partial f}{\partial x}(1, 0) \neq 0$ is invertible. By the conclusion of the previous corollary, one should be able to find a solution to the implicit equation $f(x, y) = 0$ with x getting expressed as a function of y, and y in some neighbourhood of 0. Indeed, if we let $V = (-1, 1)$, and $x = \psi(y) = (1 - y^2)^{\frac{1}{2}}$, we have that $\psi(0) = 1$, and the corollary above is verified. Note that the neighbourhood V cannot be enlarged any further for the conclusion to hold.

Exercise 1.6.7. Show that the implicit equation

$$e^x y - \sinh^2 y = 0$$

has a solution $y = f(x)$ in a neighbourhood of 0, for some C^∞ function f satisfying $f(0) = 0$.

Theorem 1.6.8 (Implicit function theorem, immersive form)**.** Let $n \leq m$ and let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be a C^r (with $r \geq 1$) map which is an immersion at $a \in U$, with $f(a) = b$. Then, there exists a neighbourhood W of b in $\mathbb{R}^m$, and a diffeomorphism $\phi : W \to \phi(W)$ such that $\phi(b) = (a, 0)$, and the composite:

$$\phi \circ f(x_1, .., x_n) = (x_1, ..., x_n, 0, .., 0) \quad \text{for all} \quad (x_1, .., x_n) \in V$$

where V is a neighbourhood of a contained in $f^{-1}(W)$.

Proof: As in the proof of the Theorem 1.6.4 above, one can absorb a permutation of coordinates in the range space $\mathbb{R}^m$ into the diffeomorphism ϕ. Thus we may permute the components $(f_1, ..., f_m)$ of f so that the square $n \times n$ submatrix:

$$\left[\frac{\partial f_i}{\partial x_j}(a)\right]_{1 \leq i, j \leq n}$$

is nonsingular. Now we construct the auxiliary map:

$$\begin{aligned} F : U \times \mathbb{R}^{m-n} &\to \mathbb{R}^m \\ (x_1, .., x_m) &\mapsto f(x_1, .., x_n) + (0, 0, .., x_{n+1}, .., x_m) \end{aligned}$$

The jacobian matrix of F is then given by:

$$DF = \begin{pmatrix} \frac{\partial f_i}{\partial x_j} & 0... & 0 \\ * & 1... & 0 \\ * & 0... & 1 \end{pmatrix}$$

where the $n \times n$ square matrix $\left[\frac{\partial f_i}{\partial x_j}\right]$ forms the top left $(n \times n)$ block in the $(m \times m)$ matrix DF above. Clearly this is non-singular at a.

By the Inverse Function Theorem 1.5.1, F is a C^r-local diffeomorphism in a neighbourhood of $(a, 0) \in U \times \mathbb{R}^{m-n}$. Hence there exists a neighbourhood W of $F(a, 0) = f(a) = b$ and a diffeomorphism $\phi : W \to \phi(W)$ such that $\phi \circ F$ is identity on $\phi(W)$. From this it follows that:

$$\phi \circ F(x_1, .., x_m) = (x_1, ..., x_m) \qquad \text{for} \;\; (x_1, .., x_m) \in \phi(W)$$

and noting that $F(x_1,..,x_n,0,...,0) = f(x_1,..,x_n)$, it follows that :

$$\phi \circ f(x_1,..,x_n) = (x_1,..,x_n,0,...,0)$$

for all $(x_1,..,x_n)$ such that $(x_1,..,x_n,0,..,0) \in \phi(W)$. Let V be a neighbourhood of a such that $V \times \{(0,..,0)\} \subset \phi(W)$. Clearly the above relation holds for $(x_1,..,x_n) \in V$. Also $f(V) = F(V \times \{(0,..,0)\}) \subset F\phi(W) = W$, so that $V \subset f^{-1}(W)$. The proposition is proved. □

1.7. **Real analytic functions and mappings.**

Definition 1.7.1. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be a smooth (C^∞) map. We say that f is *real analytic* in a neighbourhood of $a \in U$ if there exists a $R > 0$ such that the infinite series:

$$\sum_{i_1 \geq 0,..,i_n \geq 0} \frac{1}{i_1!,..,i_n!} \left(\frac{\partial^{i_1}\partial^{i_2}..\partial^{i_n} f}{\partial x_1^{i_1}\partial x_2^{i_2}..\partial x_n^{i_n}}(a) \right) (x_1 - a_1)^{i_1}...(x_n - a_n)^{i_n}$$

converges absolutely to $f(x_1,..,x_n)$ for $\|x - a\| < R$. The series above is then called the *Taylor series* of f at a. A function which is real analytic at every point of U is called a *real analytic* or C^ω-function. A map $f : U \to \mathbb{R}^m$ is called a *real analytic mapping* or *C^ω-mapping* if each component function is a real analytic function.

Example 1.7.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$. It is easy to verify that this is a C^∞ function, and that all the derivatives of f at $x = 0$ vanish. Thus f has identically vanishing Taylor series at 0. Since f is not identically zero in any neighbourhood of 0, it cannot be analytic at 0.

There are the obvious analogues of the inverse and implicit function theorems in the real analytic setting. We omit, since we won't be using them. Details can be found in §1.3 of Raghavan Narasimhan's book *Analysis on Real and Complex Manifolds*.

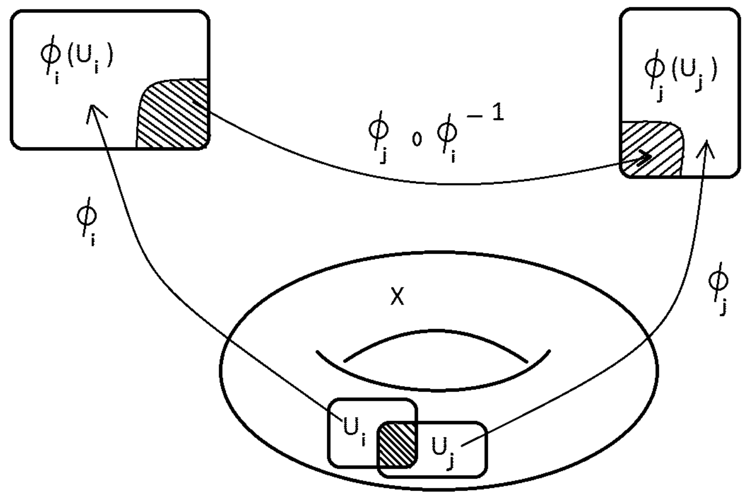

FIGURE 3. Coordinate changes on a differentiable manifold

2. Differentiable Manifolds

From now on, we shall be dealing exclusively with C^∞ or C^ω functions and maps unless otherwise stated.

2.1. **Manifolds.** We assume that the reader is familiar with several variable calculus, and the definition of a differentiable manifold. We briefly recall a few concepts to fix notation and terminology.

Definition 2.1.1. A *smooth* or C^∞ *differentiable manifold* X of dimension n is a paracompact, second countable, Hausdorff topological space with an open covering $\{U_i\}_{i\in\mathbb{N}}$ such that there exist homeomorphisms $\phi_i : U_i \to \phi_i(U_i)$ satisfying:

(i): $\phi_i(U_i)$ are open subsets of $\mathbb{R}^n$ for all $i \in \mathbb{N}$.

(ii): For each $i, j \in \mathbb{N}$, the *coordinate change* map:

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is smooth ($=C^\infty$), as a map between open subsets of $\mathbb{R}^n$.

The pair (ϕ_i, U_i) is called a *coordinate chart*, and the collection $\{(\phi_i, U_i)\}_{i\in\mathbb{N}}$ is called an *atlas*. Similarly, if all the coordinate changes $\phi_j \circ \phi_i^{-1}$ are real analytic, we call it a *real analytic* or C^ω *manifold*. Clearly every real analytic manifold is a smooth manifold.

Remark 2.1.2. Because a manifold is locally homeomorphic to Euclidean space, it is locally metrizable. Because of paracompactness, and the Smirnov Metrization Theorem (see Munkres' *Topology*), it follows that a manifold is metrizable. The second countability then becomes equivalent to having a countable dense subset.

Definition 2.1.3 (Smooth and real analytic structure)**.** We say that two smooth (resp. real analytic) atlases on a smooth (resp. real analytic) manifold X given by $\Phi := \{(\phi_i, U_i)\}$ and $\Psi := \{(\psi_j, V_j)\}$ are *compatible* if for each pair i, j such that $U_i \cap V_j \neq \phi$, we have:

$$\psi_j \circ \phi_i^{-1} : \phi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$$

is a smooth (resp. real analytic) map of open subsets of $\mathbb{R}^n$. One checks easily that compatibility of atlases on a smooth (resp. real analytic) manifold X is an equivalence relation, and an equivalence class of atlases on X is called a *smooth structure* (resp. *real analytic structure*) on X.

For all purposes, one can replace a given atlas with a more convenient compatible atlas without changing anything. For example, given any smooth or real analytic atlas $\Phi := \{(\phi_i, U_i)\}$ of a manifold X and a refinement $\mathcal{V} : \{V_j\}$ of $\mathcal{U}$, we can define a new compatible atlas $\Psi := \{(\psi_j, V_j)\}$ by setting $\psi_j : V_j \to \psi_j(V_j)$ to be the restriction of $\phi_{i(j)}$, where $U_{i(j)}$ is some pre-chosen element of $\mathcal{U} := \{U_i\}$ containing V_j. Again, by the coordinate-change conditions of Φ, this is well-defined independent of the choice of $U_{i(j)}$.

We list some basic examples of differentiable manifolds.

Example 2.1.4 (Open Subsets of Euclidean Space)**.** Clearly, by taking just one chart $U = \mathbb{R}^n$, and $\phi = \mathrm{Id}_{\mathbb{R}^n}$, $\mathbb{R}^n$ becomes a real analytic manifold of dimension n. Similarly, for any open subset $U \subset \mathbb{R}^n$, letting $i : U \to \mathbb{R}^n$ be the inclusion map, $\{(i, U)\}$ is a real analytic atlas for U, making it a real analytic manifold of dimension n. Similarly, any open subset of a smooth (resp. real analytic) manifold of dimension n is again a smooth (resp. real analytic) manifold of dimension n in a natural way, with *induced* smooth (resp. real analytic) structure.

Example 2.1.5 (Manifolds defined by equations)**.** Let $U \subset \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^m$ be a smooth (resp. real analytic) map, where $m \leq n$. Let $b \in \mathbb{R}^m$ be a *regular value* for f, i.e. that the Jacobian matrix $Df(p) := [\frac{\partial f_i}{\partial x_j}]$ is of rank m at *each point* $p \in f^{-1}(a)$. By convention, b is a regular value if $f^{-1}(b) = \phi$. Then the inverse image $X = f^{-1}(b)$, if nonempty, is a smooth (resp. real analytic) manifold of dimension $n - m$. (That is, the m equations $f_i = b_i, \;\; i = 1, .., m$ determine a manifold of dimension $n - m$, provided the Jacobian criterion above is satisfied). This is a consequence of the *implicit function theorem* as follows.

Let $X = f^{-1}(b)$, and let $a = (a_1, .., a_n)$ be any point on X. Let $b = (b_1, .., b_m)$. As noted in that Theorem 1.6.5, there is a neighbourhood U of a and a smooth (resp. real analytic) diffeomorphism $\phi : V \to U$ of a neighbourhood V of $c := (b, 0) = (b_1, .., b_m, 0, .., 0)$ such that $\phi(c) = a$ and

$$f\left(\phi(x_1, ..., x_n)\right) = (x_1, .., x_m)$$

for $(x_1, .., x_n) \in V$. From this it follows that the neighbourhood of a given by $U \cap X$ is precisely the set $\phi\left[V \cap (\{b\} \times \mathbb{R}^{n-m})\right]$. Letting W be the neighbourhood of $0 \in \mathbb{R}^{n-m}$ defined by $\{b\} \times W = V \cap (\{b\} \times \mathbb{R}^{n-m})$, the map $\psi : W \to U \cap X$ defined as $\psi(y) := \phi(b, y)$ gives a chart around a in X. Since a is an arbitrary point of X, this makes X a C^∞ (resp. C^ω) manifold of dimension $n - m$. That the coordinate changes are smooth on X follow from the fact that they are restrictions of composites of smooth (resp. real analytic) maps of open subsets of $\mathbb{R}^n$.

Example 2.1.6 (Spheres)**.** The sphere

$$S^n \subset \mathbb{R}^{n+1} = \{(x_0, .., x_n) \in \mathbb{R}^{n+1} : \; \Sigma_{i=0}^n x_i^2 = 1\}$$

is a real analytic manifold of dimension n, by applying the Example 2.1.5 defined above to the real analytic function $f(x_0, .., x_n) = \Sigma_{i=0}^n x_i^2$ and showing that 1 is a regular value of f.

Example 2.1.7 (Some Linear Groups)**.** The set of $n \times n$ matrices, denoted $M(n, \mathbb{R})$ is just Euclidean space of dimension n^2, and thus a real analytic manifold by 2.1.4 above.

(i): (*The General Linear Groups*) The set $GL(n, \mathbb{R})$ of non-singular $n \times n$ matrices is an open subset of $M(n, \mathbb{R})$ (being the inverse image of the open subset $\mathbb{R} \setminus \{0\}$ of $\mathbb{R}$ under the continuous map det) and thus a real analytic manifold of dimension n^2 by 2.1.4 above. It is a group under matrix multiplication, and is called the *real general linear group*. Similarly the *complex general linear group*, consisting of non-singular $n \times n$ matrices with complex entries, is a real analytic manifold of dimension $2n^2$.

(ii): (*The Special Linear Groups*) The *real special linear group* $SL(n, \mathbb{R})$, and the *complex special linear group* $SL(n, \mathbb{C})$ are real analytic manifolds of dimension $n^2 - 1$ and $2n^2 - 2$ respectively, by applying Example 2.1.5 above to the real analytic determinant map $\det : M(n, \mathbb{R}) \to \mathbb{R}$ and $\det : M(n, \mathbb{C}) \to \mathbb{C}$ respectively. (Use Cramer's rule to expand the determinant in terms of any row, to compute the derivative of det with respect to any entry a_{ij} and check that 1 is a regular value of det.)

(iii): (*The Orthogonal and Special Orthogonal Groups*) The real orthogonal group is defined as the subset:

$$O(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) : \ AA^t = I\}$$

It is easily verified that this is precisely the group of linear transformations of $\mathbb{R}^n$ which leave the standard Euclidean inner product in $\mathbb{R}^n$ invariant, i.e. $A \in O(n,\mathbb{R})$ iff $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all x, $y \in \mathbb{R}^n$. The fact that this is a real analytic manifold of dimension $\frac{1}{2}n(n-1)$ is seen as follows. Let $S(n,\mathbb{R})$ denote the linear subspace of $M(n,\mathbb{R})$ consisting of symmetric $n \times n$ real matrices, which is Euclidean space $\mathbb{R}^{\frac{1}{2}n(n+1)}$. Compute the derivative of the real analytic map:

$$\begin{array}{rcl} \phi : M(n,\mathbb{R}) & \to & S(n,\mathbb{R}) \\ A & \mapsto & AA^t \end{array}$$

at the point $A \in M(n,\mathbb{R})$ to be $D\phi(A)X = AX^t + XA^t$ (by using the linear path $A + sX$ starting at A). Thus, if $A \in O(n,\mathbb{R})$, we have $D\phi(A)(\frac{1}{2}YA) = Y$ for any symmetric matrix $Y \in S(n,\mathbb{R})$, and hence that $D\phi(A)$ is surjective at each point of $O(n,\mathbb{R}) = \phi^{-1}(I)$, and hence that $I \in S(n,\mathbb{R})$ is a regular value of ϕ. Thus by 2.1.5 above, $O(n,\mathbb{R})$ is a real analytic manifold of dimension $\frac{1}{2}n(n-1)$.

This group $O(n,\mathbb{R})$ has two path-connected components, and which component an orthogonal matrix A lies in is determined by whether the determinant $\det A = +1$ or -1. (This is not entirely trivial, and relies on the fact that every orthogonal matrix of determinant $+1$ can be brought, after an orthogonal change of basis, into block diagonal form with $\frac{n}{2}$ (resp. $\frac{n-1}{2}$) 2×2 blocks if n is even (resp. odd, in which case the last diagonal entry is 1). Each 2×2 block is of the form:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $\theta \in \mathbb{R}$, viz. a planar rotation. Using this fact, it is an easy exercise to show that every orthogonal matrix of determinant $+1$ can be joined by a smooth path to the identity matrix I.)

The connected component of $O(n,\mathbb{R})$ which contains all orthogonal matrices of determinant $+1$ is a normal subgroup of $O(n,\mathbb{R})$ called the *special orthogonal group* and denoted $SO(n,\mathbb{R})$. It is a real analytic manifold of dimension $\frac{1}{2}n(n-1)$. For notational convenience, we will always denote $O(n,\mathbb{R})$ (resp. $SO(n,\mathbb{R})$) as $O(n)$ (resp. $SO(n)$). We have the obvious inclusions:

$$\begin{array}{r} SO(n) \subset O(n) \subset GL(n,\mathbb{R}) \\ SO(n) \subset SL(n,\mathbb{R}) \subset GL(n,\mathbb{R}) \end{array}$$

(iv): (*The Unitary and Special Unitary Groups*) For a complex matrix $A \in M(n,\mathbb{C})$, define the adjoint of A to be $A^* = \overline{A}^t$. Define the *unitary group*:

$$U(n) = \{A \in M(n,\mathbb{C}) : \ AA^* = I\}$$

It is easily seen that this is precisely the group of (complex) linear transformations of $\mathbb{C}^n$ which leave the hermitian inner product $\langle x, y\rangle := \Sigma_{i=1}^n \overline{x}_i y_i$ on $\mathbb{C}^n$ invariant. By considering the map $A \mapsto AA^*$ of $M(n,\mathbb{C})$ into the linear subspace $H(n,\mathbb{C}) \subset M(n,\mathbb{C})$ of $n \times n$ complex hermitian matrices, noting that $H(n,\mathbb{C})$ is the Euclidean space $\mathbb{R}^{n^2}$, and applying considerations similar to the ones in (ii) above, one checks that $U(n)$ is a real analytic manifold of dimension n^2.

The group $U(n)$ is path connected for all n (again because a unitary change of basis brings a given unitary matrix into diagonal form, with each diagonal entry in $U(1)$, i.e. the circle group of complex numbers of modulus 1). The kernel of the group homomorphism $\det : U(n) \to U(1)$ is a normal subgroup $SU(n)$, called the *special unitary group*. It is a real analytic manifold of dimension $n^2 - 1$, as is seen from 2.1.5 again. Clearly $U(n)$, $SU(n)$ are subgroups of $SO(2n)$ respectively, after making the obvious identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and noting that the determinant of an $n \times n$ complex matrix considered as a $2n \times 2n$ real matrix is the absolute square of its complex determinant. In fact we have the inclusions:

$$\begin{array}{r} SU(n) \subset U(n) \subset GL(n,\mathbb{C}) \\ SU(n) \subset SL(n,\mathbb{C}) \subset GL(n,\mathbb{C}) \\ SU(n) \subset U(n) \subset SO(2n) \end{array}$$

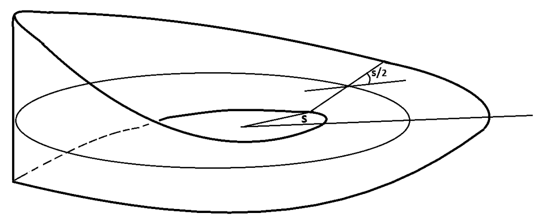

Figure 4. Open Moebius strip

Example 2.1.8 (Products)**.** Let M and N be smooth (resp. real analytic) manifolds of dimensions m and n respectively. Then $M \times N$ is a smooth (resp. real analytic) manifold of dimension $m+n$. For if $\{(\phi_i, U_i)\}_{i\in\mathbb{N}}$ and $\{(\psi_j, V_j)\}_{j\in\mathbb{N}}$ are atlases for M and N respectively, then

$$\{(\phi_i \times \psi_j, U_i \times V_j)\}_{i,j\in\mathbb{N}}$$

is an atlas for $M \times N$.

Example 2.1.9 (n-Torus)**.** The n-fold product of circles $S^1 \times S^1 ... \times S^1$ is an analytic manifold of dimension n, called the *n-torus*. The 2-torus looks like a bicycle tube and is pictured in Fig.3.

Example 2.1.10 (The Open Moebius strip)**.** Let ϕ be the map:

$$\begin{array}{rcl} \phi : \mathbb{R} \times (-1,1) & \to & \mathbb{R}^3 \\ (s,t) & \mapsto & \left(\left(2 + t\cos\frac{s}{2}\right)\cos\, s, \left(2 + t\cos\frac{s}{2}\right)\sin\, s, t\sin\frac{s}{2}\right) \end{array}$$

The image X of ϕ in $\mathbb{R}^3$ is called the *open Moebius strip.* It is pictured in Fig. 4. It is a smooth manifold of dimension 2, because $\mathbb{R} \times (-1,1)$ is a smooth manifold, and ϕ is a local homeomorphism. The reader should verify that the two charts $\phi((0,2\pi)\times(-1,1))$ and $\phi((-\pi,\pi)\times(-1,1))$ constitute an analytic atlas for X by computing the coordinate change.

A manifold of dimension 1 is called a *curve* and a manifold of dimension 2, a *surface.*

Exercise 2.1.11. (i): Prove that the subset of $\mathbb{R}^3$ given by revolving the circle of radius 1 centred at (2,0) in $\mathbb{R}^2$ about the z-axis is the 2-torus $S^1 \times S^1$. Show that the defining equation of this surface is:

$$(x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2) = 0$$

(ii): Using the map:

$$\begin{array}{rcl} (0,2\pi)\times(0,2\pi) & \to & S^1 \times S^1 \\ (s,t) & \mapsto & ((2+\cos\, s)\cos\, t, (2+\cos\, s)\sin\, t, \sin\, s)) \end{array}$$

(and similar ones defined on $(-\pi,\pi)\times(0,2\pi)$ and so on), construct an analytic atlas on the torus $T^2 = S^1 \times S^1$.

2.2. **Projective Spaces and Grassmannians.** We need to briefly recall some facts about the quotient topology. For more details, the reader is urged to consult any book on general topology, such as Munkres' *Topology*.

Let X be a topological space, and $\sim$ be an equivalence relation on X. Thus, one has a natural set of *equivalence classes*, which is denoted by $Y = X/\sim$, and the obvious quotient map $p : X \to Y$ which maps each element $x \in X$ to its equivalence class $[x] = p(x)$. There is a unique natural topology on the set Y which satisfies the following two properties:

(i): The map $p : X \to Y$ is continuous.

(ii): If $f : X \to Z$ is any continuous map which preserves the equivalence relation, (i.e. $x \sim y \;\Rightarrow\; f(x) = f(y)$), there exists a unique continuous map $g : Y \to Z$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

commute.

It is easy enough to define this topology, which is called the *quotient topology*. Just declare a set $U \subset Y$ to be open iff its inverse image $p^{-1}(U)$ is open in X. The fact that this topology satisfies both the properties listed above is an easy exercise. We remark in passing that any surjection p from a topological space X to a set Y defines an equivalence relation on X (i.e. $x \sim y$ iff $p(x) = p(y)$), and under this equivalence relation, the set Y is just the set $X/\sim$. Thus we can again topologize the range set Y with the quotient topology. So when we say $p : X \to Y$ is *"quotient map"*, it just means that p is surjective and continuous, Y has the quotient topology with respect to the equivalence relation defined by p.

Exercise 2.2.1. Show that a continuous surjection which is an open map is a quotient map. Similarly a continuous surjection which is a closed map is also a quotient map. Verify that the map:

$$\begin{aligned} p : [0, \infty) &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

is a quotient map which is neither a closed map nor an open map.

Exercise 2.2.2. Show that a continuous surjection which is a local homeomorphism at each point is a quotient map. Hence the map $t \mapsto e^{2\pi i t}$ or $\mathbb{R}$ to S^1 is a quotient map. Similarly, in the open Moebius strip in Example 2.1.10, the map ϕ is a quotient map.

Example 2.2.3 (Real projective space)**.** Let $X = S^n$, the n-dimensional sphere. Let $\sim$ be the equivalence relation defined by $x \sim y$ iff $x = \pm y$. That is, each point is equivalent to itself, and to its antipodal point. The resulting quotient space is called *real projective space of dimension* n, and denoted $\mathbb{RP}(n)$. Note that this is just the *space of all lines in* $\mathbb{R}^{n+1}$ *passing through the origin*, because each such line is determined by a unit vector on it, and the unit vector in turn is determined only upto a sign.

The quotient map $p : S^n \to \mathbb{RP}(n)$ is an open map, and if we restrict p to any open hemisphere, say of the kind:

$$U_i^+ := \{(x_0, .., x_n) \in S^n : \; x_i > 0\} \quad \text{or} \quad U_i^- := \{(x_0, .., x_n) \in S^n : \; x_i < 0\}$$

with $i = 0, 1, ..n$, then it is a homeomorphism onto its image $V_i = p(U_i^{\pm})$. These open sets then give an atlas on $\mathbb{RP}(n)$. The point $p(x_0, .., x_n)$ is denoted as $[x_0 : .... : x_n]$. This notation means that for $x_i \in \mathbb{R}$ with $\sum_{i=0}^n x_i^2 = 1$, the points: $[x_0 : ... : x_n]$ and $[-x_0 : ... : -x_n]$ are the same point in $\mathbb{RP}(n)$. The x_i are called the *homogeneous coordinates* of the point, and are indeterminate upto a common sign change of all of them.

To see that the atlas defined above is real analytic, note that open hemispheres $U_i^{\pm}$ are homeomorphic to the open n-disc $D^n \subset \mathbb{R}^n$ via the map:

$$(x_0, ..., x_n) \mapsto (x_0, ..., \widehat{x_i}, ..x_n)$$

where the hat denotes omission. Note that for a point $[x_0 : .... : x_n] \in p(U_i^+) = p(U_i^-) = V_i$, the homogeneous coordinate x_i is non-zero. Thus a homeomorphism ϕ_i from V_i to D^n, would be well defined once we scale all the homogeneous coordinates so as to make $x_i > 0$. That is:

$$\begin{aligned} \phi_i : V_i &\to D^n \\ [x_0 : .... : x_n] &\mapsto (\operatorname{sgn}(x_i))(x_0, .., \widehat{x_i}, ., x_n) \end{aligned}$$

On $V_i \cap V_j$, both homogeneous coordinates x_i and x_j are non-zero, and the coordinate change is given by:

$$\begin{aligned} \phi_i\phi_j^{-1} : \phi_j(V_i \cap V_j) &\to \phi_i(V_i \cap V_j) \\ (y_0, .., y_{j-1}, y_{j+1}, .., y_n) &\mapsto (\operatorname{sgn} y_i)(y_0, ..., \widehat{y_i}, .., (1 - \sum y_l^2)^{\frac{1}{2}}, ..y_n) \end{aligned}$$

which is clearly an analytic map since $\operatorname{sgn} y_i$ is analytic on $\phi_j(V_i \cap V_j) = D^n \setminus \{y_i = 0\}$.

Exercise 2.2.4 (Another description of $\mathbb{RP}(n)$)**.** Let $X = \mathbb{R}^{n+1} \setminus \{0\}$, and define an equivalence relation on X by $x \sim y$ iff $x = \lambda y$ for some real number $\lambda \neq 0$. Show that the quotient space $X/\sim$ is $\mathbb{RP}(n)$. Again, the equivalence class of $(x_0, .., x_n) \in X$ is the point in $\mathbb{RP}(n)$ denoted by $[x_0 : .. : x_n]$, with the understanding that these homogeneous coordinates x_i of that point are indeterminate upto a common non-zero scalar multiplier.

Exercise 2.2.5. Show that $\mathbb{RP}(1)$ is homeomorphic to S^1. (Hint: consider the map $z \to z^2$ of the circle to itself).

Example 2.2.6 (Grassmannians)**.** The grassmannian is a generalisation of projective space. We take the set of all k-dimensional $\mathbb{R}$-vector subspaces of $\mathbb{R}^n$, i.e. the k-planes in $\mathbb{R}^n$ that pass through the origin. This set is called the *grassmannian* $G_k(\mathbb{R}^n)$. We make it a real analytic manifold as follows. Equip $\mathbb{R}^n$ with the usual euclidean inner product $\langle\, , \rangle$, and for a fixed k-subspace $E \in G_k(\mathbb{R}^n)$, let us denote:

$$U_E = \{F \in G_k(\mathbb{R}^n) : \ F \cap E^\perp = \{0\}\}$$

We claim that there is a bijection of U_E onto the vector space $\hom_{\mathbb{R}}(E, E^\perp)$. By the definition of U_E, $F \in U_E$ iff the kernel $E^\perp$ of the orthogonal projection $\pi_E : \mathbb{R}^n \to E$ intersects F trivially, so $F \in U_E$ iff $\pi_{E|F} : F \to E$ (the restriction of π_E to F) is an isomorphism. Define:

$$\begin{aligned} \phi_E : U_E &\to \hom_{\mathbb{R}}(E, E^\perp) \\ F &\mapsto \pi_{E^\perp|F} \circ (\pi_{E|F})^{-1} : E \to E^\perp \end{aligned}$$

Its inverse is given by

$$\begin{aligned} \psi_E : \hom_{\mathbb{R}}(E, E^\perp) &\to U_E \\ T &\mapsto \{(x + Tx) \in E \oplus E^\perp = \mathbb{R}^n : \ x \in E\} = \text{graph of } T \end{aligned}$$

It is easily checked that ϕ_E and ψ_E are inverses of each other, and since $E \in U_E$ for each E, the sets U_E constitute a covering of $G_k(\mathbb{R}^n)$. We can put a topology on each U_E by pulling back the natural topology on the $k(n-k)$-dimensional vector space $\hom_{\mathbb{R}}(E, E^\perp)$. That is, $V \subset U_E$ is declared to be open in U_E iff $\phi_E(V)$ is open in $\hom_{\mathbb{R}}(E, E^\perp)$. We need to check that the topologies thus induced on $U_E \cap U_F$ from U_E and U_F match. This will follow from the fact (proved below) that the coordinate changes are analytic diffeomorphisms, and in particular, homeomorphisms.

This will induce the topology on $G_k(\mathbb{R}^n)$ *coherent* with the topologies just obtained on the various U_E's. That is, $V \in G_k(\mathbb{R}^n)$ is declared to be open iff $V \cap U_E$ is open in U_E for each $E \in G_k(\mathbb{R}^n)$. From the analysis below, it will follow that for each E, F, the image $\phi_E(U_E \cap U_F)$ is open in $\hom_{\mathbb{R}}(E, E^\perp)$. In particular, this forces all the $U_E's$ to be open, and $\{(\phi_E, U_E) : \ E \in G_k(\mathbb{R}^n)\}$ then defines a real analytic atlas.

So we proceed to examine the coordinate changes. Let E be the space spanned by the orthonormal basis $\{e_i\}_{i=1}^k$. Complete this to $\{e_i\}_{i=1}^n$, an orthonormal basis of $\mathbb{R}^n$. Let F be another fixed k-subspace in U_E, and let $\{f_i\}_{i=1}^k$ be an orthonormal basis for F. For $T \in \hom_{\mathbb{R}}(E, E^\perp)$, note that a basis for the k-subspace $\psi_E(T)$ is given by $\{e_i + Te_i\}_{i=1}^k$. Such a subspace $\psi_E(T)$ will belong to U_F iff the orthogonal projection π_F from it to

F is an isomorphism. That is, the projections onto F of the basis vectors $(e_i + Te_i)$ of $\psi_E(T)$, span F. That is, the $k \times k$ matrix of inner products:

$$[\alpha(T)_{ij}] := [\langle e_i + Te_i, f_j\rangle]$$

is nonsingular. The determinant of this matrix is a polynomial in the entries of $T \in \hom_{\mathbb{R}}(E, E^\perp)$, so setting it $\neq 0$ gives an open subset of $\phi_E(U_E)$. This shows that $\phi_E(U_E \cap U_F)$ is open in $\phi_E(U_E) = \hom_{\mathbb{R}}(E, E^\perp)$, so that $U_E \cap U_F$ is open in U_E, with respect to the topology just defined on U_E.

For convenience, complete the orthonormal basis $\{f_i\}_{i=1}^k$ of F to an orthonormal basis $\{f_i\}_{i=1}^n$ of $\mathbb{R}^n$. If some k-dimensional subspace P is in $U_E \cap U_F$, then it is $\psi_E(T)$ for some $T \in \hom_{\mathbb{R}}(E, E^\perp)$ and also $\psi_F(S)$ for some $S \in \hom_{\mathbb{R}}(F, F^\perp)$. We need to show that $S = \phi_F \circ \phi_E^{-1}(T)$ is a real-analytic function of T. Since $\{e_i + Te_i\}_{i=1}^k$ and $\{f_j + Sf_j\}_{j=1}^k$ both constitute bases for P, we must have a non-singular $k \times k$ matrix $[\lambda(T)_{ij}]$ satisfying:

$$(f_j + Sf_j) = \sum_{i=1}^k \lambda(T)_{ji}(e_i + Te_i) \quad \text{for } \ j = 1, ..., k \qquad (*)$$

Taking inner products of both sides with f_m, for $1 \leq m \leq k$, and noting that $\langle Sf_j, f_m\rangle = 0$ for $m, j \in \{1, 2, .., k\}$ (since $S \in \hom_{\mathbb{R}}(F, F^\perp)$, and $f_j \in F$), we get:

$$\delta_{jm} = \sum_{i=1}^k \lambda(T)_{ji} \langle e_i + Te_i, f_m\rangle$$

As noted earlier, for $T \in \phi(U_E \cap U_F)$, the matrix $[\alpha(T)_{im}] := [\langle e_i + Te_i, f_m\rangle]$ is nonsingular, so our matrix $[\lambda(T)_{ij}] = [(\alpha(T)^{-1})_{ij}]$. We substitute this in $(*)$ above, and take the inner product of both sides with $f_{k+r},\ r = 1, .., n-k$. This yields the matrix entries of S (with respect to the bases $\{f_j\}_{j=1}^k$ of F and $\{f_{k+r}\}_{r=1}^{n-k}$ of $F^\perp$) as:

$$S_{jr} = \sum_{i=1}^k (\alpha(T)^{-1})_{ji} \langle e_i + Te_i, f_{k+r}\rangle \quad \text{for } \ 1 \leq i \leq k; \ \ 1 \leq r \leq n-k$$

Since $\alpha(T)_{ij}$ is linear in the entries of T, its inverse is real analytic in T wherever $\alpha(T)$ is invertible (i.e. all over $\phi_E(U_E \cap U_F)$). This shows that $S = \phi_F \circ \phi_E^{-1}(T)$ is real analytic in T on $\phi_E(U_E \cap U_F)$.

Consequently, $G_k(\mathbb{R}^n)$ is a real analytic manifold of dimension $k(n-k)$.

Exercise 2.2.7. Show that the map $O(n, \mathbb{R}) \to G_k(\mathbb{R}^n)$ which takes an orthogonal matrix A to the $\mathbb{R}$-span of its first k columns is a continuous surjection, and hence that $G_k(\mathbb{R}^n)$ is compact.

Exercise 2.2.8. Show that the *complex projective space*:

$$\mathbb{CP}(n) := \{L \in \mathbb{C}^{n+1} : \ L \text{ a 1-dimensional } \mathbb{C} - \text{subspace of } \mathbb{C}^{n+1}\}$$

is a real analytic manifold of dimension $2n$. Similarly, show that the *complex grassmannian*:

$$G_k(\mathbb{C}^n) = \{P \in \mathbb{C}^n : \ P \text{ a k-dimensional } \mathbb{C} - \text{subspace of } \mathbb{C}^n\}$$

is a real analytic manifold of dimension $2k(n-k)$. It is a straightforward carrying over to $\mathbb{C}$ of the discussion above. Indeed, one can define a *complex manifold* of (complex dimension n) analogously to a smooth manifold, by requiring the charts to be homeomorphic to open subsets of $\mathbb{C}^n$, and the coordinate changes to be holomorphic. The manifolds above turn out to be complex manifolds of (complex) dimension n and $k(n-k)$ respectively. Since a holomorphic map of one open subset of $\mathbb{C}^n$ to another such open set is a real analytic map by regarding $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ and separating real and imaginary parts in the domain and range, a complex manifold of complex dimension n is a real analytic manifold of dimension $2n$.

We shall later see that grassmannians are examples of so called "*homogeneous spaces*" associated with the orthogonal group $O(n, \mathbb{R})$ and its closed subgroup $O(k, \mathbb{R}) \times O(n-k, \mathbb{R})$, and so natural real analytic atlases will result on them. There is yet a third way of doing it, by embedding $G_k(\mathbb{R}^n)$ as a real analytic submanifold of some high-dimensional projective space.

Exercise 2.2.9. Express the open Moebius strip of Example 2.1.10 as a quotient space of $S^1 \times (-1, -1)$ by a group action of $\mathbb{Z}_2$.

Exercise 2.2.10. Construct an analytic atlas of $S^n \subset \mathbb{R}^{n+1}$ by using stereographic projection from the north and south poles. Stereographic projection from the north pole $e_{n+1} = (0, ..., 1)$, for example, is defined as follows: Take any point $x \in S^n$ which is not the north pole. Connect this point to to the north pole by a straight line, and set $\phi(x)$ to be the unique point where this line meets the hyperplane $(e_{n+1})^{\perp} = \mathbb{R}^n$. Analogously for the south pole. Construct another analytic atlas on it by using spherical polar coordinates and trigonometric functions. Verify that they give S^n the same real analytic structure.

Exercise 2.2.11.

(i): Define two distinct C^∞ structures on $\mathbb{R}$. (*Hint:* Use the map $x \mapsto x^3$).

(ii): Do there exist two distinct analytic structures on $\mathbb{R}$ whose underlying C^∞ structures are the same? (*Hint:* Find a C^∞-diffeomorphism of $\mathbb{R}$ to itself which is not C^ω).

We remark however, in the above exercise, the new C^∞ (resp. C^ω) structure is C^∞ (resp. C^ω) *diffeomorphic* to the standard one, using precisely the maps used above.

3. Smooth mappings and tangent spaces

3.1. Smooth functions and mappings.

Definition 3.1.1 (Smooth and Analytic Mappings)**.** A continuous map $f : M \to N$ between two smooth manifolds is said to be *smooth* if for each chart (ψ, V) of N and each chart (ϕ, U) of M such that $U \cap f^{-1}(V) \neq \phi$, the composite map:

$$\phi(U \cap f^{-1}(V)) \stackrel{\phi^{-1}}{\to} U \cap f^{-1}(V) \stackrel{f}{\to} V \stackrel{\psi}{\to} \psi(V)$$

is a smooth map of open sets of euclidean spaces. That this is well defined follows from the fact that the coordinate changes of M and N are smooth. When $N = \mathbb{R}$, we call f a smooth function. A smooth map between manifolds which has a smooth inverse is called a *smooth diffeomorphism*. A smooth diffeomorphism of a neighbourhood of a point p in a manifold is called a *local diffeomorphism* at p. Clearly, there is a real analytic version of all of these notions.

It is easy to see that composites of smooth maps are again smooth by the Chain Rule. The local charts of a smooth manifold are local diffeomorphisms, by definition. Likewise, for the real analytic setting.

Example 3.1.2. Let $X = f^{-1}(a)$ be the manifold of dimension $n - m$ defined by taking the inverse image of a regular value $a \in \mathbb{R}^m$ under a smooth map $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, as described in the Example 2.1.5. Then the restriction to X of any smooth mapping $g : U \to N$, N any smooth manifold, is again smooth. This is because, by the implicit function theorem, for a $p \in X$, there is a local diffeomorphism ϕ of a neighbourhood V around p which makes the following diagram commute:

$$\begin{array}{ccc} X \cap V & \hookrightarrow & V \\ \phi \downarrow & & \downarrow \phi \\ \phi(X \cap V) & \stackrel{j}{\hookrightarrow} & \phi(V) \end{array} \tag{5}$$

where the map j is the inclusion as the first $n - m$ coordinates, which is clearly smooth. This shows that the inclusion map $i : X \to U$ is a smooth map, and so $g_{|X} = g \circ i$ is smooth. Similarly for the real analytic case.

In fact the above Example 3.1.2 can be generalised to what are called submanifolds. Since many manifolds we meet in future will be submanifolds of $\mathbb{R}^n$ (or submanifolds of open subsets of $\mathbb{R}^n$), it would be handy to conclude smoothness of various maps defined on such submanifolds by realising it as a restriction of a smooth map defined on $\mathbb{R}^n$ (or of an open subset of $\mathbb{R}^n$). We first define a submanifold.

Definition 3.1.3 (Submanifold)**.** Let M be a smooth manifold of dimension n, and $N \subset M$ be a subset. We say that N is a *k-dimensional submanifold* or *k-submanifold of M* if there exists an atlas $\{(\phi_i, U_i)\}_{i \in \Lambda}$ for M (compatible with its given C^∞ structure) and a subset $\Gamma \subset \Lambda$ such that:

(i): $N \subset \cup_{i \in \Gamma} U_i$.

(ii): $\phi_i(N \cap U_i) = (\mathbb{R}^k \times \{0\}) \cap \phi_i(U_i)$ for each $i \in \Gamma$.

There is clearly an analogous definition for the analytic ($=C^\omega$) situation.

The following exercise is an immediate consequence of the definition above:

Exercise 3.1.4.

(i): A k-submanifold N of a smooth manifold M is itself a smooth manifold of dimension k. Further, an equivalent atlas on M which satisfies (i) and (ii) of the definition above gives rise to equivalent atlases on N. Hence, even though a submanifold is defined with respect to one particular atlas, its smooth structure is well-defined.

(ii): A k-submanifold N of a smooth manifold M is locally closed in M. Give an example to show that N need not be a closed subset of M in general.

(iii): The inclusion map $i : N \hookrightarrow M$ is a smooth map. In particular, if $f : M \to Y$ is any smooth map to a smooth manifold Y, then so is $f_{|N} : N \to Y$.

(iv): If $X = f^{-1}(a)$ is the inverse image of a regular value of a smooth map $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ an open set, i.e. the situation of Example 3.1.2 above), then X is a smooth submanifold of U of dimension $n-m$. This follows from the submersive form of the Implicit Function Theorem 1.6.4, as discussed in Example 2.1.5.

(v): Let N be an n-dimensional submanifold of an n-dimensional manifold M. Show that N is an open subset of M.

Remark 3.1.5. We should remark here that some authors define a submanifold to be the image of an injective immersion (recall that an *immersion* is a smooth map whose derivative at each point of its domain is injective). That definition is less restrictive than our definition. For example, the line of irrational slope on the torus defined by:

$$N = \{(e^{it}, e^{i\alpha t}) : t \in \mathbb{R}\} \subset S^1 \times S^1$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, is the image of an injective immersion $t \mapsto (e^{it}, e^{i\alpha t})$ of $\mathbb{R}$ into the torus, but *not* a submanifold as per our definition above. It is not locally closed inside the torus, and hence violates (ii) of Exercise 3.1.4 above.

Exercise 3.1.6.

(i): Show that the open Moebius strip described earlier in Example 2.1.10 is a submanifold of $\mathbb{R}^3$. This is an example of a submanifold which is not a closed subset.

(ii): Is the spiral

$$X = \{(e^t \cos t, e^t \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$$

a submanifold of $\mathbb{R}^2$?

(iii): Show that $\mathbb{RP}(2)$ can be realised as a smooth submanifold of $\mathbb{R}^4$ by using the map:

$$\begin{aligned} f : \mathbb{RP}(2) &\to \mathbb{R}^4 \\ [x_0 : x_1 : x_2] &\mapsto (x_0^2 - x_1^2, x_0x_1, x_0x_2, x_1x_2) \end{aligned}$$

(Here $(x_0, x_1, x_2) \in S^2$ is a unit vector). Is it possible with $\mathbb{R}^4$ replaced by $\mathbb{R}^3$? (Ans: No, but the reason is non-trivial. Or rather, non-triviality of its "normal bundle" (to be defined later). Also the above submanifold $\mathbb{RP}(2)$ of $\mathbb{R}^4$ cannot be realised as the inverse image of a regular value of a smooth map $\mathbb{R}^4 \to \mathbb{R}^2$ for a similar reason.

(iv): Show that the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \mid x \mid$ is a smooth 1-dimensional manifold, but not a smooth submanifold of $\mathbb{R}^2$.

An important example of a real analytic map is:

Example 3.1.7 (The exponential map)**.** Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$, and $M(n, \mathbb{F})$ denote the vector space of $n \times n$ matrices with entries in $\mathbb{F}$, and $GL(n, \mathbb{F})$ the general linear group over $\mathbb{F}$. We define the *exponential map*:

$$\begin{aligned} \exp : M(n, \mathbb{F}) &\to M(n, \mathbb{F}) \\ A &\mapsto \exp A := \sum_{i=0}^{\infty} \frac{A^i}{i!} \end{aligned}$$

That this is a real analytic map on all of $M(n, \mathbb{F})$ is easy to check, from the absolute convergence of the series on the right over all of $M(n, \mathbb{F})$. For an $n \times n$ matrix $A \in M(n, \mathbb{F})$, define $\|A\| = \max_{i,j} |A_{ij}|$. Then it is easy to see that:

$$\|AB\| \leq n \, \|A\| \, \|B\|$$

from which it follows that $\|A^k\| \leq n^{k-1}\|A\|^k$, so that:

$$\left\| \sum_{i=0}^{k} \frac{A^i}{i!} \right\| \leq S_k$$

where S_k is the k-th partial sum of the series $e^{n\|A\|}$. Since $M(n,\mathbb{F})$ is complete with respect to this norm $\| \ \|$ (which is equivalent to the usual Euclidean norm), it follows that the series converges absolutely over all of $M(n,\mathbb{F})$. In particular, exp is a real analytic map.

Exercise 3.1.8 (One parameter subgroups)**.** Show that:

(i):

$$\lim_{t\to 0} \frac{(\exp(tA+tB)\exp(-tA)\exp(-tB) - I)}{t^2} = \frac{1}{2}[A,B]$$

where $[A,B] = AB - BA$ is the *commutator* of B and A.

(ii): $\det(\exp A) = e^{\mathrm{tr}A}$. This shows that $\exp : M(n,\mathbb{F}) \to GL(n,\mathbb{F})$. (*Hint:* It is enough to prove this for $\mathbb{F} = \mathbb{C}$, for which case it is convenient to use the density of diagonalisable matrices in $M(n,\mathbb{C})$.)

(iii): The exponential map is a group homomorphism from the additive group $\mathbb{R}A$ to a subgroup of $GL(n,\mathbb{F})$. This subgroup, often denoted $\exp tA$ is called the *one parameter subgroup* generated by A.

(iv):

$$\lim_{t\to 0} \frac{\exp(tX)A\exp(-tX) - A}{t} = [X,A]$$

Exercise 3.1.9. Let $A \in M(n,\mathbb{R})$, and set $N = \{\exp(tA) : t \in \mathbb{R}\}$ be the one parameter group generated by A. Is $N \subset GL(n,\mathbb{R})$ a submanifold in general ? (*Hint:* Consider the torus $SO(2) \times SO(2)$ as a subgroup of $GL(4,\mathbb{R})$, and use the example of Remark 3.1.5.)

Exercise 3.1.10. Show that the map $\exp : M(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is a local diffeomorphism. Explicitly write down a smooth local inverse to it. There is clearly an analogous result for the complex situation.

Exercise 3.1.11 (Surjectivity of the exponential map for $\mathbb{F} = \mathbb{C}$)**.** Show that :

(a): If B is a diagonalisable (=semisimple) element of $GL(n,\mathbb{C})$, then $B = \exp A$ for some $A \in M(n,\mathbb{C})$.

(b): If N is a nilpotent matrix in $M(n,\mathbb{C})$ (viz. $N^k = 0$ for some k), then $(I+N) = \exp A$ for some $A \in M(n,\mathbb{C})$. (*Hint:* Use the same logarithmic series you used in Exercise 3.1.10 above).

(c): Use (a) and (b) above together with the Jordan Decomposition of a matrix $A \in GL(n,\mathbb{C})$ as $A = S + N$ where S is semisimple, N is nilpotent and $[S,N] = 0$ to show that $\exp : M(n,\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective.

Remark 3.1.12. Note that the exponential map $\exp : M(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is *not surjective*. Since $\det(\exp A) = e^{\mathrm{tr}A}$, it is clear that matrices in $GL(n,\mathbb{R})$ with negative determinant cannot be in the image of exp. It is then natural to ask whether $\exp : M(n,\mathbb{R}) \to GL_+(n,\mathbb{R})$ is surjective, where

$$GL_+(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : \det A > 0\}$$

The answer is still no, for $n \geq 2$, for somewhat more subtle reasons, as outlined in the next exercise.

Exercise 3.1.13 (Non-surjectivity of $\exp : M(2,\mathbb{R}) \to GL_+(2,\mathbb{R})$)**.** Prove the following:

(i): Let $C \in GL_+(2,\mathbb{R})$ with $C = \exp B$ for some $B \in M(2,\mathbb{R})$. Then $C = A^2$ for some $A \in GL_+(2,\mathbb{R})$.

(ii): For $A \in M(2,\mathbb{R})$, we have $\mathrm{tr}(A^2) + 2\det A \geq 0$. (*Hint:* Cayley-Hamilton Theorem).

(iii): If $\det C = 1$ and $C = A^2$ for some A, then $\operatorname{tr} C \geq -2$. Hence find a *diagonal* matrix C of determinant 1 which is not a square, and hence outside the image of exp.

3.2. **Patching and smooth partitions of unity.** On a smooth manifold (though *not* on a real analytic manifold) there is a method of patching up locally defined smooth functions into globally defined ones. For this one needs "smooth partitions of unity". We first need a little lemma, whose proof will be deferred till after the construction of partitions of unity below.

Lemma 3.2.1 (Bump Functions)**.** Let $U \subset \mathbb{R}^n$ be an open set, and let $W \subset U$ be a non-empty open set such that $\overline{W} \subset U$. Then there exists a smooth function $\mu : \mathbb{R}^n \to \mathbb{R}$ such that:

(i): $\mu(x) \geq 0$ for all $x \in \mathbb{R}^n$.
(ii): $\operatorname{supp} \mu \subset U$. In particular $\mu \equiv 0$ outside U.
(iii): $\mu(x) > 0$ on W.

Recall that for a function f, we define the *support of* f as

$$\operatorname{supp} f := \overline{\{y \in M : f(y) \neq 0\}}$$

.

Proposition 3.2.2 (Partitions of Unity)**.** Let M be any smooth manifold, and $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be any open covering of M. Then there exists a family of smooth functions $\{\lambda_i\}_{i\in\Lambda}$ on M satisfying:

(i): $\lambda_i(x) \geq 0$ for all $x \in M$.

(ii): For each $x \in M$, there exists a neighbourhood W of x such that all but finitely many λ_i vanish on W.

(iii): For each $i \in \Lambda$, the *support* of λ_i is contained in U_i.

(iv): For each $x \in M$, $\sum_{i\in\Lambda} \lambda_i(x) \equiv 1$ (The sum is finite in view of (ii)).

Such a collection of functions $\{\lambda_i\}_{i\in\Lambda}$ is called a *partition of unity subordinate to the open covering* $\mathcal{U} = \{U_i\}_{i\in\Lambda}$. These are constructed below.

Proof: First notice that if the open covering $\mathcal{V} = \{V_j\}_{j\in\Gamma}$ is a refinement of the given open covering of $\mathcal{U} = \{U_i\}_{i\in\Lambda}$, (this means that for each $j \in \Gamma$ there exists an $i \in \Lambda$ such that $V_j \subset U_i$), then a partition of unity $\{\mu_j\}$ subordinate to $\mathcal{V}$ will yield a partition of unity $\{\lambda_i\}$ subordinate to $\mathcal{U}$. To see this, first define a map of indexing sets $\theta : \Gamma \to \Lambda$ by *fixing* a $\theta(j) \in \Lambda$ so that $V_j \subset U_{\theta(j)}$, for $j \in \Gamma$. Then define

$$\lambda_i := \sum_{j\in\theta^{-1}(i)} \mu_j$$

By definition, the empty sum is taken to be zero. The sum is always a finite one at each $x \in M$, since by hypothesis (ii), $\mu_j(x)$ is non-zero for at most finitely many j. The assertion (i) is clear for λ_i by the corresponding assertion for μ_j.

Now note that each μ_j will occur as a summand in *exactly one* λ_i, viz. for the unique i satisfying $i = \theta(j)$. Then $\sum_{i\in\Lambda} \lambda_i(x) = \sum_{i\in\Lambda} \sum_{j\in\theta^{-1}(i)} \mu_j(x) = \sum_{j\in\Gamma} \mu_j(x) \equiv 1$ for each $x \in M$, verifying (iv) above for the collection $\{\lambda_i\}$.

Set $A_i := \{x : \lambda_i(x) > 0\}$. By definition of λ_i, $x \in A_i$ implies $\mu_j(x) > 0$ for some $j \in \theta^{-1}(i)$. Thus

$$A_i = \bigcup_{j\in\theta^{-1}(i)} B_j$$

where $B_j := \{x : \mu_j(x) \neq 0\}$. Since $B_j \subset V_j$ for all j, and (ii) applies to μ_j, we have that B_j is a locally finite collection. Thus if $x \in M$, then there is a neighbourhood W of x such that $W \cap B_j = \phi$ for $j \notin F$, where $F \subset \Gamma$ is a finite set. Hence $F_1 := \theta(F)$ is a finite subset of Λ, and for $i \notin F_1$, $\theta^{-1}(i) \cap F = \phi$. Hence $W \cap A_i = \phi$ for $i \in F_1$. Thus A_i is also a locally finite collection. This proves (ii) for the collection λ_i.

Finally, to see (iii) for the collection λ_i, we note that the closure of an arbitrary *locally finite collection* of sets is the union of their closures, and hence

$$\operatorname{supp}\lambda_i = \overline{A_i} = \bigcup_{j\in\theta^{-1}(i)} \overline{B_j} = \bigcup_{j\in\theta^{-1}(i)} \operatorname{supp}\mu_j$$

The right hand side above is contained in $\bigcup_{j\in\theta^{-1}(i)} V_j$ which is contained in U_i. This shows (iii) for the collection $\{\lambda_i\}$, and it follows that this collection is a partition of unity subordinate to $\mathcal{U}$.

So we may, without loss of generality, refine the given covering $\mathcal{U}$ into another one such that all the U_i are coordinate charts, i.e. diffeomorphic to open subsets of $\mathbb{R}^n$ via some diffeomorphisms ϕ_i (e.g. just take the common refinement $\{U_i \cap W_j\}$, where W_j are coordinate charts). Further refinement will preserve this property.

Next, by the paracompactness of M, we may, without loss of generality, further refine this covering $\mathcal{U}$ to be locally finite (i.e., each point $x \in M$ has a neighbourhood W which meets only finitely many U_i non-trivially.) Going to a subcovering will preserve this property. Since subcoverings are refinements, earlier properties will be preserved.

Finally, we may pass to a subcovering which is countable. That is, we may assume that Λ is a countable set, because M is 2nd countable, and every open covering of M has a countable subcovering. Also by going to a subcovering, we may assume that that $\mathcal{U}$ is an irredundant covering: $U_i \not\subset \cup_{j\neq i} U_j$ for each $i \in \Lambda$ (Inductively discard all U_i's which don't satisfy this).

So, without loss of generality, we may assume that $\mathcal{U}$ satisfies:

(a): Λ is countable, say $\Lambda = \mathbb{N}$

(b): $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ is a locally finite covering of M.

(c): For each $i \in \mathbb{N}$, U_i is diffeomorphic to an open set in $\mathbb{R}^n$ via smooth diffeomorphism ϕ_i.

(d): $U_i \not\subset \cup_{j\neq i} U_j$ for each $i \in \mathbb{N}$.

By the condition (d) above it follows that the non-empty closed set

$$C_1 = \left(\cup_{j\neq 1} U_j\right)^c$$

is contained in U_1, and not contained in any other U_j for $j \neq 1$. By the fact that our space M is a metric space, and hence normal, there exists an open set $W_1 \subset U_1$ such that:

$$C_1 \subset W_1 \subset \overline{W}_1 \subset U_1$$

Again W_1 is contained only in U_1 and not contained in any U_j for $j \neq 1$ (by the corresponding fact for C_1). Further, $W_1 \cup (\cup_{j\neq 1} U_i) = M$. Suppose we have inductively defined open sets W_i for $1 \leq i \leq m$ such that:

(W1): $\phi \neq W_i \subset \overline{W}_i \subset U_i$ for $1 \leq i \leq m$.

(W2): $(\cup_{i=1}^m W_i) \cup \left(\cup_{j=m+1}^\infty U_j\right) = M$

(W3): $W_j \subset U_i$ iff $j = i$

We inductively construct W_{m+1} as follows. Consider the closed set

$$C_{m+1} = \left[(\cup_{i=1}^m W_i) \cup \left(\cup_{j=m+2}^\infty U_j\right)\right]^c$$

This set is non-empty because it contains the nonempty set $(\cup_{j\neq m+1} U_j)^c$. Also it is contained in U_{m+1}, and no other U_j. Thus there must be, by normality, an open subset W_{m+1} such that:

$$C_{m+1} \subset W_{m+1} \subset \overline{W}_{m+1} \subset U_{m+1}$$

It is left to the reader to verify that the inductive properties (W1), (W2) and (W3) are satisfied by W_{m+1}.

Now we claim that $\mathcal{W} = \{W_i\}_{i=1}^\infty$ is an open covering of M. This is because $\{U_i\}$ being a locally finite covering, for each $x \in M$, there exists an $m \in \mathbb{N}$ such that $x \notin \cup_{j=m+1}^\infty U_j$, so that (W2) implies that

$x \in \cup_{i=1}^m W_i$. Also (W1) and (W3) imply that each W_i is non-empty and contained in a unique U_j, namely U_i. This implies that $\mathcal{W}$ is also a locally finite open covering of M. Indeed, $\mathcal{W}$ is what is called a *shrinking of* $\mathcal{U}$.

By the Lemma 3.2.1 above, there exists a C^∞ function $\widetilde{\mu}_i$ on $\phi_i(U_i)$ whose support is contained in $\phi_i(U_i)$, and which is everywhere ≥ 0 and strictly positive on $\phi_i(W_i)$. Define the function:

$$\begin{array}{rcl} \mu_i : M & \to & \mathbb{R} \\ x & \mapsto & \widetilde{\mu}_i(\phi_i(x)) \quad \text{if } x \in U_i \\ & \mapsto & 0 \quad \text{if } x \notin U_i \end{array}$$

Clearly, by the support condition on $\widetilde{\mu}_i$, this function μ_i is C^∞ on all of M. Also $\mu_i(x) > 0$ for $x \in W_i$ and $\operatorname{supp} \mu_i \subset U_i$. By the fact that the set $V_i := \mu_i^{-1}((0,\infty))$ contains W_i, and that W_i is contained in a unique U_i, it follows that each V_i is contained in a unique U_i. Further, $\operatorname{supp} \widetilde{\mu}_i \subset \phi_i(U_i)$ implies $\overline{V}_i = \operatorname{supp} \mu_i \subset U_i$, verifying the condition (iii) of the proposition. Note that $\{V_i\}_{i=1}^\infty$ is also an open covering of M, since $\mathcal{W}$ is. From the local finiteness of $\mathcal{U}$, and that each V_j is contained in a unique U_j, it follows that each point $x \in M$ has a neighbourhood W such that all but finitely many μ_i are $\equiv 0$ on W. Finally, since each $x \in M$ is in some W_i, some $\mu_i(x) > 0$.

Thus we may define, for each $x \in M$:

$$\lambda_i(x) = \frac{\mu_i(x)}{\sum_{j=1}^\infty \mu_j(x)}$$

These functions are easily verified to be the required partition of unity subordinate to $\mathcal{U}$ $\square$.

Proof of Lemma 3.2.1: We first note that we may assume that the open set U is a bounded subset of $\mathbb{R}^n$. This is because there is a smooth diffeo $\phi : \mathbb{R}^n \to B(0,1)$, and this diffeomorphism will carry our open set U into the bounded open subset $\phi(U)$ of $B(0,1)$. The open set $\phi(W)$ will have closure $\overline{\phi(W)}$ contained in $\phi(U)$, and the required bump function on U will be obtained by composing the corresponding bump function on $\phi(U)$ with the diffeo ϕ.

So we assume U is a bounded set, and hence has compact closure in $\mathbb{R}^n$. Thus $\overline{W}$ is a compact subset of U. Let $0 < 2\epsilon_1 := d(\overline{W}, U^c)$ (d being euclidean distance), and $V \subset U$ be the subset:

$$V = \{x \in U : d(x, U^c) > \epsilon_1\}$$

which clearly satisfies:

$$W \subset \overline{W} \subset V \subset \overline{V} \subset U$$

Similarly, let

$$U_1 = \{x \in U : d(x, U^c) > \frac{\epsilon_1}{2}\}$$

so that $\overline{V} \subset U_1 \subset \overline{U}_1 \subset U$.

Now consider the continuous function:

$$\begin{array}{rcl} f : \mathbb{R}^n & \to & \mathbb{R}_+ \\ x & \mapsto & \dfrac{d(x, V^c)}{d(x, \overline{W}) + d(x, V^c)} \end{array}$$

This function is identically $= 1$ on W, and vanishes identically outside V. To "smoothen out" this continuous function we need to convolve it with approximate identities. We do this as follows.

Since f is continuous and of compact support (contained in $\overline{U}$), it is uniformly continuous, so there exists a $\epsilon_2 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |f(x-y) - f(x)| < \frac{1}{2} \quad \text{for } \|y\| < \epsilon_2 \qquad (*)$$

Let $\epsilon = \min(\frac{\epsilon_1}{2}, \epsilon_2)$. Consider the non-negative function:

$$\begin{array}{rcl} \phi_\epsilon : \mathbb{R}^n & \to & \mathbb{R}_+ \\ x & \mapsto & A_\epsilon \exp\left(\dfrac{-\epsilon^2}{\epsilon^2 - \|x\|^2}\right) \text{ for } \|x\| < \epsilon \\ & \mapsto & 0 \quad \text{for} \quad \|x\| \geq \epsilon \end{array}$$

where A_ϵ is a normalising constant such that

$$\int_{\mathbb{R}^n} \phi_\epsilon \, dx = 1$$

This function is compactly supported in $B(0,\epsilon)$, and as $\epsilon \to 0$, ϕ_ϵ becomes more concentrated around the origin, and its peak value at the origin gets higher. Such a function is called an "approximate identity", for reasons to become clear soon. Now consider the function (convolution):

$$\mu(x) = (f * \phi_\epsilon)(x) := \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y)dy \qquad x \in \mathbb{R}^n$$

Clearly $\mu(x) \geq 0$ for all $x \in \mathbb{R}^n$. Thus assertion (i) of our lemma follows.

Note that if $x \in U_1^c$, then for $\|y\| \leq \epsilon$ we have the inequality:

$$d(x-y, U^c) \leq d(x-y, x) + d(x, U^c) = \|y\| + d(x, U^c) \leq \epsilon + \frac{\epsilon_1}{2} \leq \epsilon_1$$

so that $x - y$ will lie outside V, so $f(x-y) = 0$. On the other hand, for $\|y\| > \epsilon$, $\phi_\epsilon(y) = 0$. Thus we have $\mu(x) = 0$ for all $x \in U_1^c$. Hence $\operatorname{supp}\mu \subset \overline{U}_1 \subset U$, verifying (ii) of our lemma.

Now we make a sup-norm estimate on μ. Using the fact that $\int \phi_\epsilon(y)dy = 1$, we have for $x \in \mathbb{R}^n$:

$$\begin{array}{rcl} |\mu(x) - f(x)| & = & \left| \displaystyle\int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_\epsilon(y)dy \right| \\ & \leq & \displaystyle\int_{\mathbb{R}^n} |f(x-y) - f(x)|\phi_\epsilon(y)dy \\ & = & \displaystyle\int_{B(0,\epsilon)} |f(x-y) - f(x)|\phi_\epsilon(y)dy \leq \frac{1}{2} \end{array}$$

by using $(*)$. Thus

$$\|\mu - f\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - f(x)| \leq \frac{1}{2}$$

Since $f \equiv 1$ on W, $\mu(x) \geq \frac{1}{2}$ for $x \in W$. Thus (iii) of our lemma is satisfied. All we need to do is check that μ is smooth. This is done by changing variables and rewriting:

$$\mu(x) = (f * \phi_\epsilon)(x) = \int_{\mathbb{R}^n} f(z)\phi_\epsilon(x-z)dz$$

differentiating under the integral sign with respect to x. The smoothness and compact support of ϕ_ϵ is used to justify this (see Rudin's *Principles of Mathematical Analyis*, Theorem 7.16). The details are left to the reader. $\square$

Remark 3.2.3.

(i): Unlike the smooth case, *real analytic bump functions do not exist*, because the zero-set of a real analytic function on $\mathbb{R}^n$ cannot have a non-empty interior, unless it is the zero function. For a real-analytic function of one variable, this is clear, because the zeroes of a non-zero real analytic function are isolated points (why?). In general, we note that if f is a real-analytic function on $\mathbb{R}^n$ vanishing on an entire neighbourhood of a point p, then by considering the restriction of the function to each hyperplane passing through p, we see that each such restriction is $\equiv 0$ (by induction), so $f \equiv 0$.

(ii): Bump functions are also sometimes called "cut-off functions".

Corollary 3.2.4 (Bump Functions on Manifolds)**.** Let M be a smooth manifold, and let W, U be open sets such that $\overline{W} \subset U$. Then there exists a smooth function $\mu : M \to [0,1]$ such that $\operatorname{supp} \mu \subset U$ and $\mu(x) = 1$ for all $x \in W$.

Proof: Using 3.2.2, construct a smooth partition of unity $\{\lambda_1, \lambda_2\}$ subordinate to the open cover $\{U_1, U_2\}$, where $U_1 := U$ and $U_2 := M \setminus \overline{W}$. Then $\mu = \lambda_1$ is the required function. □

Corollary 3.2.5 (Extending smooth functions)**.** Let M be a smooth manifold, and $U \subset M$ an open set. Let W be an open subset of M with $\overline{W} \subset U$. Let $f : U \to \mathbb{R}$ be a smooth function. Then there exists a smooth function $F : M \to \mathbb{R}$ with $F_{|W} = f_{|W}$.

Proof: First define $G : M \to \mathbb{R}$ by $G_{|U} \equiv f$ and G identically zero on U^c. Of course, G is not going to be smooth in general. However, let $V \subset U$ be an open set satisfying $\overline{W} \subset V \subset \overline{V} \subset U$. Let $\mu : M \to \mathbb{R}$ be a cut-off function as in the previous corollary, with $\mu \equiv 1$ on W and $\mu \equiv 0$ on V^c. Then $F := \mu G$ is easily verified to be the required function. Clearly $\mu \equiv 1$ on W implies $F_{|W} = G_{|W} = f_{|W}$. On U, we have $F_{|U} = \mu G = \mu f$ which is smooth on U. On the open set $(\overline{V})^c$, we have $F \equiv 0$, so it is smooth there. Since $M = U \cup (\overline{V})^c$, F is smooth on M. □

It is natural to ask whether a smooth function on a k-submanifold N of an n-manifold M will extend to a smooth function on M. The example below shows that this is false.

Example 3.2.6. Let $M = \mathbb{R}^2$ and $N = (0, \infty) \times \{0\}$, which is a 1-submanifold of M. The function $f(x) = 1/x$ is smooth on N, but does not extend to a smooth function on M, or even to $\mathbb{R} \times \{0\}$ for that matter.

However, the following is true.

Proposition 3.2.7 (Extension of a smooth function on a closed submanifold)**.** Let M be a smooth n-manifold and N be a k-submanifold of M. *Assume that N is closed in M.* Then if f is a smooth function on N, there exists a smooth function F on M such that $F_{|N} = f$.

Proof: We need a fact that follows from the *tubular neighbourhood theorem* to be proved later. This states that there is an open neighbourhood U of N in M and a smooth map $r : U \to N$ such that $r_{|N} = \mathrm{id}_{|N}$. This map r is called a *retraction*, and this fact says that N *is a smooth neighbourhood retract* in M. Indeed, this property of N does not require it to be closed. Thus the function $G := f \circ r : U \to \mathbb{R}$ is an extension of f to the neighbourhood U. Now, since we assume N to be closed, the fact that M is a normal topological space implies that there exists an open set W satisfying $N \subset W \subset \overline{W} \subset U$. The Corollary 3.2.5 above implies the existence of a smooth function $F : M \to \mathbb{R}$ with $F_{|W} = G_{|W}$. Since $N \subset W$, we have $F_{|N} = G_{|N} = f$. □

Exercise 3.2.8. Determine whether the following statement is true or false. Let M be a smooth n-manifold, and $N \subset M$ a closed smooth k- submanifold. If $f : M \to \mathbb{R}$ is a continuous function such that $f_{|N}$ is smooth, then there exists an open set U containing N such that $f_{|U}$ is smooth. (*Hint:* Look at Exercise 1.3.3)

3.3. **Tangent space to a smooth manifold.** Having defined the notion of smooth maps between manifolds, one needs to make sense of the derivative of such a smooth map. One could, of course, use local charts, and define the derivative of a map by composing with these local charts. But it isn't clear that this procedure will lead to something global. We recall that in the case of $\mathbb{R}^n$, we looked at the behaviour of f along all directions (i.e. all the directional derivatives). On a manifold M, we have to define a linear space at each point $x \in M$ which is the totality of all the "directions" in M at that point x. This is called the tangent space at x, and we need some technology to define it.

Definition 3.3.1 (Function Germs)**.** Let M be a smooth manifold, and $x \in M$ a point. We define the *germ of a smooth function at* x to be the equivalence class of a pair (U, f) where U is a neighbourhood of x and f is a smooth function $f : U \to \mathbb{R}$ under the following equivalence relation: $(U, f) \sim (V, g)$ if there exists a neighbourhood W of x with $W \subset U \cap V$ such that $f(y) = g(y)$ for all $y \in W$. The set of all germs at x, with pointwise addition and multiplication defined in the obvious manner, is clearly an $\mathbb{R}$-algebra, and is denoted $\mathcal{O}_{M,x}$, or simply $\mathcal{O}_x$ when M is understood. The constant function a defines a unique germ called the *constant germ.* Strictly speaking, we should write a germ as $[(U, f)]$, but in future, we will often denote a germ just by a letter like f, and suppress the domain of definition of its representative (U, f) and the box brackets, for notational convenience.

Remark 3.3.2 (Smooth germs vs. analytic germs)**.** There is a fundamental difference between smooth and analytic germs. An analytic germ f, say at $0 \in \mathbb{R}^n$, is uniquely determined by all the Taylor coefficients of f at 0. On the other hand, all the Taylor coefficients of the function $f(x) = \exp(-x^{-2})$ vanish at $0 \in \mathbb{R}$, but it does not represent the zero germ in $\mathcal{O}_{\mathbb{R},0}$, for it is not identically 0 on any neighbourhood of 0. Indeed, the set of all smooth germs is very much larger than the set of all real analytic germs.

From now on, we shall be dealing exclusively with the smooth situation. Many of the subsequent statements and propositions are also valid in the real-analytic situation, but we shall leave these matters as an exercise for the reader.

Exercise 3.3.3. Let M be a smooth n-manifold, and $U \subset M$ an open set. As noted in Example 2.1.4, U inherits an n-manifold structure from M. For $x \in U$, show that the natural inclusion map $\mathcal{O}_{U,x} \hookrightarrow \mathcal{O}_{M,x}$ is an isomorphism.

Exercise 3.3.4. For a smooth manifold M, there is the $\mathbb{R}$-algebra of smooth functions on M, denoted $C^\infty(M)$. For a fixed $x \in M$, there is also the natural homomorphism:

$$\begin{array}{rcl} \chi : C^\infty(M) & \to & \mathcal{O}_{M,x} \\ f & \mapsto & [f] \end{array}$$

taking the smooth function f to its germ $[f]$ at x.

(i): Show that χ is a surjective map. (*Hint:* Use the Corollary 3.2.5 above.)

(ii): Show by constructing an example that χ is not surjective in the analogous real-analytic setting.

Exercise 3.3.5. For M a smooth manifold, and $x \in M$, there is a natural *evaluation* homomorphism of $\mathbb{R}$-algebras:

$$\begin{array}{rcl} e_x : \mathcal{O}_{M,x} & \to & \mathbb{R} \\ [f] & \mapsto & f(x) \end{array}$$

where $\mathcal{O}_{M,x}$ is the $\mathbb{R}$-algebra of smooth germs at x. Show that e_x is surjective, and the kernel of this evaluation map, viz. $\mathfrak{m}_x := \ker e_x$ is a maximal ideal. Show that it is the unique maximal ideal, in $\mathcal{O}_{M,x}$. (A ring with a unique maximal ideal, e.g. a ring like $\mathcal{O}_{M,x}$, is called a *local ring*).

Exercise 3.3.6 (Pullback of a germ)**.** Let $f : M \to N$ be a smooth map of smooth manifolds. Show that the map:

$$\begin{aligned} f^* : \mathcal{O}_{N,f(x)} &\to \mathcal{O}_{M,x} \\ [g] &\mapsto [g \circ f] \end{aligned}$$

is an $\mathbb{R}$-algebra homomorphism satisfying $f^*(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$. (viz. it is a *homomorphism of local rings*). For example, the pullback map i^* for the inclusion map $i : U \subset M$ of an open subset U of M is the inverse of the map $\mathcal{O}_{U,x} \to \mathcal{O}_{M,x}$ defined in Exercise 3.3.3.

Remark 3.3.7. The following facts are obvious:

(i): For smooth maps $f : M \to N$ and $g : N \to Y$ of manifolds, $(g \circ f)^* = f^* \circ g^*$.

(ii): If (ϕ, U) is a smooth chart around a point $x \in M$, we can use this chart to make $\mathcal{O}_{M,x}$ isomorphic as an $\mathbb{R}$-algebra to $\mathcal{O}_{\mathbb{R}^n,\phi(x)}$. The pullback homomorphism on germs defined in Exercise 3.3.6 above (combined with the Exercise 3.3.3):

$$\mathcal{O}_{\mathbb{R}^n,\phi(x)} = \mathcal{O}_{\phi(U),\phi(x)} \xrightarrow{\phi^*} \mathcal{O}_{U,x} = \mathcal{O}_{M,x}$$

is the isomorphism, with inverse $(\phi^{-1})^* : \mathcal{O}_{M,x} \to \mathcal{O}_{\mathbb{R}^n,\phi(x)}$.

Hence, we first need to analyse the local rings $\mathcal{O}_{M,x}$ for $M = \mathbb{R}^n$.

Definition 3.3.8 (Directional derivative in $\mathbb{R}^n$)**.** On the local ring $\mathcal{O}_{\mathbb{R}^n,x}$, we define the action of a vector $v \in \mathbb{R}^n$ by means of the *directional derivative along* v. More precisely, let (U, f) represent a germ in $\mathcal{O}_{\mathbb{R}^n,x}$, and define:

$$\partial_v(f) := Df(x)v = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

Then it is readily verified that $\partial_v : \mathcal{O}_{\mathbb{R}^n,0} \to \mathbb{R}$ is well-defined (i.e. independent of the germ representative (U, f) chosen), $\mathbb{R}$-linear, and that it acts on the product of germs via the *Leibniz formula*:

$$\partial_v(fg) = g(0)\partial_v(f) + f(0)\partial_v(g)$$

For example, we have the derivations given by coordinate partial derivatives. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. Define the derivations $\frac{\partial}{\partial x_i}{}_{|x}$ on $\mathcal{O}_{\mathbb{R}^n,x}$, called the *coordinate partials* by the formula:

$$\left(\frac{\partial}{\partial x_i}\right)_{|x} f := \partial_{e_i}(f) = \lim_{t\to 0} \frac{f(x+te_i) - f(x)}{t} = \frac{\partial f}{\partial x_i}(x) \quad \text{ for } f \in \mathcal{O}_{\mathbb{R}^n,x}$$

This is going to be the motivation for the following definition of a tangent vector.

Definition 3.3.9 (Derivations and tangent spaces)**.** An $\mathbb{R}$-valued $\mathbb{R}$-linear map (=linear functional) on an $\mathbb{R}$-algebra which obeys the Leibniz formula is called an $\mathbb{R}$-*derivation* of that algebra. It is also clear that all such $\mathbb{R}$-derivations form an $\mathbb{R}$-vector space. The $\mathbb{R}$-vector space of all derivations of the $\mathbb{R}$-algebra $\mathcal{O}_{M,x}$ is called the *tangent space of* M *at* x. It is denoted $T_x(M)$, and elements in it are called *tangent vectors at* x.

In view of the inclusion isomorphism defined in Exercise 3.3.3, the natural map $T_x(M) \to T_x(U)$, for an open subset $U \subset M$ and $x \in U$, is therefore an isomorphism. Thus the tangent space to M at x is naturally identified with the tangent space to U at x, for any open set $U \subset M$ containing x.

Exercise 3.3.10. Prove that a tangent vector $X \in T_x(M)$ kills all constant germs in $\mathcal{O}_{M,x}$.

Definition 3.3.11 (Derivatives of smooth maps on manifolds)**.** Let $f : M \to N$ be a smooth map between manifolds. Then there is a natural linear map called the *derivative of* f at x, denoted as $Df(x) : T_x(M) \to T_{f(x)}(N)$. It is defined as $(Df(x)(X))\, h = X(f^*h)$ where h is any germ in $\mathcal{O}_{N,f(x)}$ and f^*h denotes the pullback germ $h \circ f$ at x as defined in Definition 3.3.6. We have the following:

(i): $D(\mathrm{Id}_M)(x) = \mathrm{Id}_{T_x(M)}$.

(ii): [Chain Rule for smooth maps] $D(f \circ g)(x) = Df(g(x))Dg(x)$

For notational convenience, Df is sometimes denoted f_*. That $Df(x)(X)$ is a derivation easily follows from the fact that X is a derivation and f^* is an $\mathbb{R}$-algebra homomorphism. The facts (i) and (ii) follow immediately from the definitions and the fact (i) in Remark 3.3.7.

Remark 3.3.12. It seems from (ii) above that the Chain Rule is a consequence of abstract nonsense, and doesn't need a proof! The catch is that we have called something $Df(x)$, without verifying that for the familiar situation of smooth maps between open sets of Euclidean spaces, it gives us back the old definition of derivative. This will be seen shortly in Lemma 3.3.15.

We aim to construct a natural identification between $\mathbb{R}^n$ and the tangent space $T_x(\mathbb{R}^n)$ at x, via the directional derivative as defined above. We first need the "first order Taylor formula" for a smooth function f on an open subset of $\mathbb{R}^n$.

Lemma 3.3.13 (1st-order Taylor formula)**.** Let f be a smooth function in a neighbourhood U of 0 in $\mathbb{R}^n$. Then we have:

$$f(x) = f(0) + \sum_{i=1}^{n} x_i g_i(x)$$

where $x = (x_1, .., x_n)$ is any point in a small enough ball around 0 contained in U, and g_i are smooth functions on this ball, and satisfy $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof: Let f be smooth. Let $\epsilon > 0$ be such that $B(0, \epsilon) \subset U$. For a fixed $x \in B(0, \epsilon)$, consider the function of one variable $h(t) := f(tx)$. Then this function h is defined and smooth on some open interval containing $(-1, 1)$, and by the one-variable fundamental theorem of calculus applied to it, we have:

$$\begin{aligned} f(x) - f(0) = h(1) - h(0) &= \int_0^1 \frac{dh}{dt}(t)dt \\ &= \int_0^1 \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx)x_i \right) dt \\ &= \sum_{i=1}^{n} x_i g_i(x) \end{aligned}$$

where

$$g_i(x) := \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

Then $g_i(0)$ is easily seen to be $\frac{\partial f}{\partial x_i}(0)$, and are clearly smooth because f is smooth. □

The next proposition, hopefully, explains why elements of $T_x(\mathbb{R}^n)$ are called "derivations". They are merely directional derivatives, as we see below.

Proposition 3.3.14 (The tangent space to $\mathbb{R}^n$)**.** For $x \in \mathbb{R}^n$, we have the following facts about $T_x(\mathbb{R}^n)$

(i): The set of coordinate partials at x, viz.

$$\left\{ \left(\frac{\partial}{\partial x_i} \right)_x \right\}_{i=1}^{n}$$

is a basis of $T_x(\mathbb{R}^n)$. In particular $\dim_{\mathbb{R}}(T_x(\mathbb{R}^n) = n$

(ii): The map:

$$\begin{array}{rcl}\theta : \mathbb{R}^n & \to & T_x(\mathbb{R}^n) \\ v & \mapsto & \partial_v\end{array}$$

(where ∂_v denotes the derivation at x defined by the directional derivative along v as defined in 3.3.8 above) is a *canonical* (=basis independent) linear map of $\mathbb{R}^n$ to $T_x(\mathbb{R}^n)$. Under this identification, the standard basis $\{e_i\}_{i=1}^n$ maps to the basis of coordinate partials described in (i) above.

(iii): In particular θ is a *canonical* (= basis independent) isomorphism. As a consequence, there is a canonical identification of $T_x(\mathbb{R}^n)$ with $T_y(\mathbb{R}^n)$ for all $x, y \in \mathbb{R}^n$.

Proof: We will take $x = 0$ for simplicity, leaving the general case to the reader (It merely involves generalising the Taylor formula Lemma 3.3.13 to a point $x \in \mathbb{R}^n$). Let $X \in T_0(\mathbb{R}^n)$, and $f \in \mathcal{O}_{\mathbb{R}^n,0}$. Applying the derivation X to the Taylor formula in the Lemma 3.3.13, using the Leibniz formula, the fact that derivations kill all constant germs (Exercise 3.3.10), and that $x_i(0) = 0$, we have

$$X(f) = X\left(f(0) + \sum_{i=1}^n x_i g_i\right) = \sum_{i=1}^n X(x_i) g_i(0) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(0) = \sum_{i=1}^n a_i \left(\frac{\partial}{\partial x_i}\right)_{|0} f$$

where $a_i := X(x_i)$. This shows that the coordinate partials *span* the tangent space $T_0(\mathbb{R}^n)$. They are linearly independent because on appplying them to the coordinate function germs x_i, we have:

$$\left(\frac{\partial}{\partial x_j}\right)_{|0} x_i = \delta_{ij}$$

which proves (i).

To see (ii), we have that ∂_v is a derivation by 3.3.8. The $\mathbb{R}$-linearity of the map θ is clear since $\partial_v f = Df(0)v$. Also by definition, $\partial_{e_i} = \left(\frac{\partial}{\partial x_i}\right)_{|0}$. Thus (ii) follows. (iii) is immediate from (i) and (ii). $\square$

Now we come to the verification of the assertion made in Remark 3.3.12, that our abstract nonsense definition of Df coincides with the usual one for open sets in $\mathbb{R}^n$.

Lemma 3.3.15. Let $f : U \to \mathbb{R}^m$ be a smooth map, where $U \subset \mathbb{R}^n$ is an open subset. Then $Df(x)$ as defined above in 3.3.11 coincides with the usual notion of derivative of f from advanced calculus.

Proof: Let $v \in \mathbb{R}^n$. Then for a germ $h \in \mathcal{O}_{\mathbb{R}^m,f(x)}$, we have:

$$\begin{array}{rcl} Df(x)(\partial_v)h & = & \partial_v(f^*h) = \partial_v(h \circ f) \quad \text{(by Definitions 3.3.6 and 3.3.11)} \\ & = & D(h \circ f)(v) \quad \text{(by Definition 3.3.8)} \\ & = & (Dh \circ Df)(v) = Dh(Df(v)) = \partial_{Df(v)}(h) \quad \text{(by the Chain rule and Definition 3.3.11 again)} \end{array}$$

Here all the derivatives except the first one are the usual ones in Euclidean space. This shows that $Df(x)$ applied to the derivation ∂_v is the same as the derivation $\partial_{Df(v)}$ as defined in Definition 3.3.11. Since v in $\mathbb{R}^n$ is being identified with $\partial_v \in T_x(\mathbb{R}^n)$ by (ii) of the previous Proposition 3.3.14, we see that our notion of derivative of f via derivations is the usual one. $\square$

We make the following:

Definition 3.3.16 (Coordinate partials in a chart on a manifold)**.** Define the coordinate partials with respect to the chart (ϕ, U) as:

$$\left(\frac{\partial}{\partial x_i}\right)_{|x} f := (\phi^{-1})_*(\partial_{e_i})f = \left(\frac{\partial}{\partial x_i}\right)_{|0} (\phi^{-1})^* f = \lim_{t \to 0} \frac{(f \circ \phi^{-1})(te_i) - (f \circ \phi^{-1})(0)}{t} \qquad \text{for } f \in \mathcal{O}_{M,x}$$

Here $(\phi^{-1})_*$ is as defined in Definition 3.3.11. It is customary to denote the components of $\phi(x)$ as x_i for all $x \in U$, and these x_i are called *coordinate functions around* x, and are defined only on U.

Proposition 3.3.17 (Basis for $T_x(M)$)**.** The coordinate partials of Definition 3.3.16 give *a* basis the *tangent space* $T_x(M)$.

Proof: Choose a coordinate chart (ϕ, U) around x. We can then identify (*non*-canonically) $\mathcal{O}_{M,x}$ and $\mathcal{O}_{\mathbb{R}^n,\phi(x)}$ as $\mathbb{R}$-algebras, by using pullback ϕ^*, which is an $\mathbb{R}$-algebra isomorphism by the Remark 3.3.7. Thus the map $D\phi(x) = \phi_*$ gives a linear isomorphism of $T_x(M) = T_x(U)$ and $T_x(\phi(U)) = T_x(\mathbb{R}^n)$, with inverse $(\phi^{-1})_*$. The proposition is now immediate from the earlier Proposition 3.3.14. $\square$

The extent of "non-canonicality" of the basis of coordinate partials for $T_x(M)$ can be easily described via the following exercise.

Exercise 3.3.18. Check that if (ψ, V) is another chart around the point x, with corresponding coordinate functions y_j, and if we denote the coordinate partials with respect to ψ by $\left(\frac{\partial}{\partial y_j}\right)_{|x}$, then there is the "Jacobian formula" :

$$\left(\frac{\partial}{\partial x_i}\right)_{|x} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \left(\frac{\partial}{\partial y_j}\right)_{|x}$$

where $\frac{\partial y_j}{\partial x_i}$ is the (ji)-th entry in the Jacobian matrix $D(\psi \circ \phi^{-1})(\phi(x))$.

Exercise 3.3.19. In the notation of Exercise 3.3.5, prove that the $\mathbb{R}$-vector space $T_x(M)$ of $\mathbb{R}$ derivations of $\mathcal{O}_{M,x}$ is isomorphic to the vector space:

$$\hom_{\mathbb{R}}\left(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R}\right)$$

the isomorphism taking a derivation X to the linear functional $f \mapsto X(f)$ for $f \in \mathfrak{m}_x/\mathfrak{m}_x^2$. Quite naturally, the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the *cotangent space* to M at x, and denoted by $T_x^*(M)$.

Exercise 3.3.20 (Velocity vectors and tangent vectors)**.** Let $c : (-\epsilon, \epsilon) \to M$ be a smooth map (called a *smooth curve in* M), with $c(0) = x$. The *velocity of* c *at* 0 is the map defined by:

$$c'(0)f := \left(\frac{dc}{dt}(0)\right) f = \frac{d(f \circ c)}{dt}(0)$$

for $f \in \mathcal{O}_{M,x}$. Verify that:

(i): $c'(0)$ is a derivation, and hence a tangent vector at x.

(ii): In a coordinate chart (ϕ, U) around x with $\operatorname{Im} c \subset U$ and $\phi(x) = 0$,
we have the formula $c'(0) = \sum_{i=1}^{n} \frac{dc_i}{dt}(0) \frac{\partial}{\partial x_i}$, where c_i denotes the function $(x_i \circ c)$ on U.

(iii): Every tangent vector in $T_x(M)$ is the velocity vector $c'(0)$ for some smooth curve c as above.

Proposition 3.3.21 (Derivative in local coordinates)**.** If we use local charts (ϕ, U) around x and (ψ, V) around $f(x)$, then we have natural bases for $T_x(M)$ and $T_{f(x)}(N)$ given by coordinate partials (Proposition 3.3.17). With respect to these bases, we have:

$$Df(x)\left(\frac{\partial}{\partial x_i}\right)_{|x} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(\phi(x)) \left(\frac{\partial}{\partial y_j}\right)_{|f(x)}$$

where f_j denotes the j-th component of $\psi \circ f \circ \phi^{-1}$.

Proof: Let dim $N = n$. By Definition 3.3.16 $\left(\frac{\partial}{\partial x_i}\right)_{|x} = (\phi^{-1})_*(\partial_{e_i})$, so that by Definition 3.3.11 and the chain rule,

$$\begin{aligned} Df(x)\left(\frac{\partial}{\partial x_i}\right)_{|x} &= Df(x)(\phi^{-1})_*(\partial_{e_i}) = Df(x)\circ D\phi^{-1}(\partial_{e_i}) \\ &= D(\psi^{-1})D(\psi)Df(x)\circ D\phi^{-1}(\partial_{e_i}) = D(\psi^{-1})D(\psi\circ f\circ\phi^{-1})(\partial_{e_i}) \end{aligned}$$

Since $\psi\circ f\circ\phi^{-1} = (f_1, .., f_n)$ is a map between open subsets of euclidean spaces, we know from advanced calculus that

$$D(\psi\circ f\circ\phi^{-1})(\partial_{e_i}) = \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial y_j} = \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_i}\partial_{f_j}$$

where f_j denotes the standard basis and y_j the corresponding coordinates on $\mathbb{R}^n$ respectively. Now applying the linear map $D(\psi^{-1})$ to both sides and using $D(\psi^{-1})(\partial_{f_j}) = \left(\frac{\partial}{\partial y_j}\right)_{|f(x)}$, we have the result. $\square$

Example 3.3.22 (Tangent space to a manifold defined by equations)**.** Let M be a manifold defined by the inverse image of the regular value 0 under a smooth or analytic map $f:\mathbb{R}^n \to \mathbb{R}^m$, as in Example 2.1.5. Let $x\in M$ be a point. We claim that the tangent space $T_x(M)$ is isomorphic to the vector space:

$$\ker Df(x) : T_x(\mathbb{R}^n) \to T_{f(x)}(\mathbb{R}^m)$$

(which is of dimension $(n-m)$ by the assumption that 0 is a regular value). Note that by the implicit function theorem cited in Example 2.1.5 above, we know that $T_x(M)$ is of dimension $n-m$. So if we can prove that the kernel of $Df(x)$ contains $T_x(M)$, we would have equality of the two, for dimensional reasons. But, if $X\in T_x(M)$, it follows from the definition that for each germ (U,h) at 0 in $\mathbb{R}^m$, the composite map $f^*(h) = h\circ f$ is a germ in $\mathcal{O}_{\mathbb{R}^n,x}$ which is constant $(= h(0))$ on the neighbourhood $f^{-1}(U)\cap M$ of x in M, and hence represents the constant germ $h(0)$ in $\mathcal{O}_{M,x}$. Thus any derivation in $T_x(M)$ will kill it, and hence $(Df(x)(X))h = X(h\circ f) = 0$ for all $X\in T_x(M)$. Thus $T_x(M)\subset\ker Df(x)$, and we are done.

Exercise 3.3.23 (Tangent space to S^n)**.** Using Example 3.3.22 above, show that the tangent space to S^n at $x\in S^n$ is given by the hyperplane $(\mathbb{R}x)^\perp$ orthogonal to the vector x. Geometrically, this is exactly what the tangent space should be.

Exercise 3.3.24 (Inverse and Implicit Function Theorems, Regular Values)**.** Formulate versions of the inverse and implicit function theorems for smooth maps between manifolds. If $f: M\to N$ is a smooth map, call $b\in N$ a *regular value of* f if the derivative $Df(x): T_x(M)\to T_b(N)$ is a surjection for all $x\in X := f^{-1}(b)$. Generalise Examples 2.1.5 and 3.3.22 to this more general situation.

As an application of Example 3.3.22 above, we can describe the so-called "Lie Algebras" of the classical linear groups of Example 2.1.7.

Definition 3.3.25 (Lie group, Lie algebra)**.** A smooth manifold G which is a group (with binary operation or multiplication denoted by ".") and for which the map:

$$\begin{aligned} \alpha: G\times G &\to G \\ (x,y) &\mapsto x.y^{-1} \end{aligned}$$

is smooth is called a *Lie Group*. From the definition it follows that the inversion map $x\mapsto x^{-1}$ of G, as well as the multiplication map $(x,y)\mapsto x.y$ of $G\times G$ to G are both smooth.

If G is a Lie group of dimension n, the tangent space at the identity element e, viz, $T_e(G)$ is called *the Lie algebra of* G, and often denoted $\mathfrak{g}$ (or even Lie(G).) It is a vector space of dimension n.

Example 3.3.26 (General linear group)**.** The general linear group $GL(n,\mathbb{F})$, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ is clearly a Lie group, since matrix multiplication is a smooth map from $M(n,\mathbb{F})\times M(n,\mathbb{F})$ to $M(n,\mathbb{F})$, because all entries of AB are bilinear in the entries A_{ij} and B_{kl}. The map $A\mapsto A^{-1}$ is a smooth map of the open subset $GL(n,\mathbb{F})$ to itself by Cramer's formula for A^{-1}.

As we observed after the Definition 3.3.9, the tangent space $T_x(U)$ of an open subset U of a manifold is the same as $T_x(M)$, and hence $T_e(GL(n,\mathbb{F}))=T_I(M(n,\mathbb{F}))$ for $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. For this reason, $M(n,\mathbb{F})$ is often denoted $\mathfrak{gl}(n,\mathbb{F})$.

Example 3.3.27 (Orthogonal group)**.** Let $G=O(n)$ or $G=SO(n)$. The fact that $G=O(n)$ is a Lie group follows from (iii) of Exercise 3.1.4, and the previous Example 3.3.26. Hence the open subset $SO(n)$ of $O(n)$ is also a Lie group. (Note that $SO(n)$ is $O(n)\cap\{A:\det A>0\}$, hence an open subset of $O(n)$).

Since $SO(n)$ is open in $O(n)$, the Lie algebras of both these groups are the same by the observation made after Definition 3.3.9, and this Lie algebra is denoted $\mathfrak{so}(n)$. By the Exercise 3.3.22 above, the tangent space at the identity $\mathrm{Lie}(O(n))=T_I(O(n))$ is the kernel of the map $D\phi(e):M(n,\mathbb{R})\to S(n,\mathbb{R})$ where $\phi(A)=AA^t$. From that example, we recall that $D\phi(A)X=AX^t+XA^t$, so that $D\phi(I)(X)=X+X^t$, and hence

$$\ker D\phi(e)=\{X\in\mathfrak{gl}(n,\mathbb{R}):X=-X^t\}$$

the space of *real skew-symmetric* $n\times n$ *matrices.*

Exercise 3.3.28. Verify that all the classical linear groups $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $U(n)$, $SU(n)$ (introduced in Example 2.1.7 earlier on) are all Lie groups. Prove their Lie algebras are given by:

$$\begin{aligned}
\mathrm{Lie}(SL(n,\mathbb{F})) &= \mathfrak{sl}(n,\mathbb{F})=\{X\in\mathfrak{gl}(n,\mathbb{F}):\ \mathrm{tr}\,X=0\}\quad\text{for }\ \mathbb{F}=\mathbb{R},\ \mathbb{C}\\
\mathrm{Lie}(U(n)) &= \mathfrak{u}(n)=\{X\in\mathfrak{gl}(n,\mathbb{C}):\ X=-X^*\}\\
\mathrm{Lie}(SU(n)) &= \mathfrak{su}(n)=\{X\in\mathfrak{gl}(n,\mathbb{C}):\ X=-X^*,\ \mathrm{tr}\,X=0\}
\end{aligned}$$

where $X^*:=-\overline{X}^t$, the Hermitian adjoint of X.

Exercise 3.3.29. Compute the following derivatives:

(i): $Df(I)$ where $f:GL(n,\mathbb{R})\to GL(n,R)$ is the map given by (a) $f(A)=(A^{-1})^t$ and (b) $f(A)=A^n$, where $n\in\mathbb{Z}$

(ii): $df(s,t)$ where f is one of the charts on the torus $T^2\subset\mathbb{R}^3$ described in Exercise 2.1.11.

$$\begin{aligned}
f:(0,2\pi)\times(0,2\pi) &\to S^1\times S^1\\
(s,t) &\mapsto ((2+\cos s)\cos t,(2+\cos s)\sin t,\sin s))
\end{aligned}$$

Hence determine the coordinate partials on T^2 as elements of $T_{f(s,t)}(\mathbb{R}^3)=\mathbb{R}^3$.

Exercise 3.3.30. Consider the torus $T^2\subset\mathbb{R}^3$ as described in the Exercise 2.1.11, and the smooth map $f:T^2\to\mathbb{R}$ defined by $f(x,y,z)=x$. Which points of T^2 are critical points of f, and which points of $\mathbb{R}$ are critical values of f? (An analogous example of a hyperboloid in $\mathbb{R}^3$ is pictured below in Fig.5.)

Definition 3.3.31 (Immersions and submersions of manifolds)**.** Let $f:M\to N$ be a smooth map. We recall that f is called an *immersion* if $Df(x):T_x(M)\to T_{f(x)}(N)$ is injective for all $x\in M$, and a *submersion* if $Df(x):T_x(M)\to T_{f(x)}(N)$ is surjective for all $x\in M$. As noted in Exercise 3.3.24, the inverse function theorem, as well as the immersive and submersive forms of the implicit function theorem have obvious formulations for smooth maps between manifolds.

Exercise 3.3.32. Let $f:M\to N$ be a smooth map, with dim $M\geq$ dim N. Show that:

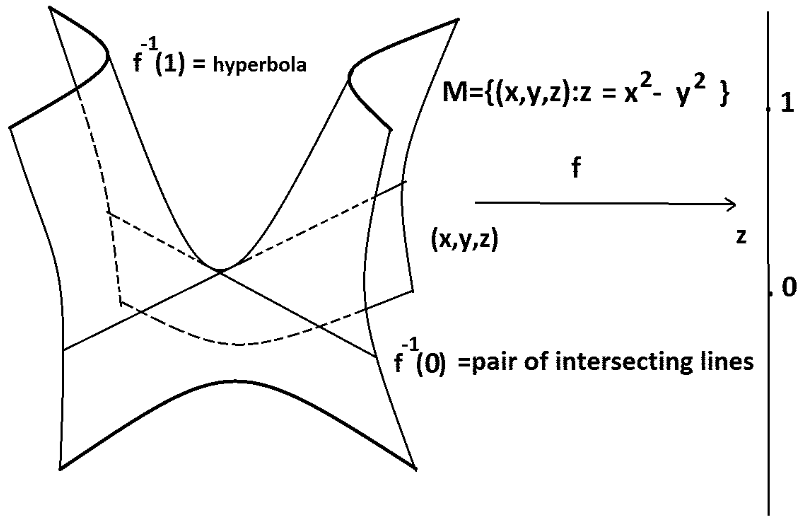

FIGURE 5. Inverse images of regular and critical values

(i): The set of critical points of f is a closed subset of M. In particular, if M *is compact*, the set of critical values of f is a closed subset of N.

(ii): Give an example to show that the last assertion of (i) above is false in general if M is non-compact.

(iii): If f is a submersion, then f is an open map. In particular, if we further assume that M is compact and N is connected, then f is surjective.

Remark 3.3.33 (Topological properties of immersions)**.** One would naturally like to formulate some analogous topological properties of immersions. Before we do that let us remark that the contrast between open and closed sets is the following:

A is an open subset of a space X iff for each point $x \in A$, there is a neighbourhood U of x in A such that $U \cap A$ is open in U. That is, the property of A being open is *"local at each point of A"*.

However, the property of A being closed is *"local at each point of X"*. Indeed, as an exercise, prove that C is closed in a topological space X iff for each open covering $\{U_\alpha\}$ of X, $C \cap U_\alpha$ is closed in U_α for each α. If A is a set such that each point $x \in A$ has a neighbourhood U in X such that $U \cap A$ is closed in U, we recall that A is merely *locally closed* in X. This is *weaker* that it being closed. For example, the set $(0, 1) \times 0$ is locally closed in $\mathbb{R}^2$, but not closed.

In view of the local nature of immersions described in the Exercise 3.3.24, all one can assert about an immersion is that it is locally a closed map. In fact, immersions can behave quite badly, as is clear from the example of the skew line on the torus described in Remark 3.1.5):

There is a final result about regular values, namely

Theorem 3.3.34 (Sard's Theorem)**.** Let $f : M \to N$ be a smooth map between smooth manifolds. Then the set of critical values of f is a nowhere dense subset of N. The set of regular values of f is a dense subset of N.

Proof: In fact, the theorem also states that the set of critical values of f, called the *critical set of f*, is a subset of measure zero in N. (Measure zero is easy to define on a manifold, by using a countable locally finite atlas, and using the notion of (Lebesgue) measure zero in $\mathbb{R}^n$. Alternatively, using a Riemannian metric, one can define a volume form and measure on a manifold, as we will see later). If dim $M < \dim N$, the critical set is precisely $f(M)$ and it is not difficult to show, using the Lipschitz property of a smooth map, that $f(M)$ is a set of measure zero in N. The proof of Sard's theorem for dim $M \geq$ dim N is more involved, and may be found in Hirsch's book *Differential Topology*, p. 69. □

Definition 3.3.35 (Embeddings)**.** An immersion $f : M \to N$ which is injective, and such that $f : M \to f(M)$ is a homeomorphism, is called an *embedding*. (Here $f(M)$ has the induced (=subspace) topology from N).

(Caution: As noted earlier in Remark 3.1.5, some authors define an embedding to be an injective immersion, which is more general than the definition here. For example the immersion of the skew line on the torus of the example of that remark above is a 1-1 immersion, but is not an embedding by our definition).

Exercise 3.3.36. Prove that $N \subset M$ is a submanifold if and only if it is the image of an embedding (see Definition 3.1.3).

The following is a deep result.

Theorem 3.3.37 (Whitney Embedding Theorem)**.** A smooth manifold of dimension n can be smoothly embedded in $\mathbb{R}^{2n}$.

For a proof of the lighter result that a smooth compact manifold can be smoothly embedded in $\mathbb{R}^{2n+1}$, see Hirsch's book *Differential Topology*, p. 24.

We remark that this bound of $2n$ is sharp. For example, $\mathbb{RP}(2)$ does not embed in $\mathbb{R}^3$. (See (iii) of the Exercise 3.1.6).

4. Vector Bundles

4.1. **Tangent and other bundles.**

Definition 4.1.1 (Tangent bundle, vector fields)**.** Let M be a smooth manifold. We consider the disjoint union of all the tangent spaces to M, viz.:

$$TM := \coprod_{x \in M} T_x(M)$$

We define the projection map $\pi : TM \to M$ to be $\pi(T_x(M)) = x$. Let (ϕ, U) be any chart of M. We note that the following *local triviality* condition holds. The following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{R}^n \\ \pi \searrow & & \swarrow \mathrm{pr}_1 \\ & U & \end{array}$$

where

$$\Phi\left(\sum_{i=1}^{n} a_i \left(\frac{\partial}{\partial x_i}\right)_{|x}\right) = (x, a_1, .., a_n)$$

$\left\{\left(\frac{\partial}{\partial x}\right)_{i|x}\right\}_{i=1}^{n}$ being the basis of $T_x(M)$ given by the coordinate partials with respect to ϕ as defined above. Note that Φ is a bijection, and since $U \times \mathbb{R}^n$ is homeomorphic to the open set $\phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$, we topologise $\pi^{-1}(U)$ by requiring Φ to be a homeomorphism. We need to check that if (ϕ, U) and (ψ, V) are two charts, then the topology induced on the overlap $\pi^{-1}(U \cap V)$ from $\pi^{-1}(U)$ by this prescription (using the chart (ϕ, U)) is the same as that induced from $\pi^{-1}(V)$ by using the chart (ψ, V).

This follows easily from the Jacobian coordinate change formula for the coordinate partials of the Exercise 3.3.18. Thus one can define the topology on TM to be the coherent topology from the family $\{\pi^{-1}(U)\}$, i.e. $W \subset TM$ is open iff $W \cap \pi^{-1}(U)$ is open in $\pi^{-1}(U)$ for all charts (ϕ, U). By definition then, TM becomes a manifold of dimension $2n$. Verify that $\pi : TM \to M$ is a smooth map. TM is called the *tangent bundle* of M, and π is called the *bundle projection.*

Definition 4.1.2 (Sections, vector fields)**.** A smooth map $s : M \to TM$ satisfying $\pi \circ s = \mathrm{Id}_M$ is called a smooth *section* or *vector field* on M. If $U \subset M$ is an open subset, and $s : U \to \pi^{-1}(U)$ is a smooth map which satisfies $\pi \circ s = \mathrm{Id}_U$, then we say that s is a *section over* U or a *vector field over* U.

For example, if (ϕ, U) is a chart on M, the coordinate partials $\frac{\partial}{\partial x_i}$ give n linearly independent smooth vector fields over U, called the *coordinate fields* on U, and every smooth vector field over U is of the form:

$$s(x) = \sum_{i=1}^{n} a_i(x) \left(\frac{\partial}{\partial x_i}\right)_{|x}$$

where a_i are smooth functions on U.

Exercise 4.1.3. Show that the tangent bundle of $S^n \subset \mathbb{R}^{n+1}$ is the space:

$$TS^n = \{(x, v) \in S^n \times \mathbb{R}^n : \langle v, x \rangle = 0\}$$

Use the two stereographic charts on S^n to explicitly check local triviality of this bundle.

Exercise 4.1.4. Here is an interesting description of the tangent bundle of S^n as a complex affine algebraic variety. Show that the set:

$$X = \{(z_0, ..., z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} z_i^2 = 1\}$$

is diffeomorphic to TS^n described in the last exercise. Denoting $x_i := \mathrm{Re}\, z_i$ and $y_i := \mathrm{Im}\, z_i$, show that the map $(z_0, .., z_n) \mapsto (1 + \sum_i y_i^2)^{-1}(x_0, .., x_n)$ gives the bundle projection $\pi : TS^n \to S^n$.

One can do various constructions with the tangent bundle. For example, one can define:

Example 4.1.5 (Cotangent bundle)**.** The *cotangent bundle* T^*M is the union:

$$\coprod_{x\in M} T_x^*(M)$$

where T_x^*M is the $\mathbb{R}$-dual of $T_x(M)$. As before, it can be given the structure of a manifold of dimension $2n$. In fact, if (ϕ, U) is a coordinate chart for M, then for each $x \in U$, a basis for $T_x^*(M)$ is given by the *coordinate differentials* $\{dx_{i|x}\}_{i=1}^n$ defined by:

$$dx_{i|x}\left(\frac{\partial}{\partial x_j}\right)_{|x} = \delta_{ij}$$

which is just the dual basis to the basis $\left\{\left(\frac{\partial}{\partial x_j}\right)_{|x}\right\}_{j=1}^n$ of $T_x(M)$ defined in the last subsection. The coordinate change from a chart (ϕ, U) (with coordinate functions x_i) to another chart (ψ, V) (with coordinate functions y_j) will lead to the change of basis given by:

$$dx_{i|x} = \sum_{j=1}^n \left(\frac{\partial x_i}{\partial y_j}(x)\right) dy_{j|x}$$

where $\left(\frac{\partial x_i}{\partial y_j}(x)\right)$ denotes the ji-th entry of the Jacobian matrix of $D(\phi\circ\psi^{-1})(\psi(x))$. (Recall that if a basis change is given by $e_j = \sum_i A_{ij} f_i$, then the dual basis changes by the formula $e_i^* = \sum_i (A^{-1})_{ij} f_j^* = \sum_i (A^{-1})^t_{ji} f_j^*$.)

Notation 4.1.6. From now on we will suppress the subscript "x" from the basis elements $dx_{i|x}$ of $T_x^*(M)$ and $\left(\frac{\partial}{\partial x_j}\right)_{|x}$ of $T_x(M)$, and simply denote them as dx_i and $\left(\frac{\partial}{\partial x_j}\right)$ respectively, for notational convenience.

Exercise 4.1.7. Write down the local formula for $(Df(x))^* dy_j$ for a smooth map $f : M \to N$, analogous to the formula in Proposition 3.3.21.

Definition 4.1.8 (Derivative of a smooth map)**.** As noted earlier in 3.3.11, a smooth map $f : M \to N$ between smooth manifolds leads in natural way to a map $Df(x) : T_x(M) \to T_{f(x)}(N)$. Putting these fibrewise maps together leads to a map:

$$Df : TM \to TN$$

which is called the *derivative of* f. That it is a smooth map, with respect to the manifold structures defined above on TM and TN follows from the formula in Proposition 3.3.21. Further, it is linear on each fibre, and is an example of what we shall later call a "bundle morphism".

Before proceeding further, it is useful to abstract the structures of the tangent and cotangent bundles, and introduce the notion of a "*smooth vector bundle*" on a manifold M.

Definition 4.1.9 (Vector bundles)**.** A *smooth vector bundle* of rank k on a smooth manifold M is a smooth manifold E of dimension $k + \dim M$ and a smooth surjection $\pi : E \to M$ such that:

(i): Each fibre $E_x = \pi^{-1}(x)$ is an $\mathbb{R}$-vector space of dimension k.

(ii): There exists an open covering $\{U_i\}_{i\in\Lambda}$ of M, and smooth maps (called *local trivialisations* or *bundle charts*):

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

making the diagram:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\Phi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{R}^k \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

commute, for each i.

(iii): For each $x \in U_i$, and each i, the restriction $\Phi_{i|E_x}$ is a vector space isomorphism to $\{x\} \times \mathbb{R}^k$.

E is called the *total space* of the bundle, and M the *base space*. π is called the *bundle projection*. Sometimes we will just call E *a vector bundle over* M, and denote it simply by E, when the projection π and the base space M are understood.

If there exists such an open covering with just a single element $\{M\}$, then the bundle is said to be a *trivial bundle*. It is denoted by ϵ^k, if it is of rank k.

A vector bundle of rank 1 is called a *line bundle*, and a bundle of rank 2 is called a *plane bundle*.

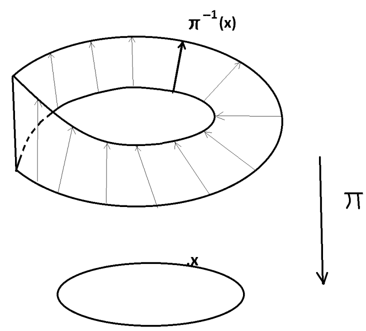

FIGURE 6. A non-trivial line bundle on S^1

Definition 4.1.10 (Sections of bundles, 1-forms). If $\pi : E \to M$ is a smooth vector bundle, a smooth map $s : M \to E$, is called a smooth *section* of the bundle if $\pi \circ s = \mathrm{Id}_M$. For $U \subset M$ an open set, a *section over* U is just a section of the restricted bundle: $\pi : \pi^{-1}(U) \to U$.

A smooth section of the cotangent bundle $s : M \to T^*M$ is called a smooth *1-form* on M. Similarly, we can define smooth 1-forms on an open set $U \subset M$ to be a section of the cotangent bundle over U. For example, for a chart (ϕ, U), the basis element dx_i defined above in 4.1.5 is a 1-form over U.

Exercise 4.1.11.

(i): Show that a smooth vector bundle $E \to M$ of rank k is trivial iff there exist k smooth sections s_i of this bundle which are everywhere linearly independent. That is, $\{s_i(x)\}_{i=1}^k$ is a linearly independent set in E_x for each $x \in M$.

(ii): Show that the tangent bundles of S^1 and S^3 are trivial.

(iii): Show that the tangent bundle of any Lie group is trivial. (See Definition 3.3.25 for the definition of *Lie group*.) Note that (ii) follows because S^1 and S^3 are Lie Groups. S^3 has a Lie group structure via the algebra of quaternions (defined by Hamilton). S^7 turns out to have non-commutative and non-associative smooth multiplication with identity and inverses via the octonion algebra (Cayley numbers). It turns out that S^1, S^3 and S^7 are the *only* spheres which have a trivial tangent bundle, by deep results in algebraic topology due to Bott and Milnor.

(iv): Show that the tangent bundle of S^{2n+1} admits a nowhere-vanishing smooth section, for all n. Show that the tangent bundle of S^{4n+3} has three everywhere linearly independent smooth sections.

(v): If $\pi : E \to M$ is a smooth vector bundle, then π is an open map. Since it is a surjection, this implies it is a quotient map.

Definition 4.1.12 (Bundle frame). Let $\pi : E \to M$ be a smooth vector bundle. A collection of smooth sections $s_i : U \to \pi^{-1}(U)$ is called a *bundle frame over* U if the collection $\{s_i(x)\}$ is a vector space basis for E_x for each $x \in U$. Using (i) of the Exercise 4.1.11 above, it is clear that a local trivialisation over U is equivalent to finding a bundle frame over U.

4.2. **Constructions with vector bundles.** There are obvious notions and constructions which carry over from vector spaces to vector bundles. In a vector bundle, a vector space basis which works simultaneously for all fibres cannot be chosen, unless the vector bundle is trivial (as seen in (i) of Exercise 4.1.11 above). However, any *basis independent* notion or construction that exists for vector spaces will carry over to a vector bundle in a fibre-wise fashion. We list a few below.

Notation 4.2.1. *All bundles below will be assumed to be smooth vector bundles. Also, if* $\pi : E \to M$ *is a vector bundle, we will often denote the restricted bundle* $\pi^{-1}(U) \to U$ *as* $E_{|U} \to U$.

Definition 4.2.2 (Vector bundle morphisms and isomorphisms). A *smooth morphism* between two smooth vector bundles $\pi_1 : E \to M$ and $\pi_2 : F \to M$ over a manifold M is a *smooth map of manifolds* $\phi : E \to F$ such that:

(i): For each $x \in M$, we have $\phi(E_x) \subset F_x$. This is equivalent to demanding that $\pi_2 \circ \phi = \pi_1$, i.e. the diagram

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & F \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

commutes.

(ii): For each $x \in M$, the map $\phi : E_x \to F_x$ is linear. It is sometimes convenient to write $\phi_{|E_x}$ as ϕ_x.

A vector bundle morphism $\phi : E \to F$ which has an inverse that is a vector bundle morphism is called an *isomorphism of vector bundles.* It is an exercise to check that this is equivalent to requiring the linear map $\phi_x : E_x \to F_x$ to be a vector space isomorphism for each $x \in M$. (The fibrewise inverses $\phi_x^{-1} : F_x \to E_x$ define the map $\phi^{-1} : F \to E$. To prove the smoothness of ϕ^{-1} is then a local problem, where local frames and the Cramer formula for the inverse of a matrix come to the rescue).

A *sub-bundle* $E \to M$ of a bundle $F \to M$ is a bundle with an inclusion $E \hookrightarrow F$ which is a morphism of bundles.

Proposition 4.2.3 (Quotient bundles). Given a smooth vector bundle $F \to M$ of rank n, and a sub-bundle $E \to M$ of rank k, it is possible to define the rank $(n-k)$ *quotient bundle* $F/E \to M$, whose fibre at x is the quotient space F_x/E_x. There is also a natural quotient morphism of bundles $p : F \to F/E$ given fibrewise by the natural vector space quotient maps $p_x : F_x \to F_x/E_x$.

Proof: We define $F/E := \coprod_{x \in M} F_x/E_x$, and $\pi : F/E \to M$ the obvious bundle projection. We will first construct local trivialisations for this bundle over open sets U_α covering M, which will produce natural topologies on $\pi^{-1}(U_\alpha)$. By construction, the bundle coordinate changes on overlaps $U_\alpha \cap U_\beta$ will be smooth, resulting in compatibility of topologies on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$, and hence a coherent topology on F/E will result. (Analogous to the definition of the topology on the tangent bundle in Definition 4.1.1).

The idea of finding a frame for F/E over a neighbourhood of any point $x \in M$ is to start out with a local frame s of E on a trivialising neighbourhood of x, and extend this to a frame for F *all over* some (possibly smaller) neighbourhood of x. Then the new elements added will furnish a frame for F/E all over this smaller neighbourhood.

So let $x \in M$, and let $\{s_i\}_{i=1}^k$ be a local frame for E over an open neighbourhood U of x, coming from some trivialisation $\Psi : E_{|U} \to U \times \mathbb{R}^k$. By shrinking U if necessary, we can assume that $F_{|U}$ is also trivial, via a trivialisation $\Phi : F_{|U} \to U \times \mathbb{R}^n$. Consider the k linearly independent vectors $\{f_i := \Phi(s_i(x))\}_{i=1}^k$ of $\mathbb{R}^n$. We can extend this to a basis $\{f_i\}_{i=1}^n$ of $\mathbb{R}^n$. Now note that for any other point $y \in U$, $\Phi(s_i(y))$ is a linearly independent set of k vectors in $\mathbb{R}^n$, and using our basis vectors $\{f_i\}_{i=1}^n$ of $\mathbb{R}^n$, we may write:

$$\Phi(s_i(y)) = \sum_{j=1}^n A_{ji}(y) f_j \quad \text{for} \quad i = 1,..,k$$

where A_{ji} is a $(n \times k)$ matrix of functions defined and smooth over U. Note that at the point x, and for $1 \le i, j \le k$, the top $(k \times k)$ block of smooth functions $A_{ij}(x) = \delta_{ij}$, by the construction of f_j. So, by the smoothness of the $(k \times k)$ top block matrix $[A_{ij}]_{1 \le i,j \le k}$ (and determinant map), there is a neighbourhood $W \subset U$ of x such that the determinant of this top $(k \times k)$ block is non-zero for all $y \in W$. So the $(n \times n)$ matrix:

$$B(y) := \begin{bmatrix} A_{11}(y) & .. & A_{1k}(y) & 0 & ...0 \\ A_{21}(y) & .. & A_{2k}(y) & 0 & ...0 \\ .. & .. & .. & .. & \\ A_{k1}(y) & .. & A_{kk}(y) & 0 & ...0 \\ A_{k+1,1}(y) & .. & A_{k+1,k}(y) & 1 & ....0 \\ .. & .. & .. & .. & 1.. \\ A_{n1}(y) & .. & A_{nk}(y) & 0 & ....1 \end{bmatrix}$$

(with lower right hand block being the identity matrix of size $(n-k)$) is non-singular for all $y \in W$.

This means that for all $y \in W$, the set $\{\Phi(s_1(y)), .., \Phi(s_k(y)), f_{k+1}, .., f_n\}$ is also a basis for $\mathbb{R}^n$. Set $s_j(y) := \Phi_y^{-1}(f_j)$ for $j = k+1, .., n$, where $\Phi_y : F_y \to \mathbb{R}^n$ is the restriction of Φ to the fibre F_y. Then, all over the neighbourhood W, $\{s_j\}_{j=1}^n$ is a smooth frame for $F_{|W}$, with $\{s_i\}_{i=1}^k$ spanning $E_{|W}$. Clearly, the frame $\{s_j(y)\}_{j=k+1}^n$ spans a (smoothly varying) $(n-k)$-dimensional vector space complement to E_y in each fibre F_y, for all $y \in W$. Hence it is a bundle frame which gives the trivialisation for the quotient bundle $(F/E)_{|W}$ over W. One now needs to check that on overlaps the coordinate changes resulting from these trivialisations are smooth, which we leave as an exercise. □

Definition 4.2.4 (Direct sums, Hom, tensor and exterior products)**.** Given vector bundles E and F on M, there is the *direct sum* $E \oplus F$ (also called the *Whitney sum*) of E and F. Its fibre over $x \in M$ is $E_x \oplus F_x$. Similarly, one defines the *tensor product bundle* $E \otimes F$. Another construction is the bundle $\hom_{\mathbb{R}}(E, F)$ over M, whose fibre at $x \in M$ is the vector space $\hom_{\mathbb{R}}(E_x, F_x)$. The bundle E^* denotes the bundle $\hom_{\mathbb{R}}(E, \epsilon^1)$, and is called the *dual bundle of* E. Finally, given a vector bundle E, one can form the r-th *exterior power bundle* $\wedge^r E$, whose fibre over $x \in M$ is the the r-th exterior power $\wedge^r E_x$ of E_x.

The bundle charts for all of the constructed bundles above all come out naturally from those of the given bundles. In fact, the simplest thing to do is to use Definition 4.1.12 and construct local frames of the various bundles being defined, out of local frames of the given bundles (as we did in the case of the quotient bundle above). For example, if we choose U to be a small enough open set such that $E_{|U}$ and $F_{|U}$ are both trivialised via frames $\{s_i\}_{i=1}^k$ and $\{t_j\}_{j=1}^m$ respectively, then the bundle $E \oplus F$ has a frame consisting of the $k+m$ elements $\{(s_i, 0)\}_{i=1}^k \cup \{(0, t_j)\}_{j=1}^m$.

Exercise 4.2.5.

(i): Giving a smooth morphism of smooth vector bundles $E \to F$ is equivalent to giving a smooth section of the bundle $\hom_{\mathbb{R}}(E, F)$.

(ii): As for vector spaces, the bundle $\hom_{\mathbb{R}}(E, F)$ is isomorphic to the vector bundle $E^* \otimes F$.

(iii): Show that for any *line bundle* E, the bundle $\hom_{\mathbb{R}}(E, E)$ is a trivial bundle. (Construct a nowhere vanishing section).

Example 4.2.6 (Tautological bundles over grassmannians)**.** Let ϵ^n be the trivial bundle of rank n over the grassmannian $G_k(\mathbb{R}^n)$. That is, the bundle:

$$p : \epsilon^n (= G_k(\mathbb{R}^n) \times \mathbb{R}^n) \to G_k(\mathbb{R}^n)$$

where p is projection to the first factor. Now we consider the subset:

$$\gamma_n^k := \{(P, v) \in \epsilon^n : v \in P\}$$

and define $\rho : \gamma_n^k \to G_k(\mathbb{R}^n)$ to be the restriction of p to E. Clearly the fibre $\rho^{-1}(P)$ is precisely the k-dimensional subspace P of $\mathbb{R}^n$. That is, the fibre over the *point* $P \in G_k(\mathbb{R}^n)$ is the *vector space* $P \subset \mathbb{R}^n$. One needs to produce local trivialisations for this bundle. Again, using the notation of Example 2.2.6, for a fixed $E \in G_k(\mathbb{R}^n)$, there is the map:

$$\begin{aligned} \Phi_E : \rho^{-1}(U_E) &\to U_E \times E \\ (P, v) &\mapsto (P, \pi_{E|P}(v)) \end{aligned}$$

which makes sense because $\rho^{-1}(U_E)$ is precisely the set:

$$\{(P, v) : \pi_{E|P} : P \to E \text{ is an isomorphism and } v \in P\}$$

If one uses the smooth identification of $\phi_E : U_E \to \hom_{\mathbb{R}}(E, E^\perp)$ via the chart ϕ_E of Example 2.2.6, with $\psi_E = \phi_E^{-1}$ as described there, then $\Phi_E^{-1}(\psi_E(T), w) = (\psi_E(T), w + Tw)$, which is certainly smooth and an isomorphism on the fibres (i.e. from $\mathbb{R}^k = E$ to the subspace $P = \operatorname{graph} T = $ fibre $\rho^{-1}(P)$ of $P \in G_k(\mathbb{R}^n)$.) This shows that Φ_E is smooth. Hence we have bundle trivialisations for γ_n^k over the open sets U_E.

The tautological bundle γ_n^1 on the space $G_1(\mathbb{R}^n) = \mathbb{RP}(n-1)$ is often denoted γ^1.

Exercise 4.2.7. Let $\gamma_n^{k,\perp}$ denote the bundle on $G_k(\mathbb{R}^n)$ whose fibre at $P \in G_k(\mathbb{R}^n)$ is $P^\perp$. Show that $\gamma_n^k \oplus \gamma_n^{k,\perp} = \epsilon^n$.

Exercise 4.2.8 (Moebius Bundle on S^1)**.** Show that γ_2^1, the tautological line bundle on $\mathbb{RP}(1) \simeq S^1$ is not a trivial bundle. (In fact, verify that this bundle (illustrated in Fig.6) is homeomorphic to the Moebius strip, as defined in Example 2.1.10.)

Proposition 4.2.9 (Tangent bundle of grassmannians and projective spaces)**.** The following hold:

(i): The tangent bundle of the grassmannian $T(G_k(\mathbb{R}^n))$ is isomorphic to the bundle $\hom_{\mathbb{R}}(\gamma_n^k, \gamma_n^{k,\perp})$.

(ii): The following relation holds for the tangent bundle of $\mathbb{RP}(n)$:

$$T\,\mathbb{RP}(n) \oplus \epsilon^1 \simeq \gamma^{1*} \oplus \gamma^{1*} \oplus ...\gamma^{1*} \quad (n+1) \text{ copies}$$

where $\gamma^{1*} := \hom(\gamma^1, \epsilon)$ is the dual bundle of γ^1. (This dual of the tautological line bundle on projective space is called the *canonical line bundle*).

Proof: We adhere to the notation introduced in Example 2.2.6. In particular, recall the coordinate chart diffeomorphism $\phi_E : U_E \to \hom(E, E^\perp)$, and its inverse $\psi_E := \phi_E^{-1}$.

We note that there is the structure of a *homogeneous space* on $G_k(\mathbb{R}^n)$. More precisely, fix a k-plane $P \in G_k(\mathbb{R}^n)$, called the *identity coset*. Then for any other $E \in G_k(\mathbb{R}^n)$, there exists a (non-unique) element $A \in O(n)$ such that $E = A.P$ where $A.P := \{Av : v \in P\}$. Indeed, we may write $G_k(\mathbb{R}^n) = O(n)/O(k) \times O(n-k)$ as a space of left cosets, where $O(k) \times O(n-k) = O(P) \times O(P^\perp)$ denotes the subgroup of $O(n)$ which fixes the k-dimensional plane P, called the *stabiliser* or *isotropy subgroup* of P, and denoted $\operatorname{Stab}(P)$.

For $B \in O(n)$, let us denote the automorphism $E \mapsto B.E$ of $G_k(\mathbb{R}^n)$ as L_B, called *left-translation by B*.

Now for any general $E \in G_k(\mathbb{R}^n)$, we define the isomorphism $\theta_E : T_E(G_k(\mathbb{R}^n) \to \hom(E, E^\perp)$ by setting $\theta_E := D\phi_E(E)$, where $\phi_E : U_E \to \hom(E, E^\perp)$ is a chart on $G_k(\mathbb{R}^n)$ as defined in Example 2.2.6. Our aim is to show that all these fibrewise θ_E's glue up to a bundle isomorphism $\theta : T(G_k(\mathbb{R}^n)) \to \hom(\gamma_n^k, \gamma_n^{k,\perp})$, thus establishing (i).

Claim: Let $A \in O(n)$, and $E = A.P$, for $E, P \in G_k(\mathbb{R}^n)$. Then the following diagram commutes:

$$\begin{array}{ccc} T_P(G_k(\mathbb{R}^n)) & \xrightarrow{\theta_P} & \hom(P, P^\perp) \\ \downarrow DL_A(P) & & \downarrow \mathrm{Ad}A \\ T_E(G_k(\mathbb{R}^n)) & \xrightarrow{\theta_E} & \hom(E, E^\perp) \end{array}$$

Here $\mathrm{Ad}A$ is the map defined by $\mathrm{Ad}A(S) = A \circ S \circ A^{-1} = A \circ S \circ A^t$. Note that since $A \in O(n)$, A also maps $P^\perp$ to $E^\perp$ isomorphically.

Proof of Claim: For let $T \in \hom(P, P^\perp)$. Then recall from Example 2.2.6 that

$$\psi_P(T) = \phi_P^{-1}(T) = \mathrm{graph}T = \mathrm{span}_{\mathbb{R}}(e_i \oplus Te_i)$$

if $\{e_i\}_{i=1}^k$ is an orthonormal frame for P. Thus, for $A \in O(n)$, we have:

$$\begin{aligned} L_A(\psi_P(T)) &= A.\mathrm{span}_{\mathbb{R}}(e_i \oplus Te_i) = \mathrm{span}_{\mathbb{R}}(Ae_i \oplus ATe_i) \\ &= \mathrm{span}_{\mathbb{R}}(f_i \oplus ATA^t f_i) = \mathrm{graph}(ATA^t) = \psi_E(\mathrm{Ad}A(T)) \end{aligned}$$

where $f_i := Ae_i$ is an orthonormal frame for $A.P = E$, and $\mathrm{Ad}A(T) \in \hom(E, E^\perp)$. Now this equation $L_A \circ \psi_P = \psi_E \circ \mathrm{Ad}A$ may be rewritten as $\phi_E \circ L_A = \mathrm{Ad}A \circ \phi_P$, since $\phi_E^{-1} = \psi_E$ etc. Noting that $T \mapsto \mathrm{Ad}A(T)$ is linear in T, it is its own derivative, we take the derivative of this last equality at P to get :

$$D\phi_E(L_A P) \circ D\phi_P(P) = \mathrm{Ad}A \circ D\phi_P(P)$$

which proves the claim, since $L_A P = A.P = E$, and $\theta_P = D\phi_P(P)$, $\theta_E = D\phi_E(E)$ by definition. $\square$.

To resume the proof of (i), we define the vector bundle morphism $\theta : T(G_k(\mathbb{R}^n)) \to \hom(\gamma_n^k, \gamma_n^{k,\perp})$ by setting θ on the fibre of E to be $\theta_E : T_E(G_k(\mathbb{R}^n) \to \hom(E, E^\perp)$. We need to show that this is a smooth morphism. By applying suitable left translations, and using the Claim above, it is enough to establish smoothness in the neighbourhood U_P of the identity coset P. We will need the following Lie-theoretic fact (viz. local triviality of "principal bundles":)

Fact: Let $p : O(n) \to G_k(\mathbb{R}^n)$ be the natural quotient map defined by $p(A) = A.P$. Then, there exists a smooth section to p over U_P, viz. a smooth map $\sigma : U_P \to p^{-1}(U_P)$ satisfying $p \circ \sigma = \mathrm{Id}$. We can further arrange $\sigma(P) = I$, the identity in $O(n)$.

Let $\pi : T(G_k(\mathbb{R}^n)) \to G_k(\mathbb{R}^n)$ and $\rho : \hom(\gamma_n^k, \gamma_n^{k,\perp}) \to G_k(\mathbb{R}^n)$ denote the bundle projections. Because of the Claim above, the following diagram commutes:

$$\begin{array}{ccc} U_P \times T_P(G_k(\mathbb{R}^n)) & \xrightarrow{\mathrm{Id}\times\theta_P} & U_P \times \hom(P, P^\perp) \\ \downarrow DL_\sigma(P) & & \downarrow \mathrm{Ad}\sigma \\ \pi^{-1}(U_P) = \coprod_{E \in U_P} T_E(G_k(\mathbb{R}^n)) & \xrightarrow{\theta} & \coprod_{E \in U_P} \hom(E, E^\perp) = \rho^{-1}(U_P) \end{array}$$

Here the first vertical arrow is the map $(E, v) \mapsto DL_{\sigma(E)}(P)v$, and this last vector lies in $T_E(G_k(\mathbb{R}^n)$, since $L_{\sigma(E)}P = \sigma(E).P = p(\sigma(E)) = E$ by the Fact cited above. The right vertical arrow is the map $(E, T) \mapsto \mathrm{Ad}(\sigma(E))T$, which lies in $\hom(E, E^\perp)$. Since the top horizontal map and vertical maps are clearly smooth diffeomorphisms, the bottom horizontal map is also smooth. This proves that θ is smooth, and an isomorphism of vector bundles since it is an isomorphism on each fibre. This shows (i)

To see part (ii), note that by (i) for $k = 1$, we have:

$$T(\mathbb{RP}(n)) \simeq \hom(\gamma^1, \gamma^{1,\perp})$$

Noting from (iii) of Exercise 4.2.5 that $\hom(\gamma^1, \gamma^1) = \epsilon^1$, and taking the direct sum of this on both sides:

$$\begin{aligned} T(\mathbb{RP}(n)) \oplus \epsilon^1 &\simeq \hom(\gamma^1, \gamma^{1,\perp}) \oplus \hom(\gamma^1, \gamma^1) \simeq \hom(\gamma^1, \gamma^{1,\perp} \oplus \gamma^1) \\ &\simeq \hom(\gamma^1, \epsilon^{n+1}) \simeq \oplus_{i=1}^{n+1} \hom(\gamma^1, \epsilon^1) \end{aligned}$$

which proves (ii) of the proposition. $\square$

4.3. **Pullbacks, transition functions.**

Definition 4.3.1 (Pullback)**.** Let $\pi : E \to B$ be a smooth vector bundle of rank k, and let $f : X \to B$ be a smooth map of manifolds. Consider the subset:

$$f^*E := \{(x,v) \in X \times E : \ f(x) = \pi(v)\}$$

Define the map $p : f^*E \to X$ by $p(x,v) = x$ (i.e. the restriction of the projection $X \times E \to X$). This bundle is called the *pullback* of the bundle $E \to X$ by f. Its fibre over $x \in X$ is precisely $E_{f(x)} = \pi^{-1}(f(x))$. If we let $\widetilde{f}$ denote the map $(x,v) \mapsto v$, then we have a commutative square:

$$\begin{array}{ccc} f^*E & \xrightarrow{\widetilde{f}} & E \\ \downarrow p & & \downarrow \pi \\ X & \xrightarrow{f} & B \end{array}$$

To show that it is locally trivial is an easy exercise (in fact if (Φ_i, U_i) is a bundle chart for E, then $\Phi_i \circ \widetilde{f}$ will trivialise f^*E over $f^{-1}(U_i)$).

Exercise 4.3.2.

(i): Convince yourself that pullbacks commute with all the operations on vector bundles defined above (e.g. tensor products, Whitney sums, homs, etc.).

(ii): If $f : X \to B$ is the constant map, then f^*E is a trivial bundle.

(iii): If there is a commutative square:

$$\begin{array}{ccc} F & \xrightarrow{g} & E \\ \downarrow p & & \downarrow \pi \\ X & \xrightarrow{f} & B \end{array}$$

where $p : F \to X$ is another vector bundle, and g is a smooth map which is linear on each fibre F_x, then there is a unique smooth vector bundle morphism $\theta : F \to f^*E$ (of bundles on X) such that $\widetilde{f} \circ \theta = g$. (This is called the *universal property* of pullbacks. Any bundle having this property with respect to f and E will become isomorphic to f^*E, by virtue of the "uniqueness" of the morphism θ in the definition).

(iv): If $f : M \to N$ is a smooth map of smooth manifolds, then there is a unique vector bundle morphism: $TM \to f^*TN$ (as bundles on M) which is denoted by f_*. If f is an immersion, this will realise TM as a sub-bundle of f^*TN. If f is a submersion, it will realise f^*TN as a quotient of TM.

(v): Let $f : S^n \to \mathbb{RP}(n)$ be the natural quotient map ($f(x_0, .., x_n) = [x_0 : ... : x_n]$). Compute the bundles $f^*\gamma^1$ and $f^*T\mathbb{RP}(n)$ on S^n.

There is another description of vector bundles, which is of great value in doing computations with them. This is by means of the "transition functions" of that bundle.

Definition 4.3.3 (Transition functions)**.** Let $\pi : E \to B$ be a smooth vector bundle of rank k, with bundle charts $\{(\Phi_i, U_i)\}$. Then for $x \in U_i \cap U_j$, we have the following diagram:

$$\begin{array}{ccccc} (U_i \cap U_j) \times \mathbb{R}^k & & \xrightarrow{\Phi_i \circ \Phi_j^{-1}} & & (U_i \cap U_j) \times \mathbb{R}^k \\ & \searrow p_1 & & \swarrow p_1 & \\ & & U_i \cap U_j & & \end{array}$$

where both p_1's are projections onto the first factor. Since the top arrow $\Phi_i \circ \Phi_j^{-1}$ maps $\{x\} \times \mathbb{R}^k$ linearly and isomorphically to itself, this means that this map must be $(x,v) \mapsto (x, g_{ij}(x)v)$, where $g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$ is a smooth map. The functions $\{g_{ij} : U_i \cap U_j \neq \phi\}$ are called the *transition functions* of the bundle $\pi : E \to B$.

Exercise 4.3.4.

(i): Show that the transition functions satisfy:

$$\begin{array}{ll} \text{(a)} & g_{ij}(x)g_{ji}(x) = \mathrm{Id}_{U_i \cap U_j} \text{ for all } x \in U_i \cap U_j; \quad g_{ii} = \mathrm{Id}_{U_i} \text{ for all } i \\ \text{(b)} & g_{ij}(x)g_{jk}(x)g_{ki}(x) = \mathrm{Id}_{U_i \cap U_j \cap U_k} \text{ for all } x \in U_i \cap U_j \cap U_k \end{array}$$

These are called the *cocycle conditions.*

(ii): Show that if there is an open covering $\{U_i\}$ of a manifold B, together with a collection of smooth functions $g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$, which satisfy the cocycle conditions (a) and (b) of (i) above, then there is a unique smooth bundle $\pi : E \to B$ whose transition functions are $\{g_{ij}\}$.

(iii): Compute the transition functions of all the vector bundles encountered so far. Write down the transition functions of all the bundles that were constructed out of some given bundles (e.g. Whitney sums, tensor products, exterior powers, homs, duals, pullbacks, etc.)

(iv): Let $\pi : E \to B$ be a smooth bundle of rank k, and let $\{(\Phi_i, U_i)\}$ be its bundle charts. Show that giving a smooth section s of this bundle is equivalent to giving a collection of smooth functions $s_i : U_i \to \mathbb{R}^k$ satisfying $s_i(x) = g_{ij}(x)s_j(x)$ for all $x \in U_i \cap U_j$.

(v): Let $E \to B$ and $F \to B$ be two smooth vector bundles of rank k. Without loss of generality, we can assume there is an open covering $\{U_i\}$ of B such that both bundles are trivialised over these opens. Let g_{ij} and h_{ij} be their transition functions with respect to these trivialisations. Show that E and F are isomorphic iff there exist smooth functions (called "intertwiners") $s_i : U_i \to GL(k, \mathbb{R})$ (for all i) such that:

$$g_{ij}(x)s_j(x) = s_i(x)h_{ij}(x) \quad \text{for all } x \in U_i \cap U_j$$

Use this to formulate a criterion for a bundle to be trivial, in terms of its transition functions.

4.4. **Tensors, differential forms.**

Definition 4.4.1 (Tensors and and differential forms)**.** In view of the above, we can construct various associated bundles to the tangent bundle. For example the (k,l)-tensor power bundle $\pi : (\otimes^k TM) \otimes (\otimes^l T^*M) \to M$. This is called the *bundle of tensors of type* (k,l). Its smooth sections are called *tensor fields of type* (k,l). For example, tensor fields of type $(1,0)$ are just smooth vector fields. Smooth sections of the k-th exterior power bundle $\pi : \Lambda^k T^*M \to M$, are called *differential k-forms* or simply *k-forms* on M.

Remark 4.4.2 (Contravariant and covariant notation)**.** In classical books on geometry, a tensor field of type (k,l) was said to be "contravariant of type k and covariant of type l."

Example 4.4.3 (Local descriptions of tensors and forms)**.** If (ϕ, U) is a coordinate chart on a smooth manifold of dimension n, then the bundles TM and T^*M get trivialised over U. Their local frames (=basis sections) over U arising from this chart are $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ and $\{dx_i\}_{i=1}^n$. The associated bundle $\pi : (\otimes^k TM) \otimes (\otimes^l T^*M) \to M$ then acquires the trivialisation over U, given by the frame:

$$\left\{ \frac{\partial}{\partial x_{i_1}} \otimes \frac{\partial}{\partial x_{i_2}} \otimes ... \frac{\partial}{\partial x_{i_k}} \otimes dx_{j_1} \otimes dx_{j_2} \otimes ... dx_{j_l} : 1 \le i_r \le n \quad 1 \le j_s \le n \right\}$$

Thus if α is a tensor field of type (k,l), we have the following local expression for α over U:

$$\alpha = \sum \alpha^{i_1 i_2 .. i_k}_{j_1 j_2 ... j_l} \left(\frac{\partial}{\partial x_{i_1}} \otimes \frac{\partial}{\partial x_{i_2}} \otimes ... \frac{\partial}{\partial x_{i_k}} \otimes dx_{j_1} \otimes dx_{j_2} \otimes ... dx_{j_l} \right)$$

where $\alpha^{i_1 i_2 .. i_k}_{j_1 j_2 ... j_l}$ is a smooth real valued function on U. (In classical books, the superscripts $i_1, .., i_k$ were called the *contravariant indices*, whereas the subscripts $j_1, .., j_l$ were called *covariant indices*).

We must always bear in mind that the expression in local frames for a tensor is purely local, depending on a coordinate chart. However, another coordinate chart (ψ, V) with coordinate functions y_i yielded the transformation formulas (see Exercise 3.3.18) on $U \cap V$:

$$\frac{\partial}{\partial x_i} = \sum_p \frac{\partial y_p}{\partial x_i} \frac{\partial}{\partial y_p}$$

which led to the transformation formula for 1-forms (see Example 4.1.5):

$$dx_j = \sum_m \frac{\partial x_j}{\partial y_m} dy_m$$

for the dual basis $\{dx_j\}_{j=1}^n$ of $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^n$ From these, it easily follows that:

$$\widetilde{\alpha}^{p_1 \dots p_k}_{m_1 \dots m_l} = \sum_{i_1,..,i_k,j_1,..,j_l} \left(\frac{\partial y_{p_1}}{\partial x_{i_1}} \dots \frac{\partial y_{p_k}}{\partial x_{i_k}} .. \frac{\partial x_{j_1}}{\partial y_{m_1}} \dots \frac{\partial x_{j_l}}{\partial y_{m_l}} \right) \alpha^{i_1 i_2 .. i_k}_{j_1 j_2 \dots j_l}$$

where $\widetilde{\alpha}^{p_1 \dots p_k}_{m_1 \dots m_l}$ denotes the components of α with respect to the coordinate bases arising from (ψ, V), namely the basis

$$\left\{ \frac{\partial}{\partial y_{p_1}} \otimes \frac{\partial}{\partial y_{p_2}} \otimes \dots \frac{\partial}{\partial y_{p_k}} \otimes dy_{m_1} \otimes dy_{m_2} \otimes \dots dy_{m_l} : 1 \le p_r \le n \quad 1 \le m_s \le n \right\}$$

For k-forms, we trivialise the bundle $\wedge^k(T^*M)$ by the frame:

$$\{dx_I := dx_1 \wedge dx_{i_2} \dots \wedge dx_{i_k} : i_1 < i_2 \dots < i_k; \quad 1 \le i_r \le n\}$$

We can thus express a k-form ω over the open set U as:

$$\omega = \sum_I \omega_I dx_I$$

where $I = (i_1, .., i_k)$ is a multi-index with $i_1 < i_2 < .. < i_k$, and ω_I is a smooth real-valued function on U. If one uses another chart (ψ, V), the corresponding formulas for the new components of ω on $U \cap V$ are:

$$\widetilde{\omega}_J = \sum_J A_{JI} \omega_I$$

where A_{JI} is the determinant of the $k \times k$-minor of the jacobian matrix $\left[\frac{\partial x_i}{\partial y_j}\right]$ formed by the rows $i_1, .., i_k$ and columns $j_1, .., j_k$.

Definition 4.4.4. The vector space of all smooth k-forms on the manifold M will be denoted by $\Lambda^k(M)$. Note that $\Lambda^0(M) = C^\infty(M)$ is the space of all smooth functions on M. Clearly, by pointwise scalar multiplication, $\Lambda^k(M)$ is a module over $\Lambda^0(M)$.

4.5. **Exterior derivative.**

Definition 4.5.1 (Differential of a smooth function)**.** Note that for any open set $U \subset M$, and a smooth function $f : U \to \mathbb{R}$, there is a natural 1-form on U denoted by df, satisfying: $df(X) = X(f)$ for a tangent vector $X \in T_x(M)$ and $x \in U$. It is called the *differential of* f.

Definition 4.5.2 (Exterior derivative)**.** We define the *exterior derivative* operator

$$d : \Lambda^k(M) \to \Lambda^{k+1}(M)$$

by first defining it on $\Lambda^0(M)$. An element of $\Lambda^0(M)$ is a smooth function f on M, and we define the 1-form df by:

$$df(X) := Df(x)(X) = X(f)$$

for a tangent vector $X \in T_x(M)$ (as in the definition 4.5.1). Now one extends it to $\Lambda^k(M)$ by requiring the following condition:

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

where $\deg \omega = p$ if ω is a p-form (the condition of a *skew-derivation*). It can be readily checked that this definition makes sense. Indeed, there is a local formula for d, in the exercise below, which uniquely determines the operator d globally.

Exercise 4.5.3.

(i): Let U be a coordinate patch corresponding to the local chart (ϕ, U), and let $\{x_i\}$ denote the corresponding coordinate functions. As remarked earlier in Example 4.4.3, the coordinate 1-forms dx_i form a basis of $\Lambda^1(U)$, and thus a basis for $\Lambda^k(U)$ is given by

$$\{dx_I := dx_{i_1} \wedge dx_{i_2} \wedge .. \wedge dx_{i_k}\}_I$$

where $I = (i_1, .., i_k)$ ranges over all k-multi-indices with $1 \leq i_1 < i_2 < ... i_k \leq n$. Show that if $\omega = \sum_I \omega_I dx_I$ is the expansion of ω over U, with ω_I smooth functions, then $d\omega$ is given by:

$$d\omega = \sum_{j,I} \frac{\partial \omega_I}{\partial x_j} dx_j \wedge dx_I$$

Verify that this formula is valid in any coordinate chart.

(ii): As a particular case, it is instructive to understand the exterior derivative in three dimensions. From vector calculus, we recall the notions of grad, curl and div on vector fields on 3-space. These are actually just particular cases of exterior derivatives, as will be seen below.

For $U \subset \mathbb{R}^3$ an open set, $\Lambda^0(U) = C^\infty(U)$, the vector space of smooth functions on U. Since global coordinates x_1, x_2, x_3 are available on U, we see that :

$$\begin{aligned}
\Lambda^1(U) &= \oplus_{i=1}^3 C^\infty(U) dx_i \\
\Lambda^2(U) &= C^\infty(U) dx_2 \wedge dx_3 \oplus C^\infty(U) dx_3 \wedge dx_1 \oplus C^\infty(U) dx_1 \wedge dx_2 \\
\Lambda^3(U) &= C^\infty(U) dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}$$

Using part (i) above show that for $\omega = \sum_{i=1}^3 \omega_i dx_i \in \Lambda^1(U)$, we have

$$d\omega = \left(\frac{\partial \omega_3}{\partial x_2} - \frac{\partial \omega_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial \omega_1}{\partial x_3} - \frac{\partial \omega_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

For $\omega = \omega_1 dx_2 \wedge dx_3 + \omega_2 dx_3 \wedge dx_1 + \omega_3 dx_1 \wedge dx_2 \in \Lambda^2(U)$, we have:

$$d\omega = \left(\sum_{i=1}^3 \frac{\partial \omega_i}{\partial x_i}\right) dx_1 \wedge dx_2 \wedge dx_3$$

These are the familiar formulas for grad, curl and divergence from advanced calculus, if we identify $\Lambda^1(U)$ and $\Lambda^2(U)$ with vector fields on U, and $\Lambda^0(U)$ and $\Lambda^3(U)$ with $C^\infty(U)$.

(iii): Prove that the composite:

$$\Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \xrightarrow{d} \Lambda^{k+2}(M)$$

is the zero map. In view of (ii) above, we recover the advanced calculus identities curl $\circ$ grad $=0$ and div $\circ$ curl $= 0$ in the case of $U \subset \mathbb{R}^3$ an open set.

4.6. **Pullbacks of differential forms.**

Definition 4.6.1. If $f : M \to N$ is a smooth map, we have already introduced the bundle morphism $f_* : TM \to f^*TN$, which comes (fibrewise) from the derivative $Df(x) : T_x(M) \to T_{f(x)}(N) = (f^*TN)_x$ (see (iv) of Exercise 4.3.2). By taking duals, there will be a bundle morphism (of bundles over M):

$$f^* : f^*T^*N \to T^*M$$

which, upon taking exterior powers will yield a vector bundle morphism:

$$\Lambda^k(f^*) : f^*(\Lambda^k T^*N) \to \Lambda^k T^*M$$

If $\omega \in \Lambda^k(N)$ is a k-form on N, it is a smooth section of $\Lambda^k T^*N$, so it results in a section of $f^*(\Lambda^k T^*N)$. We apply the above vector bundle morphism $\Lambda^k(f^*)$ to this pullback, and get a section of $\Lambda^k T^*M$, i.e. an element of $\Lambda^k(M)$. This form is called the *pullback* of ω, and denoted by $f^*\omega$, for notational convenience.

The only way to fathom the jargon above is to do the following:

Exercise 4.6.2.

(i): Show that for a 0-form (=function) $g : N \to \mathbb{R}$, $f^*g = g \circ f$. Also show that $f^*dg = d(f^*g)$.

(ii): Show that for smooth maps $f : M \to N$, $g : N \to X$, we have $(g \circ f)^* = f^* \circ g^*$, by using the corresponding property for pullbacks of bundles, duals etc. Also note that $\mathrm{Id}^* = \mathrm{Id}$. Thus if $f : M \to N$ is a diffeomorphism, then f^* is an isomorphism from $\Lambda^k(N)$ to $\Lambda^k(M)$ for all k.

(iii): Show that for differential forms ω, τ on N, we have $f^*(\omega \wedge \tau) = f^*\omega \wedge f^*\tau$

(iv): Show that for a smooth map $f : M \to N$, and a differential k-form ω on N, we have $d(f^*\omega) = f^*d\omega$.

(v): [Local formula for pullback] Let f be as above, and let (ϕ, U) (with coordinate functions $x_1, .., x_m$) be a chart on M, and (ψ, V) a chart (with coordinate functions $y_1, .., y_n$) on N such that $f(U) \subset V$. Represent a k-form ω on the chart V as $\omega_{|V} = \sum_{|I|=k} \omega_I dy_I$, where $\omega_I(y_1, .., y_n)$ are smooth functions. Show that $f^*\omega$ is represented on U by:

$$(f^*\omega)(x) \sum_{|I|=k} (\omega_I(f(x_1, .., x_m))df_I \quad \text{for} \ \ x \in U$$

where, for $I = (i_1, .., i_k)$, df_I denotes the k-form $df_{i_1} \wedge df_{i_2} \wedge .. \wedge df_{i_k}$ and f_i denotes the i-th component of $\psi \circ f \circ \phi^{-1}$

Definition 4.6.3 (Restricting a differential form)**.** Let N be a smooth manifold, M a smooth submanifold of N, and $i : M \hookrightarrow N$ the smooth inclusion map. If ω is a smooth k-form on N, the pullback form $i^*\omega$ on M is called *the restriction of* ω to M. It is denoted $\omega_{|M}$.

Exercise 4.6.4. Consider the torus T^2 as a submanifold of $\mathbb{R}^3$ as in Exercise 2.1.11. Let $\omega = dy \wedge dz$. Compute the restricted form $\omega_{|T^2}$ in the chart defined in (ii) of that exercise.

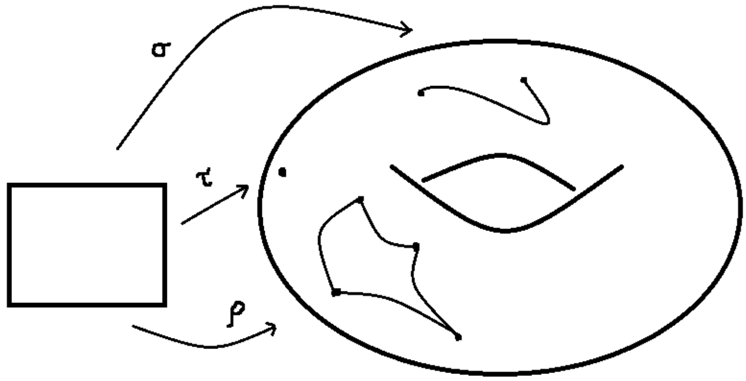

FIGURE 7. Regular 2-cubes on a torus

4.7. **Integration of differential forms.** The reason for introducing differential forms is to globalise the notion of integration from euclidean space to manifolds. For more details on this, the reader is urged to consult the book *Calculus on Manifolds* by M. Spivak.

Definition 4.7.1 (Regular k-cubes and k-chains)**.** Let M be a smooth manifold and let $\sigma : U \to M$ be a smooth map, where $U \subset \mathbb{R}^k$ is an open set containing the unit k-cube $[0,1]^k$. Then the map σ is called a *regular k-cube* in M. The free abelian group on all the regular k-cubes is called the group of *regular k-chains.* A regular k-chain is by definition a finite formal linear combination:

$$\sum_{i=1}^{m} n_i \sigma_i \quad n_i \in \mathbb{Z}$$

where σ_i are regular k-cubes in M. Note that the *image* of a regular k-cube is to be distinguished from a regular k-cube. For example, if σ is the constant map, then the image of this regular k-cube is just a point.

For the $(k-1)$-cube $[0,1]^{k-1}$ in $\mathbb{R}^{k-1}$, we define the *face maps* or *degeneracy maps*:

$$\begin{aligned} i_j^0 : [0,1]^{k-1} &\to [0,1]^k \\ (t_1, .., t_{k-1}) &\mapsto (t_1, .., 0, ..., t_{k-1}) \quad \text{0 at the } j\text{-th place} \end{aligned}$$

which may be called the *j-th back-face map.* Similarly,

$$\begin{aligned} i_j^1 : [0,1]^{k-1} &\to [0,1]^k \\ (t_1, .., t_{k-1}) &\mapsto (t_1, .., 1, ..., t_{k-1}) \quad \text{1 at the } j\text{-th place} \end{aligned}$$

which may be called the *j-th front-face map.*

Definition 4.7.2 (Boundaries of regular cubes and chains)**.** If σ is a regular k-cube in a manifold M, then $\sigma_j^0 := \sigma \circ i_j^0$ and $\sigma_j^1 := \sigma \circ i_j^1$ are regular $(k-1)$ cubes. We define the *boundary* of σ to be the chain:

$$\partial\, \sigma := \sum_{j=1}^{k} (-1)^{j-1} (\sigma_j^1 - \sigma_j^0)$$

For a regular k-chain $c = \sum_i n_i \sigma_i$, we define $\partial\, c := \sum_i n_i \partial \sigma_i$.

The reader should do some pictures to convince herself that the signs have been put in to give it the correct geometric meaning.

Exercise 4.7.3 (Boundary of a boundary)**.** Verify that $\partial \circ \partial = 0$

If ω is a smooth k-form on U, an open subset of $\mathbb{R}^k$ containing $[0,1]^k$, then we may write, by 4.4.3:

$$\omega = f(x_1,..,x_k)dx_1 \wedge ... \wedge dx_k$$

where f is a smooth function. Let τ^k denote the identity regular k-cube in $\mathbb{R}^k$. We define:

$$\int_{\tau^k} \omega = \int_0^1 ... \int_0^1 f(x_1,..,x_k)dx_1 dx_2..dx_k$$

Definition 4.7.4 (Integration on chains)**.** Let σ be a regular k-cube in a manifold M, and $\omega \in \Lambda^k(M)$ a k-form. Since $\sigma : U \to M$ is a smooth map, where U is a neighbourhood of $[0,1]^k$ in $\mathbb{R}^k$, the pullback $\sigma^*\omega$ is a k-form on U (see definition 4.6.1). Thus, we define

$$\int_\sigma \omega := \int_{\tau^k} \sigma^*\omega$$

where the right side is defined by the foregoing remarks. Finally, if $c = \sum_i n_i\sigma_i$ is a regular k-chain, we define:

$$\int_c \omega := \sum_i n_i \int_{\sigma_i} \omega$$

Proposition 4.7.5 (Stokes' Theorem)**.** Let M be a smooth manifold, ω a $k-1$-form, and c a k-chain on M. Then

$$\int_c d\omega = \int_{\partial c} \omega$$

Proof: We first prove the result for τ^k and a $(k-1)$-form ω on a neighbourhood U of $[0,1]^k$ in $\mathbb{R}^k$. In terms of the basis $(k-1)$-forms, we may write

$$\omega = \sum_{j=1}^k (-1)^{j-1}\omega_j dx_1 \wedge .. \wedge \widehat{dx_j} .. \wedge dx_k$$

(where the hat, as usual, denotes omission). Hence:

$$d\omega = \left(\sum_{j=1}^k \frac{\partial \omega_j}{\partial x_j} \right) dx_1 \wedge ... \wedge dx_k$$

Then, successively using the Fundamental Theorem of Calculus on the j-th variable, we have

$$\begin{aligned}
\int_{\tau^k} d\omega &= \int_0^1 .....\int_0^1 \left(\sum_{j=1}^k \frac{\partial \omega_j}{\partial x_j} \right) dx_1 dx_2 ... dx_k \\
&= \sum_{j=1}^k (-1)^{j-1} \int_0^1 ... \int_0^1 (-1)^{j-1}(\omega_j(x_1,..,1,..x_k) - \omega_j(x_1,..,0,..x_k))dx_1...\widehat{dx_j}..dx_k \\
&= \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 ... \int_0^1 ((i_j^1)^*\omega - (i_j^0)^*\omega \right) = \int_{\partial\tau^k} \omega
\end{aligned}$$

the last line following from the fact that $(i_j^0)^*(dx_1 \wedge ... \wedge \widehat{dx_l} ... \wedge dx_k) = 0$ for $j \neq l$, so that $(i_j^0)^*\omega = (-1)^{j-1}(\omega_j \circ i_j^0)dx_1...\widehat{dx_j}..dx_k$, and likewise for $(i_j^1)^*$.

Now for a general regular k-cube σ in a manifold M, we have

$$\begin{aligned}\int_\sigma d\omega &= \int_{\tau^k} \sigma^* d\omega = \int_{\tau^k} d(\sigma^*\omega) \\ &= \int_{\partial\tau^k} \sigma^*\omega = \sum_{j=1}^{k} (-1)^{j-1} \left(\int_{i_1^j} \sigma^*\omega - \int_{i_0^j} \sigma^*\omega \right) \\ &= \sum_{j=1}^{k} (-1)^{j-1} \left(\int_{\sigma\circ i_1^j} \omega - \int_{\sigma\circ i_0^j} \omega \right) = \int_{\partial\sigma} \omega\end{aligned}$$

Finally, for regular k-chains, we obviously have the result by linearity. □

Exercise 4.7.6. Consider the $(n-1)$-form

$$\sum_j (-1)^{j-1} x_j dx_1 \wedge ...\widehat{dx_j}.. \wedge dx_n$$

on $\mathbb{R}^n$, and let ω be the restriction (=pullback under inclusion) of this form to S^{n-1}. Compute $\int_{S^{n-1}} \omega$. (Hint: Express the sphere as the boundary of a suitable regular n-chain).

4.8. **Orientability of bundles, manifolds.**

Remark 4.8.1 (Triviality of real line bundles)**.** A line bundle $\pi : E \to B$ is trivial iff there exists a bundle atlas $\{(\Phi_i, U_i)\}$ for E such that the transition functions $\{g_{ij}\}$ corresponding to this trivialisation are all positive. Only if is clear, since a trivial bundle can be trivialised with just one chart, for which $g_{11} = 1$. Conversely, if there exists such a system of charts with $g_{ij}(x) > 0$ for all $x \in U_i \cap U_j$, all i, j, we may write $g_{ij}(x) = \exp(h_{ij}(x))$ for some smooth functions $h_{ij} : U_i \cap U_j \to \mathbb{R}$. The cocycle conditions on g_{ij} imply:

(a): $h_{ii} = 0$ for all i; $h_{ij}(x) = -h_{ji}(x)$ for all $x \in U_i \cap U_j$.
(b): $h_{ij}(x) + h_{jk}(x) + h_{ki}(x) = 0$ for all $x \in U_i \cap U_j \cap U_k$.

Extend each h_{ij} by zero outside $U_i \cap U_j$, and continue to denote them by h_{ij}. Note that the condition (b), i.e. $h_{ij} + h_{jk} + h_{ki} = 0$ may no longer hold at all points, but only at points in $U_i \cap U_j \cap U_k$.

Now take a partition of unity $\{\lambda_i\}$ subordinate to the covering $\{U_i\}$, with supp$\lambda_i \subset U_i$ for all i, and define:

$$t_i(x) = \sum_k \lambda_k(x) h_{ik}(x) \quad \text{for } x \in U_i$$

Note that this is a finite sum. It is easy to check that these functions t_i are smooth on U_i. Since supp $\lambda_k \subset U_k$, for all $x \in U_i \cap U_j$, we have $\lambda_k(x)(h_{ij}(x)+h_{jk}(x)+h_{ki}(x)) = 0$. This can be rewritten as $\lambda_k(x)(h_{ik}(x)-h_{jk}(x)) = \lambda_k(x)h_{ij}(x)$ for all $x \in U_i \cap U_j$. Then for $x \in U_i \cap U_j$, we have:

$$t_i(x) - t_j(x) = \sum_k \lambda_k(x)(h_{ik}(x) - h_{jk}(x)) = \sum_k \lambda_k(x) h_{ij}(x) = h_{ij}(x)$$

Thus the collection of functions $s_i := \exp\, t_i$ satisfy $s_i(x) = g_{ij}(x)s_j(x)$, and define a nowhere vanishing section of E. (See (iv) of 4.3.4).

Definition 4.8.2. We say that a vector bundle $\pi : E \to B$ of rank k is *orientable* if the top exterior power of this bundle, namely $\pi : \Lambda^k E \to B$ is a trivial line bundle. This is clearly equivalent to asking (see Exercise 4.3.4, part (iii) and the Remark 4.8.1 above) that there exist a set of bundle charts so that the transition functions for this line bundle, viz. $\det[g_{ij}(x)] > 0$ for all $x \in U_i \cap U_j$, and all i, j. Clearly, a line bundle is trivial iff it is orientable.We say that a manifold M is an *orientable manifold* if its tangent bundle is orientable. This is equivalent to requiring the existence of a compatible atlas so that the determinants of the jacobians $\det\left[\frac{\partial x_i}{\partial y_j}\right]$ of all the coordinate changes are positive at all points of $U_i \cap U_j$, for all i, j.

Example 4.8.3 (Tautological bundle on $\mathbb{RP}(n)$)**.** We claim that the tautological bundle γ^1_{n+1} on $\mathbb{RP}(n)$ is not an orientable (=trivial) bundle for all n. Note that since the restriction of a trivial bundle to a submanifold is also trivial, it is enough to show that the bundle γ^1_2 on $\mathbb{RP}(1)$ is not a trivial bundle. The easiest way to do this is to note that the removal of the zero-section $\mathbb{RP}(1)\times\{0\}$ from the trivial bundle $\mathbb{RP}(1)\times\mathbb{R}$ will disconnect it into the two connected components $\mathbb{RP}(1)\times(0,\infty)$ and $\mathbb{RP}(1)\times(-\infty,0)$. If there were a bundle isomorphism from γ^1_2 to the trivial bundle $\mathbb{RP}(1)\times\mathbb{R}$, it would carry the zero-section to the zero-section, and so if we denote by Z the zero- section of γ^1_2, the space $\gamma^1_2\setminus Z$ would be disconnected.

Recall that the fibre $\gamma^1_{2,x}$ over $x=[x_0:x_1]\in\mathbb{RP}(1)$ is the line given by x, i.e. the set:

$$\{(x,v)\in\mathbb{RP}(1)\times\mathbb{R}^2 : v_0x_1=v_1x_0\}$$

We claim that $\gamma^1_2\setminus Z$ is connected. There are two bundle charts (Φ_0,U_0) and (Φ_1,U_1) where

$$U_i=\{[x_0:x_1]:x_i\neq 0\}\quad i=0,1$$

Both U_0 and U_1 are homeomorphic to $\mathbb{R}$, as we have seen before, via the maps $[x_0:x_1]\mapsto\frac{x_1}{x_0}$ and $\frac{x_0}{x_1}$ respectively. The bundle charts are:

$$\begin{array}{rcl}\Phi_i:\gamma^1_{2|U_i} & \to & U_i\times\mathbb{R}\\ ([x_0:x_1],(v_0,v_1)) & \mapsto & ([x_0:x_1],v_i)\end{array}$$

for $i=0,1$. Since $v_0=\frac{x_0}{x_1}v_1$ on $U_0\cap U_1$, this shows that for $[x_0,x_1]\in U_0\cap U_1\simeq\mathbb{R}^*:=\mathbb{R}\setminus\{0\}$, (after identifying U_0 and U_1 with $\mathbb{R}$ as described above), the coordinate change map is:

$$\Phi_0\circ\Phi_1^{-1}(t,\mu)=\left(\frac{1}{t},t\mu\right)$$

Thus $\gamma^1_2\setminus Z$ is homeomorphic to two copies of $\mathbb{R}\times\mathbb{R}^*$, with the point (t,μ) in the first copy identified with $(\frac{1}{t},t\mu)$ in the second copy, for $t\neq 0$. We leave it as an easy exercise for the reader to show that this space is connected.

Exercise 4.8.4.

(i): Show that the open Moebius strip is not an orientable manifold. (If the tangent bundle of the Moebius strip were orientable, its restriction E to the central circle of the Moebius strip would also be orientable. Now, the tangent bundle T of this central circle is trivial, so orientable, and hence the quotient bundle E/T would also be orientable. However, it is easy to check that this quotient bundle E/T on the central circle ($\simeq\mathbb{RP}(1)$) is isomorphic to the line bundle γ^1_2, which is non-trivial by the above example.)

(ii): Show that a projective space $\mathbb{RP}(n)$ is an orientable manifold iff n is odd.

(iii): Show that for every manifold M, its tangent bundle is an orientable manifold.

(iv): Show that a complex manifold is an orientable manifold.

4.9. **Integration again.** Suppose M is a smooth *orientable* n-dimensional manifold, and ω is an n-form on M. We would like to be able to define $\int_M\omega$. One possibility is to "express" M as a regular n-chain, and apply the definitions about integrations on chains. This would use the rather non-trivial fact that an orientable n-manifold can be so expressed. An easier method is to use partitions of unity, coordinate charts, and apply our knowledge of integration on $\mathbb{R}^n$.

So assume that ω is an n-form of *compact support* (i.e. vanishes outside a compact subset $K\subset M$). For example, if M is compact, then all n-forms on M are compactly supported. Let $\{(\phi_i,U_i)\}$ be a smooth oriented atlas for M, viz. *such that* $\det(D(\phi_i\circ\phi_j^{-1})$ *is everywhere positive on* $U_i\cap U_j$ *for all* i,j. Let $\{\lambda_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$. Then $\lambda_i\omega$ is a compactly supported n-form, with support contained inside U_i. Then the pullback $(\phi_i^{-1})^*(\lambda_i\omega)$ is a compactly supported n-form on the open subset $\phi_i(U_i)$ of $\mathbb{R}^n$. We can extend this to a compactly supported n-form (by zero) on all of $\mathbb{R}^n$. It is therefore of the form $f\,dx_1\wedge dx_2..\wedge dx_n$ for some smooth compactly supported function f on $\mathbb{R}^n$. Thus we define the integral of this form on $\mathbb{R}^n$ to be, as before, the multiple integral $\int_{\mathbb{R}^n} f(x_1,..,x_n)dx_1...dx_n$ which makes sense by the compact support of f. Thus we finally define:

$$\int_M \omega = \sum_i \int_{\mathbb{R}^n} (\phi_i^{-1})^*(\lambda_i \omega)$$

Note that the support of ω being compact, will intersect only finitely many of the supports of λ_i, because the latter collection is locally finite from the definition of partitions of unity. Thus the sum on the right is a finite sum.

Proposition 4.9.1. Let M be an orientable, smooth n-manifold, ω a compactly supported n-form on it. Then the definition of $\int_M \omega$ is independent of the choice of oriented atlases and partitions of unity.

Proof: We first note that if $\omega = f(x_1,..,x_n)dx_1 \wedge ... \wedge dx_n$ is a compactly supported form on an open set U of $\mathbb{R}^n$, and $\theta : U \to \theta(U)$ is a smooth diffeomorphism *which preserves orientation* (viz. $\det D\theta(x) > 0$ for all $x \in U$), then

$$\begin{aligned}
\int_{\mathbb{R}^n} \omega &= \int_U f(x_1,..,x_n)dx_1...dx_n \\
&= \int_{\theta(U)} f(\theta^{-1}(\theta_1,..,\theta_n))(\det D\theta)^{-1} d\theta_1,..,d\theta_n \\
&= \int_{\theta(U)} (\theta^{-1})^*(\omega) \\
&= \int_{\mathbb{R}^n} (\theta^{-1})^*(\omega)
\end{aligned}$$

where we have used the change of variable formula from integral calculus, and (v) of Exercise 4.6.2.

Now let $\{(\phi_i, U_i)\}$ and $\{(\psi_j, V_j)\}$ be two countable atlases, both of whose coordinate change Jacobians are everywhere positive. Thus, $\det D(\phi_i \circ \psi_j^{-1})$ will have the same sign for all i, j (on $U_i \cap V_j$). Without loss of generality, assume these are all positive (otherwise compose each ψ_j with a reflection in $\mathbb{R}^n$). Let $\{\lambda_i\}$ and $\{\mu_j\}$ be partitions of unity subordinate to the coverings $\{U_i\}$ and $\{V_j\}$ respectively. Then, since $\sum_i \lambda_i = \sum_j \mu_j \equiv 1$, we have:

$$\begin{aligned}
\int_M \omega &= \sum_i \int_{\mathbb{R}^n} (\phi_i^{-1})^*(\lambda_i \omega) \\
&= \sum_i \sum_j \int_{\mathbb{R}^n} (\phi_i^{-1})^*(\lambda_i \mu_j \omega) \\
&= \sum_i \sum_j \int_{\mathbb{R}^n} (\psi_j^{-1})^*(\phi_i^{-1} \circ \psi_j)^*(\lambda_i \mu_j \omega) \\
&= \sum_i \sum_j \int_{\mathbb{R}^n} (\psi_j^{-1})^*(\lambda_i \mu_j \omega) \quad \text{by our remark in the first paragraph} \\
&= \sum_j \int_{\mathbb{R}^n} (\psi_j^{-1})^*(\mu_j \omega)
\end{aligned}$$

which proves our assertion. □

After the definition of Riemannian metrics in the sequel, we shall see how a natural everywhere non-vanishing section of $\Lambda^n(M)$ called *the volume form* gets defined on an orientable Riemannian manifold. Then every other n-form will be a multiple of this volume form by a smooth function, and integration of n-forms will be possible without using partitions of unity.

4.10. **Closed and Exact forms, de Rham's Theorem.** This subsection contains a very important result, the de Rham Theorem. However, most of the results are stated without proof. Good references for this material are R. Bott and L. Tu's book *Differential Forms in Algebraic Topology* and S. I. Goldberg's *Curvature and Homology*.

Definition 4.10.1 (Closed and exact forms)**.** Let M be a smooth manifold. We say that a differential form $\omega \in \Lambda^k(M)$ is *closed* if $d\omega = 0$. We say it is an *exact form* if $\omega = d\beta$ for some $\beta \in \Lambda^{k-1}(M)$. Since $d \circ d = 0$ by (iii) of Exercise 4.5.3, it is clear that an exact form is closed.

Example 4.10.2 (Punctured Plane)**.** Let $M = \mathbb{R}^2 \setminus \{0\}$, the punctured plane. Consider the differential 1-form $\omega \in \Lambda^1(M)$ given by :

$$\omega = \frac{-ydx + xdy}{(x^2 + y^2)}$$

It is easily verified that $d\omega = 0$, i.e. that ω is a *closed* form. However, we claim that ω *is not exact.* For let $\sigma : [0, 1] \to M$ be the regular 1-cube defined by $\sigma(t) = (\cos 2\pi t, \sin 2\pi t)$. It is easily computed that $\sigma^*\omega = 2\pi dt$, and that $\partial\sigma = 0$. Thus:

$$\begin{aligned} \int_\sigma \omega &= \int_0^1 \sigma^*\omega \\ &= \int_0^1 (2\pi dt) = 2\pi \end{aligned}$$

If ω were exact, we would have $\omega = df$ for some 0-form (=smooth function) f on M. Then Stokes Theorem 4.7.5 would imply that $\int_\sigma \omega = \int_\sigma df = \int_{\partial\sigma} f = 0$ since $\partial\sigma = 0$. Thus ω *is a closed form on the punctured plane which is not exact.* This fact is of crucial importance in potential theory in physics. If we translate the above into advanced calculus, this examples says that if there is a "source" (or "sink") at the origin of the plane, the curl free ("irrotational") vector field $v(x, y) = \left(\frac{-y}{(x^2+y^2)}, \frac{x}{(x^2+y^2)}\right)$ on the punctured plane is not conservative, i.e. is not grad f for a "potential function" f.

On the other hand if we took $M = \mathbb{R}^2$, we have learned in elementary vector calculus that a curl-free (irrotational) vector field (viz. a closed 1-form, after replacing $\frac{\partial}{\partial x}$ by dx and $\frac{\partial}{\partial y}$ by dy) on M is indeed the gradient of a smooth potential function (i.e. is an exact form). We recall that this potential function is given by taking setting $f(x, y) := \int_\sigma \omega$ where σ is *any* regular 1-cube beginning at some fixed base point (say $(0, 0)$) and ending at (x, y). (i.e. $f(x, y)$ is the "work done" by the vector field in moving a unit charge from $(0, 0)$ to (x, y) along *any path whatsoever.* So the fundamental difference between the two scenarios hinges around the fact that whereas *all paths* joining $(0, 0)$ to (x, y) on the plane are somehow equivalent, the same is untrue for the punctured plane! A topological difference is manifesting itself analytically. We will quantify this in the rest of this subsection.

Definition 4.10.3 (Cycles, boundaries, homology)**.** Let M be a smooth manifold. A *standard k-simplex* in $\mathbb{R}^k$ is defined as:

$$\Delta^k := \text{convex hull of } \{0, e_1, .., e_k\} \quad \text{in } \mathbb{R}^k$$

By a *singular k-simplex* in M, we mean a smooth map $\sigma : U \to M$, where U is a neighbourhood of the *standard k-simplex* Δ^k in $\mathbb{R}^k$. We denote the free abelian group on the set of singular k-simplices in M as $C_k(M, \mathbb{Z})$. An element of this group is called a *singular k-chain*, and there is a boundary operator:

$$\partial : C_k(M, \mathbb{Z}) \to C_{k-1}(M, \mathbb{Z})$$

defined by setting $\partial(\sigma) = \sum_{i=0}^k (-1)^i \sigma \circ \epsilon_i$, where ϵ_i is the i-th *face operator*, putting Δ^{k-1} as the face opposite to the i-th vertex of Δ^k. In other words, if we denote the vertices of Δ^k as $p_0 = 0, p_1 = e_1, ..., p_k = e_k$, then $\epsilon_i(p_j) = p_j$ for $j < i$, and $\epsilon_i(p_j) = p_{j+1}$ for $j \geq i$, and extension by convexity to Δ^{k-1}. (All this is very much in analogy with what we did with regular k-cubes above).

We have therefore have the *chain complex of regular chains on M*:

$$....C_{k+1}(M, \mathbb{Z}) \xrightarrow{\partial} C_k(M, \mathbb{Z}) \xrightarrow{\partial} C_{k-1}(M, \mathbb{Z})...$$

which just means a sequence of (free) abelian groups satisfying $\partial \circ \partial = 0$ (by redoing Exercise 4.7.3, for singular chains). The singular k-chains whose boundary is zero are called *k-cycles* viz.

$$Z_k(M) := \ker(\partial : C_k(M, \mathbb{Z}) \to C_{k-1}(M, \mathbb{Z})) = \text{abelian group of } k\text{-cycles}$$

The k-chains which are in the image of ∂ are called k-boundaries, viz.

$$B_k(M) := \mathrm{Im}\,(\partial : C_{k+1}(M, \mathbb{Z}) \to C_k(M, \mathbb{Z})) = \text{abelian group of } k\text{-boundaries}$$

Because $\partial \circ \partial = 0$, we have $B_k(M) \subset Z_k(M)$, i.e., all boundaries are cycles. *However, not all cycles are boundaries* (as we will see in Exercise 4.10.9 below) and we denote the quotient:

$$H_k(M, \mathbb{Z}) := Z_k(M)/B_k(M)$$

as the *k-th integral homology group of M*. An element of $H_k(M, \mathbb{Z})$ is denoted as an equivalence class $[c]$, and called a *homology class*, where c is a cycle. Note that $[c] = [c']$ iff $c - c' = \partial b$, for some $b \in C_{k+1}(M, \mathbb{Z})$, and in this case we say the cycles c and c' are *homologous.* Note that $B_k(M)$ and $C_k(M)$ (being subgroups of free abelian groups) are free, but $H_k(M, \mathbb{Z})$ need not be free in general. (For example, $H_1(\mathbb{RP}(2), \mathbb{Z})$ turns out to be $\mathbb{Z}_2$.)

Remark 4.10.4. The definitions above can be made for any topological space X, with a singular k-simplex being defined as a *continuous map* $\sigma : \Delta^k \to X$. It turns out that in the situation where X is a manifold, requiring singular simplices to be smooth maps certainly reduces the group of chains, cycles and boundaries, but causes no difference in homology. This is proved by constructing smoothing operators which produce chain homotopy equivalences between the smooth and continuous singular chain complexes.

We now state a couple of propositions without proof. Their proofs maybe found in standard algebraic topology books, such as *Algebraic Topology* by E.H. Spanier.

Proposition 4.10.5 (Homology of euclidean space)**.** The homology groups of $\mathbb{R}^n$ are as follows:

$$H_0(\mathbb{R}^n, \mathbb{Z}) = \mathbb{Z}, \quad H_i(\mathbb{R}^n, \mathbb{Z}) = 0 \quad \text{for } i > 0$$

Proposition 4.10.6 (Homotopy Invariance of Homology)**.** Let X and Y be topological spaces which are homotopy equivalent. That is, there are continous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the map Id_X, and $f \circ g$ is homotopic to Id_Y. Then $H_k(X, \mathbb{Z})$ and $H_k(Y, \mathbb{Z})$ are isomorphic for all k. In particular, if X and Y are homeomorphic, all their homology groups agree.

Exercise 4.10.7 (Singular homology of a point)**.** Let $X = \{p\}$, a single point. Show that the singular chain complex of X is given as:

$$...C_{2i+1}(X, \mathbb{Z}) \xrightarrow{0} C_{2i}(X, \mathbb{Z}) \xrightarrow{\mathrm{Id}} C_{2i-1}(X, \mathbb{Z}) \to ..... \to C_1(X, \mathbb{Z}) \xrightarrow{0} C_0(X, \mathbb{Z})$$

where $C_i(X, \mathbb{Z}) = \mathbb{Z}$ for all $i \geq 0$. Hence $H_i(X, \mathbb{Z}) = 0$ for $i > 0$, and $H_0(X, \mathbb{Z}) = \mathbb{Z}$. So the Proposition 4.10.5 follows from the homotopy invariance Proposition 4.10.6, since $\mathbb{R}^n$ is homotopically equivalent to $\mathbb{R}^0 = \{0\}$.

Let $\sigma : U \to M$ be a smooth map, where U is an open neighbourhood of Δ^k. In analogy with regular k-cubes, we can again define the integral of a k-form ω on a singular k-simplex σ by setting:

$$\int_\sigma \omega := \int_{\Delta^k} \sigma^* \omega$$

where the right hand integral is again a k-fold multiple integral in euclidean space over the domain $\Delta^k \subset \mathbb{R}^k$. As expected, we have the:

Proposition 4.10.8 (Stokes Theorem for singular chains)**.** Let M be a smooth n-manifold, and $c \in C_k(M, \mathbb{Z})$ a singular k-chain in M. Then, for $\omega \in \Lambda^{k-1}(M)$, we have:

$$\int_{\partial c} \omega = \int_c d\omega$$

Exercise 4.10.9. Use the Stokes Theorem 4.10.8 above to show that the singular 1-simplex σ defined in Example 4.10.2 above is a 1-cycle which is not a boundary, and hence that $H_1(\mathbb{R}^2 \setminus \{0\}, \mathbb{Z}) \neq 0$.

Thus we see that, in view of Proposition 4.10.6 that $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$ *are not homotopically equivalent.* Furthermore, this topological phenomenon is being detected by potential theory! So what is the precise connection between differential forms and homology? This is the content of the rest of this discussion.

Definition 4.10.10 (Cocycles, coboundaries, deRham complex)**.** Let M be a smooth manifold. There is a "co-chain complex" of $\mathbb{R}$-vector spaces:

$$... \to \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \to ....$$

called the *de Rham complex.* This just means that it is a sequence of $\mathbb{R}$-vector spaces in which the composite of two successive d's is zero. We define the subspaces:

$$\begin{aligned} Z^k(M) &= \ker\left(d : \Lambda^k(M) \to \Lambda^{k+1}(M)\right) = \mathbb{R}\text{-vector space of closed } k\text{-forms, or } k\textit{-cocycles} \\ B^k(M) &= \operatorname{Im}\left(d : \Lambda^{k-1}(M) \to \Lambda^k(M)\right) = \mathbb{R}\text{-vector space of exact } k\text{-forms, or } k\textit{-coboundaries} \end{aligned}$$

Since $d \circ d = 0$, we have $B^k(M) \subset Z^k(M)$. However, they are not equal, as we saw from the 1-form ω on the puctured plane in Example 4.10.2. The quotient vector space:

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

is called the *k-th de Rham cohomology of M.* It is an $\mathbb{R}$-vector space, and by its definition, depends on the smooth structure of M.

Theorem 4.10.11 (de Rham)**.** Let M be a smooth n-manifold. There is a natural (=functorial with respect to smooth maps) $\mathbb{R}$-linear map:

$$\theta : H^k_{dR}(M) \to \hom_{\mathbb{Z}}(H_k(M, \mathbb{Z}), \mathbb{R})$$

which is an isomorphism of $\mathbb{R}$-vector spaces. It is defined, via integration, as:

$$\theta([\omega])([c]) := \int_c \omega$$

That θ above is well-defined, independent of choice of representatives of the cohomology class $[\omega]$ or homology class $[c]$ is an easy consequence of Stokes Theorem 4.10.8 and that ω (resp. c) is closed (resp. a cycle). A proof of the de Rham theorem, however, is quite involved and may be found in S. I. Goldberg's book *"Curvature and Homology"*, Appendix A.

The power of this theorem is that it establishes that the de Rham cohomology, which is defined by the smooth structure of a manifold is in fact a homotopy invariant. (See Proposition 4.10.6 above). So, for example, the de Rham cohomologies of the circle and the punctured plane are identical, even though they are not even manifolds of the same dimension. Another example is the fact we knew for $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ from vector calculus namely:

Proposition 4.10.12 (Poincare Lemma)**.** For euclidean spaces we have:

$$H^0_{dR}(\mathbb{R}^n) = \mathbb{R}; \quad H^k_{dR}(\mathbb{R}^n) = 0 \quad \text{for} \quad k \geq 1$$

Proof: For $n = 0$, we see that the result is obvious. Now we can use the homotopy equivalence of $\mathbb{R}^n$ and $\mathbb{R}^0$ (Proposition 4.10.6 stated above) and the de Rham Theorem to conclude the result for any $\mathbb{R}^n$. For a direct proof, see Appendix A, §A.6 of S.I. Goldberg's *Curvature and Homology.* □

4.11. **Additional Exercises.**

Exercise 4.11.1. Consider the map:

$$\begin{array}{rcl} f : G_k(\mathbb{R}^n) & \to & G_{n-k}(\mathbb{R}^n) \\ P & \mapsto & P^\perp \end{array}$$

Show (using local coordinate charts) that f is smooth, and compute $Df(P)$ as a linear map from $T_P(G_k(\mathbb{R}^n) = \hom(P, P^\perp)$ to $T_{f(P)}(G_{n-k}(\mathbb{R}^n) = \hom(P^\perp, P)$. (*Hint:* Consider adjoints of linear maps with respect to the inner product $\langle -, - \rangle$)

Exercise 4.11.2. Compute the transition functions for the tautological line bundle $\gamma^1 \to \mathbb{RP}(n)$, which are trivialised over the open sets $V_i = p(U_i^+) = p(U_i^-)$ defined in Example 2.2.3

Exercise 4.11.3.

(i): Let M be a smooth manifold and let $\alpha \in \Lambda^p(M)$ be a closed form, and $\beta \in \Lambda^q(M)$ be an exact form. Show that $\alpha \wedge \beta \in \Lambda^{p+q}(M)$ is an exact form.

(ii): Show that the singular 1-simplices on the torus (see Example 2.1.11) given by $\sigma, \tau : [0,1] \to T^2$ with $\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$ and $\tau(t) = (2 + \cos 2\pi t, 0, \sin 2\pi t)$ are 1-cycles which are not boundaries of any 2-chains. Show that σ and τ are not homologous.

(iii): Show that the restriction of the 1-form ω defined in Example 4.10.2 to the half slit plane $V = \mathbb{R}^2 \setminus ([0, \infty) \times \{0\})$ is an exact form.

5. Metrics and Connections

5.1. Riemannian Metrics.

Definition 5.1.1 (Riemannian Metric). Let X be a smooth manifold, and let $\pi : E \to X$ be a smooth vector bundle on X. A *smooth Riemannian metric* on this bundle is a smooth section g of the tensor product bundle $E^* \otimes E^* \to X$ such that $g(x) \in E_x^* \otimes E_x^*$ is a positive definite real valued bilinear form on E_x, for each $x \in X$. It may be thought of as the family $\{g(x)\}$ of positive definite inner products on E_x, varying smoothly with $x \in X$. For the sake of convenience, we will denote $g(x)(X, Y)$ as $\langle X, Y\rangle_x$ or even $\langle X, Y\rangle$ for $X, Y \in E_x$.

A smooth Riemannian metric on the smooth tangent bundle $\pi : TX \to X$ of X is said to be a *smooth Riemannian metric on* X. A pair (X, g), where X is a smooth (or analytic) Riemannian manifold and g a Riemannian metric on X is called a *Riemannian manifold.* Usually g is suppressed in the notation when it is understood. Note that a smooth manifold can have many Riemannian metrics.

Since $\{dx_i\}_{i=1}^n$ forms a local basis for the space of 1- forms on a chart $U \subset X$, we may write g locally as:

$$g(x) = \sum_{i,j} g_{ij}(x) dx_i \otimes dx_j \quad x \in U$$

where $[g_{ij}(x)]$ is a positive definite symmetric matrix for each $x \in U$, and each g_{ij} is a smooth function on U. The right hand side is sometimes written as $\sum_{i,j} g_{ij}(x) dx_i dx_j$ for notational convenience. Clearly the matrix g_{ij} is computed by $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle_x$.

Proposition 5.1.2 (Existence of Smooth Riemannian Metrics). Let $\pi : E \to X$ be a smooth vector bundle on a smooth manifold X. Then there exists a smooth Riemannian metric on it.

Proof: We first note that if $E \to X$ is a trivial bundle, i.e. if $E = X \times V$ for some real vector space V, and $\pi = \mathrm{pr}_1$, then we may take a positive definite inner product h on V and define g to be the constant section $g(x) = h$ for all $x \in X$. For a general bundle, choose a system of bundle charts $\{(\Phi_i, U_i)\}$ such that the restricted bundles $\pi : E_{|U_i} \to U_i$ are all trivial. Let h_i be Riemannian metrics on these trivial bundles, and let $\{\lambda_i\}$ be a partition of unity subordinate to the open covering $\{U_i\}$, with $\operatorname{supp} \lambda_i \subset U_i$ (see Proposition 3.2.2). Extend the sections h_i by 0 outside U_i, and continue to denote these (discontinuous) sections of $E^* \otimes E^* \to X$ by h_i. Verify that $\sum_i \lambda_i h_i$ is a Riemannian metric on the bundle E, i.e. a smooth section of the bundle $E^* \otimes E^* \to X$. $\square$.

Note that in general, a Riemannian metric on any bundle will induce a Riemannian metric on all subbundles. Note that due to the lack of analytic partitions of unity, the above procedure for constructing a smooth partition of unity flops for the analytic case, and analytic Riemannian metrics on analytic vector bundles and analytic manifolds have to be constructed by other (global) means.

Proposition 5.1.3 (Kernel and image of a constant rank bundle morphism). Let $\theta : E_1 \to E_2$ be a smooth morphism of two smooth vector bundles over a manifold M. If the rank of the map $\theta_x : E_{1,x} \to E_{2,x}$ is constant, independent of $x \in M$, then there are natural smooth vector bundles $\ker \theta$ (resp. $\mathrm{Im}\ \theta$) whose fibre at $x \in M$ is $\ker \theta_x$ (resp. $\mathrm{Im}\ \theta_x$). The quotient bundle $E_2/\mathrm{Im}\ \theta$ is called Coker θ. If $\ker \theta$ (resp. Coker θ) is a bundle of rank 0, we call θ a *monomorphism* (resp. *epimorphism*) of vector bundles.

Proof: Since $\operatorname{rank} \theta_x$ (say $= k$) is independent of $x \in M$, each fibre $\ker \theta_x$ of $\ker \theta$ is a vector space of dimension $m - k$, where $m = \operatorname{rank} E_1$. Likewise each fibre $\mathrm{Im}\, \theta_x$ of $\mathrm{Im}\, \theta$ is a k-dimensional vector space. We need to trivialise both $\ker \theta$ and $\mathrm{Im}\, \theta$ by finding smooth frames for both in some neighbourhood of each point of M. So let $a \in M$, and let U be a neighbourhood of a on which both E_1 and E_2 are trivial. Let $\{s_1, ..., s_m\}$ be a smooth frame of $E_{1|U}$. Since $\{s_i(a)\}_{i=1}^m$ is a basis for $E_{1,a}$, the vectors $\{\theta(s_i(a))\}_{i=1}^m$ span $\mathrm{Im}\, \theta_a$, which is a vector space of dimension k, by the rank assumption on θ. By renumbering s_i if necessary, assume that $\{\theta(s_i(a))\}_{i=1}^k$ is a basis for $\mathrm{Im}\, \theta_a$. Since θ_x is smooth in x, it follows (by shrinking U if needed) that the set $\Sigma_x := \{\theta(s_i(x))\}_{i=1}^k$ is a linearly independent set in $E_{2,x}$ for all $x \in U$. Because $\mathrm{Im}\, \theta_x$ is of dimension k, this set Σ_x must be a spanning set, indeed a basis for it, for each $x \in U$. This provides a frame over U for the bundle $\mathrm{Im}\,\theta$, which is therefore locally trivial.

This also implies that $\theta(s_j(x))$ for $j = k+1, .., m$, being elements of $\mathrm{Im}\,\theta_x$, must be expressible as linear combinations of elements in the basis Σ_x. That is

$$\theta(s_j(x)) = \sum_{i=1}^{k} \lambda_{ij}(x)\theta(s_i(x)) \quad \text{for } \; j = k+1, k+2, ..., m$$

where λ_{ij} are smooth in x, because $\theta(s_l(x))$ are smooth in x for all $l = 1, .., m$.

Hence $\theta\left(s_j(x) - \sum_{i=1}^{k} \lambda_{ij}(x)s_i(x)\right) = 0$ for $j = k+1, .., m$. Setting $\sigma_j(x) := s_j(x) - \sum_{i=1}^{k} \lambda_{ij}(x)s_i(x)$, we get $(m-k)$ smooth sections $\{\sigma_j\}_{j=k+1}^{m}$ lying in the kernel of θ. The linear independence of the set $\{\sigma_j(x)\}_{j=k+1}^{m}$ is immediate from that of the set $\{s_j(x)\}_{j=k+1}^{m}$ for $x \in U$. Thus we have a local frame for $\ker\,\theta$ in U. Hence $\ker\,\theta$ is a smooth locally trivial bundle of rank $(m-k)$. $\square$

Definition 5.1.4 (Bundle exact sequences)**.** We say a sequence of bundle morphisms:

$$E_1 \xrightarrow{\theta} E_2 \xrightarrow{\phi} E_3$$

is an *exact sequence of bundles* if $\mathrm{Im}\,\theta = \ker\,\phi$. We say a sequence:

$$0 \to E_1 \xrightarrow{\theta} E_2 \xrightarrow{\phi} E_3 \to 0$$

is a *short exact sequence of vector bundles* if every pair of successive morphisms is an exact sequence. This clearly means that θ is a monomorphism, ϕ is an epimorphism, and $\mathrm{Im}\,\theta = \ker\,\phi$.

Exercise 5.1.5. Riemannian metrics on bundles have some immediate consequences.

(i): Let E_2 be a Riemannian vector bundle, and let:

$$0 \to E_1 \xrightarrow{\theta} E_2 \xrightarrow{\phi} E_3 \to 0$$

be a short exact sequence of vector bundles. Show that there is a vector bundle morphism $\psi : E_3 \to E_2$ such that $\phi \circ \psi = \mathrm{Id}_{E_3}$. Conclude that $E_2 \simeq E_1 \oplus E_3$. (Every short exact sequence of smooth bundles *splits*, since every smooth bundle E_2 is a Riemannian bundle).

(ii): Let E, F be two smooth vector bundles with Riemannian metrics g and h respectively. Define a bilinear form $\langle\,,\,\rangle$ on the bundle $\hom(E, F)$ by:

$$\langle T, S\rangle := \mathrm{trace}_E(T^*S)$$

where T^* is defined by the equation $g(T^*f, e) = h(f, Te)$. Verify that this is a Riemannian metric on the bundle $\hom(E, F)$.

Example 5.1.6 (Riemannian metric on the Sphere)**.** We recall from the Exercise 4.1.3 that the tangent bundle of S^n is the bundle:

$$TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : \; \langle x, v\rangle = 0\}$$

This makes it a "sub-bundle" of the trivial bundle $S^n \times \mathbb{R}^{n+1}$, on which we have the obvious constant Riemannian bundle metric from the Euclidean metric on $\mathbb{R}^{n+1}$. This induces the metric on TS^n by defining $\langle (x, v), (x, v')\rangle_x = \langle v, v'\rangle$ where the right side is the Euclidean inner product.

Lemma 5.1.7 (Riemannian metric on submanifolds of $\mathbb{R}^n$)**.** Let $M \subset \mathbb{R}^n$ be a submanifold of dimension m, and let $\phi : U \subset M$ be a smooth local chart (parametrisation) of M on an open set $\phi(U) \subset M$. Then, with respect to the coordinate system $(u_1, .., u_m)$ on $\phi(U)$ coming from the coordinate chart $(\phi^{-1}, \phi(U))$, the Riemannian metric induced from $\mathbb{R}^n$ is given by:

$$g(\phi(u_1, ..., u_m)) = \sum_{i,j=1}^{m} \left\langle \frac{\partial \phi}{\partial u_i}, \frac{\partial \phi}{\partial u_j} \right\rangle du_i \otimes du_j$$

where $\langle \, , \, \rangle$ denotes the euclidean inner product on $\mathbb{R}^n$.

Proof: Clearly, the coordinate basis vector $\frac{\partial}{\partial u_i}$ is, by definition $D\phi(\frac{\partial}{\partial u_i})$, which is nothing but $\frac{\partial \phi}{\partial u_i}$. Thus

$$g_{ij} = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle = \left\langle \frac{\partial \phi}{\partial u_i}, \frac{\partial \phi}{\partial u_j} \right\rangle$$

and this proves the assertion. $\square$

Exercise 5.1.8.

(i): Using the usual spherical polar coordinates (θ, ϕ) on S^2 you constructed in Exercise 2.2.10, write down the matrix $[g_{ij}(\theta, \phi)]$ on coordinate charts of S^2, for the metric induced from $\mathbb{R}^3$.

(ii): Use the parametrisation of Exercise 2.1.11 (ii), and compute the matrix $[g_{ij}(s, t)]$ of the induced euclidean metric from $\mathbb{R}^3$ on the torus. Do the same for the Moebius strip of Example 2.1.10.

Example 5.1.9 (Riemannian metric on the Grassmannian)**.** Let $M = G_k(\mathbb{R}^n)$, and $P \in M$ a point on it. By Example 2.2.6 and Proposition 4.2.9 (i), the tangent space $T_P(M)$ is naturally identified with the vector space $\hom_{\mathbb{R}}(P, P^\perp)$. If $X : P \to P^\perp$ is a linear map, we note that there is an *adjoint* of this map $X^* : P^\perp \to P$, which is defined by:

$$\langle Xx, y\rangle = \langle x, X^* y\rangle \quad \text{for } x \in P,\ y \in P^\perp$$

where $\langle \, , \, \rangle$ is the euclidean inner product on $\mathbb{R}^n$. We define for tangent vectors $X,\ Y \in T_P(M)$, the inner product

$$\langle X, Y\rangle_P := g(P)(X, Y) = \text{trace}_P(X^* Y) = \sum_{i=1}^{k} \langle Xe_i, Ye_i\rangle$$

where e_i is any orthonormal basis of the k-plane P. This is clearly symmetric and positive definite on $T_P(M)$. We need to show that $g(P)$ is a smooth function of P. We recall the trivialisation constructed for the bundle $\rho : \hom(\gamma_n^k, \gamma_n^{k,\perp}) \to G_k(\mathbb{R}^n)$ from the proof of Proposition 4.2.9.

$$U_P \times \hom(P, P^\perp) \stackrel{Ad\sigma}{\to} \rho^{-1}(U_P) = \coprod_{E \in U_P} \hom(E, E^\perp)$$

where $\sigma : U_P \to O(n)$ is a smooth section of the map (principal bundle) $p : O(n) \to G_k(\mathbb{R}^n)$ over the open set U_P. We also recall from the Claim in the proof of 4.2.9 that for $E = A.P \in G_k(\mathbb{R}^n)$, the derivative DL_A gives an isomorphism between $T_P(G_k(\mathbb{R}^n)$ and $T_E(G_x(\mathbb{R}^n)$ which corresponds to the isomorphism $\text{Ad}A : \hom(P, P^\perp) \to \hom(E, E^\perp)$. Now we have the following:

Claim: The inner product defined above is *homogeneous*. That is, if $A \in O(n)$, then for $E = A.P \in G_k(\mathbb{R}^n)$, we have: $\langle \text{Ad}A(X), \text{Ad}A(Y)\rangle_E = \langle X, Y\rangle_P$.

Proof of Claim: Clearly, if we choose an orthonormal basis $\{e_i\}$ of $E = A.P$, we have:

$$\begin{aligned} \langle \text{Ad}A(X), \text{Ad}A(Y)\rangle_E &= \text{tr}_E(AX^*A^tAYA^t) = \text{tr}_E(AX^*YA^t) = \sum_i \langle AX^*YA^te_i, e_i\rangle \\ &= \sum_i \langle YA^te_i, XA^te_i\rangle = \sum_i \langle Yf_i, Xf_i\rangle = \text{tr}_P(X^*Y) = \langle X, Y\rangle_P \end{aligned}$$

since the vectors $f_i = A^te_i$ constitute an orthonormal basis for $A^tE = A^{-1}E = P$. This proves the Claim. $\square$

Now getting back to the proof that the metric defined above is smooth, we note that if we fix any basis $\{f_i\}$ of P, then by the first paragraph above, $E \mapsto \mathrm{Ad}\sigma(E)(f_i)$ will be a smooth frame of $\hom(\gamma_n^k, \gamma_n^{k,\perp})$ all over U_P. Also the matrix $[g_{ij}]$ of the inner product with respect to this frame will be constant $(= \langle f_i, f_j\rangle_P)$ over U_P. This shows that $\langle ., .\rangle$ defined above is smooth.

Definition 5.1.10 (Isometry)**.** If $\pi_i : E_i \to M$, with $i = 1, 2$ are two vector bundles with bundle metrics g_i respectively, then a bundle isomorphism $\theta : E_1 \to E_2$ is called a *bundle isometry* if:

$$g_2(\theta X, \theta Y) = g_1(X, Y) \text{ for all } \quad X, Y \in E_{1,x} \quad \text{and all } x \in M$$

A diffeomorphism $f : M_1 \to M_2$ is called an *isometry* if the bundle map $f_* : TM_1 \to f^*TM_2$ (defined in (iv) of 4.3.2) is a bundle isometry. (Note that the pullback of a bundle with a Riemannian metric acquires a Riemannian metric in a tautological way). An immersion $i : M \to N$ of Riemannian manifolds is called a *Riemannian immersion* if the bundle morphism $i_* : TM \to i^*TN$ satisfies $g_M(x)(X, Y) = g_N(x)(i_*X, i_*Y)$ for all $X, Y \in T_xM$, all $x \in M$. A Riemannian immersion which is an embedding is called a *Riemannian embedding.*

Definition 5.1.11. Let G be a Lie group acting smoothly from the left (resp. analytically) on a smooth manifold M. That is, there is a smooth map:

$$\begin{aligned} \mu : G \times M &\to M \\ (g, x) &\to gx \end{aligned}$$

satisfying $g(g'x) = (gg')x$ for all $g, g' \in G$, $x \in M$ and $e.x = x$ for all $x \in M$. Further assume that the action is *transitive*, i.e. for each $x \in M$, the *orbit of* x, viz $Gx = \{gx : g \in G\}$, is all of M. By left translation L_g we mean the map $x \mapsto gx$, which is a diffeomorphism of M. We say that a Riemannian metric h is *left G-homogeneous* if L_g is an isometry with respect to h, for each $g \in G$. Similarly, we may define right actions, and right G-homogeneity.

Example 5.1.12 (Grassmannians)**.** We saw in Example 5.1.9 above that the Riemannian metric we constructed on $G_k(\mathbb{R}^n)$ was a homogeneous metric with respect to the natural left action of $O(n)$ on it.

Exercise 5.1.13 (Homogeneity of the Riemannian metric on S^n)**.** Note that $G = O(n)$ acts by left multiplication on S^{n-1}. Verify that the Riemannian metric on TS^{n-1} constructed in Example 5.1.6 is homogeneous with respect to this group action.

Example 5.1.14 (Invariant metrics on Lie groups)**.** Since a Lie group acts smoothly on itself by both left (resp. right translations), and the tangent bundle of a Lie group is trivial by part (iii) of the Exercise 4.1.11 using left (resp. right) translations, we can put a left (resp. right) homogeneous (=left or right invariant) metric on a Lie group by putting a positive definite bilinear form $\langle\, ,\rangle$ on the tangent space $T_e(G)$ and left (resp. right) translating it to all points of G. Note that for a Lie group, there is a *unique* left (resp. right) translation L_g (resp R_g) taking e to a point g, so these trivialisations, and invariant metrics are well-defined. This left-invariant (resp. right invariant) metric is the unique metric on G which makes each left translation (resp. right translation) an isometry.

A metric on G which is both left and right invariant is called a *bi-invariant* metric. This is the unique metric making each left *and* right translation an isometry.

We remark that a left (resp. right) G-invariant metric on G need not be right (resp. left) invariant in general. We shall see below a condition for a metric $\langle\, ,\rangle$ on the Lie algebra $\mathfrak{g} := T_e(G)$ to give rise to a bi-invariant metric. It is purely a condition on the Lie algebra.

Remark 5.1.15 (Criterion for left invariant metric to be bi-invariant)**.** Clearly, since $L_g \circ \mathrm{Ad}(g^{-1}) = R_g$, a left-invariant metric on G is bi-invariant iff it is invariant under the adjoint action Ad of G on itself (Recall $Ad(g)h := ghg^{-1} = (L_g \circ R_{g^{-1}})h$ for $g, h \in G$). Since the metric is left-invariant, and there is the formula:

$$L_{Ad(h)g} \circ Ad(h) = (Ad(h) \circ L_g) \quad \forall \ g, h \in G$$

it follows that global Ad-invariance over G is equivalent to demanding that the linear map $D(Ad(g))(e) : \mathfrak{g} \to \mathfrak{g}$ (which is denoted $(Ad(g))_*$ for ease of notation) preserves the inner product on $\mathfrak{g} = T_e(G)$.

Example 5.1.16 (Left-invariant metric which is not bi-invariant)**.** Here is an example of a left-invariant metric which is not bi-invariant. Consider $G = GL(2, \mathbb{R})$, with $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ being its Lie algebra. Verify that the bilinear form

$$\langle X, Y \rangle := \mathrm{tr}(XY^t)$$

is symmetric, in fact positive definite, and defines an inner product on $\mathfrak{g}$, called the *Hilbert-Schmidt* inner product (=the Euclidean inner product if we treat $\mathfrak{g}$ as Euclidean 4-space). If we define a left-invariant metric by left-translating as above, we get a Riemannian metric on G. This Riemannian metric is not right invariant, because $\langle ., . \rangle$ is not Ad-invariant. For let:

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

from which it follows that $Z := Ad(g)X = gXg^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ from which it follows that $\langle X, X \rangle = 1 \neq \langle Ad(g)X, Ad(g)X \rangle = 2$.

Proposition 5.1.17. If G is a compact Lie group, then it has a bi-invariant metric.

Proof: Let $d\mu$ denote the right-invariant Haar-measure on G, and let $(\, , \,)$ denote any positive definite inner product on $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$ define:

$$\langle X, Y \rangle := \int_G ((Ad(g))_* X, Ad(g)_* Y) d\mu(g)$$

Since a smooth function of G is being integrated on a compact set, this expression makes sense. Since $Ad(g)Ad(h) = Ad(gh)$ and $d\mu(g) = d\mu(gh)$, it is easily verified that $\langle ., . \rangle$ is a positive definite Ad-invariant inner product on $\mathfrak{g}$. Extending it to a left-invariant metric by left-translations as in 5.1.14 above gives a bi-invariant metric on G. $\square$

We shall be dealing with homogeneous manifolds and metrics in greater detail in a later subsection.

5.2. **Arc length on a manifold.** On a Riemannian manifold, there is a natural method of measuring distances. Let M be a smooth or analytic manifold, with a Riemannian metric $\langle \, , \, \rangle$, whose restriction to $T_x(M)$ is denoted $\langle \, , \, \rangle_x$.

Definition 5.2.1. We define a *piecewise smooth curve* $c : [0, 1] \to M$ *joining* x *and* y in M to be a continuous curve which is smooth when restricted to each sub-interval $[a_i, a_{i+1}]$ of some finite partition of $[0, 1]$ (with $0 = a_0 < a_1 < ... < a_N = 1$), and such that $c(0) = x, \ c(1) = y$. For such a piecewise smooth curve, the velocity vector $\frac{dc}{dt} := c_* \left(\frac{d}{dt}\right) = Dc \left(\frac{d}{dt}\right)$ makes sense at all points $c(t)$ except when $t = a_i$. One then defines the *arc length* of c by:

$$l(c) := \sum_{i=1}^{N-1} \int_{a_i}^{a_{i+1}} \left\langle \frac{dc}{dt}(s), \frac{dc}{dt}(s) \right\rangle_{c(s)}^{\frac{1}{2}} ds$$

Exercise 5.2.2. Show that the arc length of the composite of two piecewise smooth curves (which is also piecewise smooth) is the sum of the arc lengths of the two curves.

Definition 5.2.3 (Distance on a Riemannian manifold)**.** For $x, y \in M$, define:

$$d(x,y) = \inf\{l(c) : \ c \ \text{ is a piecewise smooth curve joining } x \text{ to } \ y\}$$

The reason we chose piecewise smooth curves in the definition above is that the composite of two smooth curves need not be a smooth curve. It is obvious that $d(x,x) = 0$ and $d(x,y) = d(y,x)$. The triangle inequality is proved as follows. Let x, y and $z \in M$. Choose $\epsilon > 0$. Then, by definition of infimum, there exists a piecewise smooth curve c_1 (resp. c_2) joining x to y (resp. y to z) with length $l(c_1) < d(x,y)+\epsilon$ (resp. $l(c_2) < d(y,z)+\epsilon$). Then for the piecewise smooth curve $c := c_1 * c_2$ which is the composite of the two, the Exercise 5.2.2 above implies that:

$$l(c) = l(c_1) + l(c_2) < d(x,y) + d(y,z) + 2\epsilon$$

Since $c(0) = x$ and $c(1) = z$, we have $d(x,z) \leq l(c) < d(x,y) + d(y,z) + 2\epsilon$. Since ϵ is arbitrary, the triangle inequality follows. The fact that $d(x,y) = 0$ implies $x = y$, as well as the fact that the metric topology from this distance function defined on M is the same as the given topology on it will follow after we discuss geodesics.

Remark 5.2.4. Note that there may not exist any piecewise smooth curve joining x to y whose arc length is equal to $d(x,y)$. For example, if we take $M = \mathbb{R}^2 \setminus \{0\}$, with the induced euclidean metric, then there is no piecewise smooth curve joining $x = (1,0)$ to $y = (-1,0)$ whose arc length is $2 = d(x,y)$. This depends on something called "geodesic completeness", which turns out to be equivalent to the completeness of M as a metric space with the distance d above. It will be proved after the discussion on geodesics.

Remark 5.2.5. It is clear that if a Lie group G is acting smoothly on a smooth manifold, and $\langle\, ,\, \rangle$ is a Riemannian metric on M which is left G-homogeneous, then the metric d defined above is also G-homogeneous (or G-invariant), that is each L_g acts as an isometry with respect to the distance d.

5.3. **Volumes on Riemannian manifolds.**

Our starting point for the discussion of volume is the following:

Proposition 5.3.1 (Euclidean volume)**.** Let P_n be the parallelopiped in $\mathbb{R}^n$ spanned by the vectors $v_1, .., v_n \in \mathbb{R}^n$. Then the volume of P_n (with respect to the usual euclidean inner product $\langle -, - \rangle$ on $\mathbb{R}^n$) is given by:

$$\mathrm{vol}(P_n) = |\det[v_{ij}]|$$

where $v_{ij} := \langle v_i, e_j \rangle$, and $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$.

Proof: We recall that P_n is defined as the convex hull of the vectors $\epsilon_1 v_1 + ... + \epsilon_n v_n$ where $\epsilon_i \in \{0,1\}$. We can prove the above formula by induction. For $n = 1$, the volume is the length of v_1, which implies the claim. By induction, assume that the formula is correct in $\mathbb{R}^{n-1}$. We can assume without loss that an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ containing the span of $v_1, .., v_{n-1}$ is $\mathbb{R}^{n-1} \times \{0\}$. This only involves an orthogonal change of basis, which does not change lengths and volumes. Now, the volume of P_n in $\mathbb{R}^n$ is the product of the volume of P_{n-1} (spanned by $v_1, .., v_{n-1} \in \mathbb{R}^{n-1} \times \{0\}$) and the length of the perpendicular from v_n to the "base" P_{n-1}, namely $|\langle v, e_n \rangle|$. So, by induction:

$$\mathrm{vol}(P_n) = \mathrm{vol}(P_{n-1}) \times |\langle v_n, e_n \rangle|$$

Since $\langle v_i, e_n \rangle = 0$ for $i = 1, 2, .., n-1$, the last column of the $n \times n$ matrix $[\langle v_i, e_j \rangle]$ consists of the (n,n)-entry $\langle v_n, e_n \rangle$. Hence, by the induction hypothesis, the absolute value of $\det[v_{ij}]$ is precisely the right hand side of the above equation, which establishes the induction step. □

More generally, if g is any positive definite inner product on a vector space V of dimension n, then the volume of a parallelopiped spanned by $\{v_i\}_{i=1}^n$ will be $|\det[v_{ij}]|$, where $v_i = \sum_j v_{ij} e_j$, with respect to the g-*orthonormal basis* $\{e_i\}_{i=1}^n$. If we further assume that $\{e_i\}$ defines positive orientation on V, the signed volume of this parallelopiped is $\det[v_{ij}]$. Thus it makes geometric sense to make the following:

Definition 5.3.2 (Volume form on an inner product space)**.** The n-form (called the *volume form* on V (with positive definite inner product $g = \langle -, - \rangle$), is defined as

$$\omega := e_1^* \wedge e_2^* \wedge ... \wedge e_n^*$$

where $(e_1, .., e_n)$ is a positively oriented orthonormal basis for V. When applied to an n-tuple of vectors $(v_1, .., v_n)$, ω computes the *signed volume* of the parallelopiped spanned by $\{v_i\}$, the sign depending on whether the tuple $(v_1, .., v_n)$ is positively or negatively oriented with respected to the orthonormal basis $(e_1, .., e_n)$.

If f_i is an arbitrary basis, and $g_{ij} := g(f_i, f_j)$, then we write $e_i = \sum_l A_{il} f_l$, and note that the relation $g(e_i, e_j) = \delta_{ij}$ leads to the relation $AgA^t = Id$, from which it follows that $\det A = (\det g_{ij})^{-1/2}$. Now since $e_i^* = \sum A_{li}^{-1} f_l^* = \sum (A^{-1})^t_{il} f_l$, we have:

$$\begin{aligned} \omega : &= e_1^* \wedge e_2^* \wedge ... \wedge e_n^* = (\det A)^{-1} (f_1^* \wedge f_2^* \wedge ... \wedge f_n^*) \\ &= (\det g_{ij})^{\frac{1}{2}} (f_1^* \wedge f_2^* \wedge ... \wedge f_n^*) \end{aligned}$$

Now let M be an orientable Riemannian manifold. By definition, there is a smooth atlas $\{(\phi_i, U_i)\}$ for M such that $\det D(\phi_i \circ \phi_j^{-1}) > 0$ for all i and j. In this case $\Lambda^n(T^*M)$ is trivial, and it would be convenient to have an everywhere positive section of this bundle whose restriction to each fibre $\Lambda^n(T_x^*M)$ gives the volume form on T_xM in accordance with the Definition 5.3.2 above. This is done as follows:

Definition 5.3.3 (Volume form on a Riemannian manifold)**.** Let (ϕ, U) be a coordinate chart on a Riemannian manifold M, with Riemannian metric $\langle \, , \, \rangle$. Let $[g_{ij}] = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ be the matrix of the metric in the resulting coordinate system. Then the *volume form* dV is given on U by the formula:

$$dV = g^{\frac{1}{2}} \, (dx_1 \wedge ... \wedge dx_n)$$

where $g := \det [g_{ij}]$. It is easy to check that this form is globally well-defined, independent of charts, and everywhere positive. (The orientability of M comes in when we want to verify this, for if $A = D(\phi_i \circ \phi_j^{-1})$ gives a change of the basis $\{\frac{\partial}{\partial x_m}\}$, the quantity $g^{\frac{1}{2}}$ changes by $|\det A|$).

Exercise 5.3.4. Let $\pi : E \to M$ be a smooth vector bundle, and let $g := \langle \, , \, \rangle$ be a Riemannian inner product on it. Then show that if $\{e_i\}_{i=1}^k$ is a local orthonormal frame (viz. orthonormal sections on some open set U), we can define a metric on the m-th exterior power $\Lambda^m E$ by requiring the set

$$\{e_I := e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_m} : I = (i_1 < i_2 ... < i_m), \;\; 1 \leq i_r \leq n\}$$

to be orthonormal all over U. Show that this indeed defines a Riemannian metric (which we continue to denote by $\langle -, - \rangle$) satisfying the formula:

$$\langle v_1 \wedge v_2 \wedge ... \wedge v_m, w_1 \wedge w_2 \wedge ... \wedge w_m \rangle = \det [\langle v_i, w_j \rangle]$$

Note that with this definition, the volume form on an orientable Riemannian manifold is the section of $\Lambda^n(T^*M)$ which is of unit length at all points, i.e. is an orthonormal frame of this trivial line bundle. (Exercise: Why is this inner-product defined on $\Lambda^m E$ positive definite? Note that this is equivalent to saying that $\det[\langle v_i, v_j \rangle] > 0$ if $\{v_i\}$ is any set of m linearly independent vectors. In the case of $m = 2$, for example, we recover the Cauchy-Schwartz inequality from positivity of this determinant.)

5.4. **Vector fields, trajectories.** Before we examine more examples of homogeneous metrics, we would like to introduce some notions which are crucial to the sequel. The first is the existence theorem on ordinary differential equations, and the definition of the exponential mapping on Lie groups.

Definition 5.4.1 (Integral curves or trajectories)**.** Let M be a smooth manifold, $U \subset M$ an open subset, and let X a smooth vector field on U. A smooth curve, i.e. a smooth map $c : (-\epsilon, \epsilon) \to U$ is said to be an *integral curve* or *trajectory* of X if:

$$\frac{dc}{dt}(t) := Dc(t)\left(\frac{d}{dt}\right) = X(c(t)) \quad \forall \quad t \in (-\epsilon, \epsilon)$$

If $c(0) = x \in U$, we say it is an integral curve *through* x.

We note here the fundamental existence theorem for ordinary differential equations.

Theorem 5.4.2 (Picard's Existence and Uniqueness Theorem for ODE's)**.** Let X be a smooth vector field on an open subset $U \subset \mathbb{R}^n$. Then:

(i): (*Existence*) For each $x \in U$, there exists an $\epsilon > 0$ (depending on x) and a smooth map $c_x : (-\epsilon, \epsilon) \to U$ such that:

$$\begin{array}{ll} (a) & \dfrac{dc_x(t)}{dt} = X(c_x(t)) \quad \text{for all} \quad t \in (-\epsilon, \epsilon) \\ (b) & c_x(0) = x \end{array}$$

Such a c_x is called a *trajectory* or *integral curve* of the vector field X.

(ii): (*Uniqueness*) If $\widetilde{c} : (-\delta, \delta) \to U$ is another smooth map satisfying (a) and (b) of (i) above, then $\widetilde{c} \equiv c_x$ on $(-\eta, \eta)$ where $\eta = \min\{\epsilon, \delta\}$.

(iii): If $K \subset U$ is a compact set, then there exists a *uniform* $\epsilon_K > 0$ depending only on K such that $c_x : (-\epsilon_K, \epsilon_K) \to U$ is defined for all $x \in K$, and satisfies (a) and (b) of (i) above.

(iv): c_x is continuous in the variable x. That is, if $x_n \to x$ for x_n, $x \in U$, and c_{x_n}, c_x are all defined on say $[-\delta, \delta]$, then $c_{x_n} \to c_x$ uniformly on $[-\delta, \delta]$.

Proof: Choose an open set $W \subset U$ such that $\overline{W} \subset U$, $\overline{W}$ is compact, and $x \in W$. Since X is smooth, both X and DX are continuous maps, so that there is a constant C such that:

$$\begin{aligned} \|X(x)\| &\leq C \quad \text{for all} \quad x \in \overline{W} \\ \|X(x) - X(y)\| &\leq C\,\|x - y\| \quad \text{for all} \quad x, y \in \overline{W} \end{aligned} \tag{6}$$

(The second equation follows from the mean value theorem, or 1st order Taylor Theorem, applied to the smooth map X.)

Choose $\epsilon > 0$ so that for the C as in (6), we have:

$$C\epsilon \leq \frac{1}{2} \quad \text{and} \quad \overline{B(x, C\epsilon)} \subset W \tag{7}$$

Now consider the space:

$$\mathcal{B}_x := \{c : [-\epsilon, \epsilon] \to \overline{W} \ : c \ \text{continuous}, \ \ c(0) = x\}$$

Define a metric on $\mathcal{B}_x$ by:

$$d(c_1, c_2) = \sup_{t \in [-\epsilon, \epsilon]} \|c_1(t) - c_2(t)\|$$

It is easily checked that $\mathcal{B}_x$ is a complete metric space with respect to this metric.

If c satisfies (a) and (b) of statement (i) of our proposition, i.e. if:

$$\frac{dc}{dt}(s) = X(c(s)) \quad \text{and} \quad c(0) = x$$

then it would follow, from the Fundamental Theorem of Calculus, that:

$$c(s) = x + \int_0^s X(c(t))dt \tag{8}$$

and conversely, if c satisfies the above integral equation, it would satisfy the ODE above, and the condition $c(0) = x$.

So consider the integral operator T on $\mathcal{B}_x$ defined by

$$(Tc)(s) = x + \int_0^s X(c(t))dt$$

Note that since $\|(Tc)(s) - x\| \leq \int_0^s \|X(c(s))\| \, ds \leq C\epsilon$, and $B(x, C\epsilon) \subset W$ by the equation (7) above, hence $(Tc)(s) \in \overline{W}$ for $s \in [-\epsilon, \epsilon]$. Also Tc is obviously continuous by its definition, and $(Tc)(0) = x$, hence $Tc \in \mathcal{B}_x$ as well. Finally, we note that $c \in \mathcal{B}_x$ satisfies (a) and (b) of (i) iff it satisfies the integral equation (8) above, i.e. iff c_x is a fixed point of T.

We now claim that T is a contraction mapping of the complete metric space $\mathcal{B}_x$. This is because for any $c, \widetilde{c} \in \mathcal{B}_x$ and any $s \in [-\epsilon, \epsilon]$ we have:

$$\begin{aligned}
d(Tc, T\widetilde{c}) &= \sup_{s \in [-\epsilon, \epsilon]} \|Tc(s) - T\widetilde{c}(s)\| \leq \sup_{s \in [-\epsilon, \epsilon]} \int_0^s \|X(c(t)) - X(\widetilde{c}(t))\| \, dt \\
&\leq C \int_0^s \|c(t) - \widetilde{c}(t)\| \, dt \leq C\epsilon \, d(c, \widetilde{c}) \\
&\leq \frac{1}{2} d(c, \widetilde{c})
\end{aligned}$$

from the equations (6) and (7) above. Thus, by applying the Banach Contraction Mapping Theorem 1.5.2, we have a fixed point c, and this is the solution sought.

Since this fixed point is unique, it follows that this c is the unique solution to the ODE above satisfying $c(0) = x$. Let us denote it by c_x, to signify that $c_x(0) = x$.

Note that for all $t \in [-\epsilon, \epsilon]$, $c_x(t) \in \overline{W}$ since $c_x \in \mathcal{B}_x$ by its construction. *A fortiori*, we have $c_x([-\epsilon, \epsilon]) \subset U$, since $\overline{W} \subset U$.

That c_x is smooth in the variable s for a fixed x follows from the integral relation:

$$c_x(s) = x + \int_0^s X(c_x(t))dt$$

because the continuity of c_x makes the integrand on the right continuous, so the relation above makes c_x a C^1 function of s, which again by the relation above, makes c_x C^2, and so on by induction (this inductive process of assuming it is C^k, and concluding it is C^{k+1} is known as "bootstrapping".)

The uniqueness in part (ii) is clear, since both c_x and $\widetilde{c}$ satisfy (a) and (b) of (i) on $[-\eta, \eta]$, they both will satisfy the same integral equation (8) for $s \in [-\eta, \eta]$.

For (iii), we note that if $K \subset U$ is any compact set, and we let $W \subset U$ be an open set such that $K \subset W \subset \overline{W} \subset U$, and let $\delta := d(K, W^c) > 0$, then $\overline{B(x, C\epsilon)} \subset W$ *for all* $x \in K$ provided we choose $C\epsilon < \frac{\delta}{2}$ say. Thus we may use such a W and such an $\epsilon > 0$ in (7), and the proof above will now apply to all $x \in K$.

For (iv), i.e. continuity in the variable x, also follows easily from the integral relation (8).

For if δ is chosen so that $C\delta < \frac{1}{2}$, we have for $s \in [-\delta, \delta]$, and d denoting the sup-distance introduced earlier on the interval $[-\delta, \delta]$:

$$\begin{aligned}
\|c_x(s) - c_y(s)\| &= \left\| \left(x + \int_0^s X(c_x(t))dt \right) - \left(y + \int_0^s X(c_y(t))dt \right) \right\| \\
&\leq \|x - y\| + C\delta d(c_x, c_y) \quad \text{for all} \quad s \in [-\delta, \delta] \\
\Rightarrow d(c_x, c_y) &\leq \|x - y\| + \frac{1}{2} d(c_x, c_y) \\
\Rightarrow d(c_x, c_y) &\leq 2\,\|x - y\|
\end{aligned}$$

by using (6) in the second line, and hence continuity in x is clear. The proposition follows. □

Smoothness in the variable x is, however, a very delicate matter, and the reader is referred to Serge Lang's *Differential Manifolds* p.66-82, for a complete proof. Also Lang's Analysis II, p. 126-138. □

Corollary 5.4.3. Let X and U be as in the Theorem 5.4.2 above. Assume that X is compactly supported, i.e., there exists a compact set $K \subset U$ such that $X(x) = 0$ for all $x \notin K$. Then, for all $x \in \mathbb{R}^n$, there exists a unique $c_x : \mathbb{R} \to \mathbb{R}^n$ satisfying (a) and (b) of (i) of the Theorem 5.4.2. A vector field all of whose trajectories can be defined for all $t \in \mathbb{R}$ is called a *complete vector field*, and this corollary says that a compactly supported vector field is complete.

Proof: Extend the vector field from U to a smooth vector field on all of $\mathbb{R}^n$ by setting $X(x) = 0$ for $x \notin K$. Continue to call it x.

For $x \in K^c$, the curve $c_x : \mathbb{R} \to \mathbb{R}^n$ defined by $c_x(t) = x$ for all $t \in \mathbb{R}$ is clearly the unique trajectory with initial point $x \in K^c$.

For $x \in K$, by (iii) of the Theorem 5.4.2, there is a uniform $\epsilon > 0$ such that the trajectory $c_x : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is defined for all $x \in K$. Thus, combining with the first para, we have $c_x : (-\epsilon, \epsilon) \to \mathbb{R}^n$ defined *for all* $x \in \mathbb{R}^n$.

For any such solution c_x with $x \in \mathbb{R}^n$, let us extend it to a solution $\widetilde{c} : (-\epsilon, 3\epsilon/2) \to \mathbb{R}^n$ as follows:

$$\begin{aligned}
\widetilde{c}(t) &= c_x(t) \;\; \text{for} \;\; t \in (-\epsilon, \epsilon) \\
&= c_{c_x(\epsilon/2)}(t - \epsilon/2) \quad \text{for} \;\; t \in (\epsilon/2, 3\epsilon/2)
\end{aligned}$$

It is easy to check by the uniqueness part (ii) of Theorem 5.4.2 that the two definitions above match on the overlap $(\epsilon/2, \epsilon)$, so that $\widetilde{c}$ is smooth. Proceeding in this manner inductively, we have it defined on all of $\mathbb{R}$. (Exercise: Is $c_x(\mathbb{R}) \subset U$ for all $x \in U$? Same question with U replaced by K?) □

Remark 5.4.4. Consider the vector field $X(t) \equiv 1 (= 1.d/dt)$ on the open interval $(0, 1)$. Then for $x \in (0, 1)$, we have $c_x(t) = x + t$, and thus $c_x(t) \in U$ only for $t \in (-x, 1 - x)$. Thus a uniform ϵ cannot be chosen for all $x \in U$.

Remark 5.4.5. The uniqueness part breaks down completely if we do not assume that X is smooth. For, define the (continuous) vector field $X(x) = x^{2/3}$ on the interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. For an $\alpha \in [0, \epsilon]$, consider the curve:

$$\begin{aligned}
c^\alpha(s) &= 0 \;\; \text{for} \;\; t \in (-\epsilon, \alpha] \\
&= (1/3)^3 (t - \alpha)^3 \quad \text{for} \;\; t \in (\alpha, \epsilon)
\end{aligned}$$

Then it is readily checked that for each $\alpha \in [0, \epsilon]$, the curve c^α satisfies $c^\alpha(0) = 0$ and $\frac{dc^\alpha(t)}{dt} = X(c^\alpha(t))$ for all $t \in (-\epsilon, \epsilon)$. Of course for $\alpha \neq \beta$, $c^\alpha \neq c^\beta$, so there is at least a one parameter family of (in fact C^1) solutions with the same initial condition, and uniqueness breaks down quite badly. Also note that the solution c^0 which is non-vanishing in any neighbourhood of 0 shows that even though $X(0) = 0$, there is a solution with initial point $x = 0$ which does *not* stay put at 0 for any $t > 0$ (unlike the constant solution $c^\epsilon \equiv 0$ on $(-\epsilon, \epsilon)$).

Definition 5.4.6. Note that by the uniqueness part (ii) of the Theorem 5.4.2, if c_x is defined on any family of open intervals I_α with $0 \in I_\alpha$ for each α, then it is also defined on $I = \cup_\alpha I_\alpha$, because the solutions will always match on any intersection $I_\alpha \cap I_\beta$ by uniqueness. (Check!). Hence there is a *maximal* open interval $J(x)$ containing 0 such that $c_x : J(x) \to U$ is defined. For example, in the case of compactly supported vector fields, we saw in Corollary 5.4.3 that $J(x) = \mathbb{R}$ for all $x \in U$.

Proposition 5.4.7. Let X and U be as in Theorem 5.4.2. Then the subset:

$$\Omega := \{(x, \lambda) \in U \times \mathbb{R} : \lambda \in J(x)\}$$

is an open subset of $U \times \mathbb{R}$ (and hence open in $\mathbb{R}^n \times \mathbb{R}$).

Proof: Let $(x, \mu) \in \Omega$, viz, $\mu \in J(x)$. Since $J(x)$ is open, choose $\lambda > \mu$ such that $[0, \lambda] \subset J(x)$. Then $c_x : [0, \lambda] \to U$ is certainly defined, since $[0, \lambda] \subset J(x)$ by definition. Since $K := c_x([0, \lambda])$ is a compact subset of U, there exists a relatively compact open set W such that

$$K \subset W \subset \overline{W} \subset U$$

Indeed, take $\delta = \frac{1}{2} d(K, U^c)$, and let $W = \{y \in U : d(y, K) < \delta\}$. One can, by the part (iii) of Theorem 5.4.2 choose a uniform $\epsilon > 0$ such that $c_z : [-\epsilon, \epsilon] \to U$ is defined for all $z \in W$.

Find a partition $0 = t_0 < t_1 < t_1 < \ldots . t_m = \lambda$ so that $t_{i+1} - t_i < \epsilon$ for all i, and set $y_i := c_x(t_i)$. By the sup-norm estimate in Theorem 5.4.2,

$$d(c_z, c_x) = \sup_{0 \leq s \leq \epsilon} \|c_x(s) - c_z(s)\| \leq 2 \|x - z\|$$

for all $x, z \in \overline{W}$, so there is a neighbourhood N_0 around $x = c_x(0) = c_x(t_0)$ such that $\|c_z(t_1) - c_x(t_1)\| < \frac{1}{2}\delta$ for all $z \in N_0$. Since $c_x(t_1) \in K$, it follows that $d(c_z(t_1), K) < \frac{1}{2}\delta < \delta$, so that $c_z(t_1) \in W$ for $z \in N_0$. Thus for $z_1 := c_z(t_1)$, we have $c_{z_1}(t)$ defined for all $t \in [-\epsilon, \epsilon]$. But, by uniqueness:

$$c_{c(z_1)(t)}(s) = c_z(s + t)$$

so that we have c_z defined on $[0, t_2] \subset [0, t_1 + \epsilon]$ for all $z \in N_0$. Now, again, there is a neighbourhood N_1 of $y_1 = c_x(t_1)$ such that $c_z(t_2) \in W$ for all $z \in N_1$. Shrinking N_0 if necessary to ensure that $c_z(t_1) \in N_1$ for all $z \in N(0)$, we can repeat the above argument to show that for $z \in N_0$, we have c_z defined on $[0, t_3] \subset [0, t_2 + \epsilon]$. Continuing finitely many times, we have for some neighbourhood N_0 of x that c_z is defined on $[0, t_m] = [0, \lambda]$ and lies in W for all $z \in N_0$. Thus the neighbourhood $N_0 \times (\mu - \delta, \lambda)$ of (x, μ) is contained in Ω, for every $\delta > 0$, and Ω is open. $\square$

Exercise 5.4.8. Show that for X and U as in Theorem 5.4.2, if $c_x(t_1) = c_x(t_2)$ for some $t_1 \neq t_2 \in J(x)$, then c_x is a *periodic trajectory*, viz.

$$c_x(t) = c_x(t + t_2 - t_1)$$

for all t, and in particular, $J(x) = \mathbb{R}$.

Definition 5.4.9. Let X and U be as in the statement of the Theorem 5.4.2, and let Ω be the open set defined in the Proposition 5.4.7. Then define:

$$\begin{aligned} \Phi : \Omega &\to U \\ (x, t) &\mapsto c_x(t) \end{aligned}$$

This map is called the *flow* of the vector field X. $\Phi(x, t)$ is often denoted $\Phi_t(x)$. Clearly $\Phi_0 = \Phi(-, 0) = \mathrm{Id}_U$.

Exercise 5.4.10. Show that $\Phi_{t+s}(x) = (\Phi_t \circ \Phi_s)(x)$, whenever both sides make sense. In particular, if X is a complete vector field, then $\Phi_t : U \to U$ is defined *for all* $t \in \mathbb{R}$, and is a diffeomorphism with inverse Φ_{-t}.

The Theorem 5.4.2 above globalises to manifolds as follows:

Theorem 5.4.11 (Existence and Uniqueness of Flows on Manifolds)**.** Let M be a smooth manifold and X be a smooth compactly supported vector field on M. (For example, if M is itself compact, then X can be any smooth vector field on M). Then there exists a smooth map $\Phi : M \times \mathbb{R} \to M$ satisfying:

(i): $\Phi(x, 0) = x$ for all $x \in M$.

(ii): For each $x \in M$, the smooth curve $c_x(t) := \Phi(x, t)$ is a trajectory for X satisfying $c_x(0) = x$. In particular, $\Phi(x, t) = x$ for $x \notin \operatorname{supp} X$

(iii): $\Phi(-, t) \circ \Phi(-, s) = \Phi(-, s + t)$ for all $s, t \in \mathbb{R}$. Thus each $\Phi(-, t)$ is a smooth diffeomorphism, with inverse $\Phi(-, -t)$.

The family $\{\Phi(-, t)\}$ is called the *one-parameter family of diffeomorphisms* or *flow* corresponding to the vector field X.

Proof: Let $K \subset M$ denote the compact support of the vector field X. We first make the following:

Claim: Let $x \in K$. There exists a neighbourhood $U(x)$ of x diffeomorphic to an open set in $\mathbb{R}^n$, and open subsets $V(x) \subset W(x) \subset \overline{W(x)} \subset U(x)$, with $x \in V(x)$, and such that:

(i): There is a smooth map $\Phi_x : U(x) \times [-\epsilon_x, \epsilon_x] \to U(x)$ with $\Phi_x(V(x) \times [-\epsilon_x, \epsilon_x]) \subset W(x)$.

(ii): For each $y \in V(x)$, the curve $c_y(t) := \Phi_x(y, t)$ lies entirely in $W(x)$ for $t \in [-\epsilon_x, \epsilon_x]$, and is a trajectory for the vector field X.

Proof of Claim: Let $U(x)$ be a coordinate chart around $x \in K$, which is therefore diffeomorphic to an open subset of $\mathbb{R}^n$. Let $W(x)$ be a relatively compact neighbourhood of x satisfying $\overline{W(x)} \subset U(x)$. Let λ be a smooth bump function with $\operatorname{supp} \lambda \subset U(x)$, and with $\lambda \equiv 1$ on $W(x)$. Consider the smooth vector field $Y := \lambda X$, which has compact support $\operatorname{supp} Y \subset K \cap \operatorname{supp} \lambda \subset U(x)$. By (iii) of Theorem 5.4.2, and the Definition 5.4.9, there is an $\epsilon_x > 0$ and a smooth flow:

$$\Phi_x : U(x) \times [-\epsilon_x, \epsilon_x] \to U(x)$$

which is a flow for Y all over $U(x)$.

By reducing ϵ_x if necessary, we may assume that the entire trajectory for $t \in [-\epsilon_x, \epsilon_x]$ through x lies within $W(x)$, viz. $\Phi_x(\{x\} \times [-\epsilon_x, \epsilon_x]) \subset W(x)$. This means that $\Phi_x^{-1}(W(x))$ is an open neighbourhood of $\{x\} \times [-\epsilon_x, \epsilon_x]$. By the tube lemma in topology, there is an open neighbourhood $V(x)$ of x such that $V(x) \times [-\epsilon_x, \epsilon_x] \subset \Phi_x^{-1}(W(x))$. This means that both assertions of (i) are satisfied.

Since $Y = \lambda X \equiv X$ on $W(x)$, and for $y \in V(x)$, $\Phi_x(y, t) \in W(x)$ for all $t \in [-\epsilon_x, \epsilon_x]$. it follows that the trajectory $c_y(t) := \Phi_x(y, t)$ for Y is a trajectory for X whenever $y \in V(x)$. This proves (ii) and the Claim is established. $\square$

To resume the proof of the theorem, we have an open covering of K by the open sets $V(x)$ of the Claim above, for $x \in K$. By the compactness of K that $K \subset \cup_{i=1}^m V(x_i)$, for some $x_i \in K$. Let $\epsilon = \min_{1 \le i \le m} \epsilon_{x_i}$. Define the map

$$\begin{array}{rcl} \Phi : M \times [-\epsilon, \epsilon] & \to & M \\ (x, t) & \mapsto & x \quad \text{for } x \notin K \\ (x, t) & \mapsto & \Phi_{x_i}(x, t) \quad \text{for } x \in V(x_i) \end{array}$$

This is well defined because, by (ii) of the Claim above, Φ_{x_i} and Φ_{x_j} are both flows for X on the intersection $V(x_i) \cap V(x_j)$, and hence agree there by uniqueness part (ii) of the last theorem 5.4.2. Now we extend Φ to $M \times \mathbb{R}$ exactly as we did in Corollary 5.4.3. The last group theoretic property of the family $\Phi_t = \Phi(-, t)$ follows from Exercise 5.4.10. $\square$

5.5. **Lie algebras, exponential mapping.**

Definition 5.5.1 (Commutator of vector fields)**.** Let M be a manifold, and let X, Y two smooth vector fields on an open subset $U \subset M$. We define another vector field denoted $[X, Y]$ on U by the prescription:

$$[X, Y](x)f := X(x)(Y(f)) - Y(x)(X(f)) \quad \text{for } x \in M, \ f \in \mathcal{O}_{M,x}$$

Note that $X(f)$ and $Y(f)$ smooth functions on some neighbourhood of x, and are thus elements of $\mathcal{O}_{M,x}$. Hence the tangent vectors $Y(x)$ and $X(x)$ can act on them. The local formula (iv) of the Exercise 5.5.2 below will show that this definition doesn't depend on germ representatives chosen etc.

Exercise 5.5.2. For vector fields X, Y, Z on an open subset $U \subset M$, show that:

(i): $[X, Y] = -[Y, X]$ (anticommutativity)

(ii): $[X, fY] = f[X, Y] + X(f)Y$ for a smooth function f on M.

(iii): $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi Identity)

(iv): Using smooth charts, show that if $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$ in the local basis of coordinate fields, then the local expression for $[X, Y]$ is:

$$[X, Y] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

and hence $[X, Y]$ is a smooth vector field if X, Y are smooth vector fields.

(v): If $f : M \to N$ is a smooth diffeomorphism, then $[f_* X, f_* Y] = f_*[X, Y]$. (Note that though $f_*(X)$ makes sense for a tangent vector for any smooth map f, it doesn't make sense for X a vector field, unless f is a diffeomorphism.)

(vi): For the coordinate vector fields $\frac{\partial}{\partial x_i}$ defined on U (for some chart (ϕ, U)), we have: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for all $1 \leq i, j \leq n$.

Definition 5.5.3 (Lie algebra, invariant vector fields)**.** We recall from Definition 3.3.25 the *Lie algebra* of a Lie group G, viz. the tangent space at the identity $T_e(G)$, and denoted Lie(G) or $\mathfrak{g}$. Given a tangent vector $X \in \mathfrak{g}$, one can produce the *left invariant vector field* generated by X, denoted $\widetilde{X}$, by the formula:

$$\widetilde{X}(x) = DL_x(e)(X) = (L_x)_*(e)(X)$$

It clearly satisfies $\widetilde{X}(L_y(x)) = X(yx) = L_{y*}(x)(\widetilde{X}(x))$, and also these are all the vector fields satisfying this condition. Such vector fields are called *left invariant vector fields* on G. Similarly, one can define right invariant vector fields. Thus there is a 1-1 linear isomorphism between $\mathfrak{g}$ and the $\mathbb{R}$-vector space of all left (resp. right) invariant vector fields on a Lie group. Similarly, we can define *left and right invariant* 1-forms on G, and these are in corespondence with the dual $\mathfrak{g}^*$ of the Lie algebra. As a consequence, one sees that the tangent and cotangent bundles of a Lie group are trivial bundles.

From the definition of left invariant vector fields and (v) of Exercise 5.5.2 above, it follows that the commutator of two left invariant vector fields is a left invariant vector field, and likewise for right invariant vector fields. Thus the tangent space $\mathfrak{g} = T_e(G)$ is a *real Lie algebra*, i.e. is an $\mathbb{R}$-vector space with an $\mathbb{R}$-bilinear pairing $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

(i): $[X, Y] = -[Y, X]$

(ii): $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (*Jacobi identity*).

If one takes the vector space $\chi(M)$ of all smooth vector fields on a smooth manifold, and define $[\,,\,]$ to be the commutator (of vector fields), $\chi(M)$ will become an infinite-dimensional Lie algebra.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We again recall from Examples and Exercises 3.3.26, 3.3.27, and 3.3.28 that the Lie algebra of $GL(n, \mathbb{F})$ is $\mathfrak{gl}(n, \mathbb{F})$, and the Lie algebra of $SL(n, \mathbb{F})$ is the space of all trace zero elements in $\mathfrak{gl}(n, \mathbb{F})$, denoted $\mathfrak{sl}(n, \mathbb{F})$. The the Lie algebra of $O(n)$ (and $SO(n)$) is the set of all trace zero skew-symmetric real $(n \times n)$-matrices $\mathfrak{o}(n)$. Finally, the Lie algebras of $U(n)$ is $\mathfrak{u}(n)$, the space of all skew-hermitian complex $(n \times n)$-matrices, whereas that of $SU(n)$ is $\mathfrak{su}(n)$, the set of trace zero elements of $\mathfrak{u}(n)$.

Theorem 5.5.4. Let G be a Lie group. Then there exists a smooth mapping:

$$\exp : \mathbb{R} \times \mathfrak{g} \to G$$

such that for a fixed element $X \in \mathfrak{g}$, the curve $t \mapsto \exp(tX)$ is a group homomorphism from the additive group of reals to the group G. Furthermore, this smooth curve is a trajectory of the left invariant vector field $\widetilde{X}$ generated by X.

Proof: Let $X \in \mathfrak{g}$. By applying cutoff functions, local charts and the Theorem 5.4.2 above, one can find a smooth curve $c : (-\epsilon, \epsilon) \to G$ such that $c(0) = e$, and such that for $t \in (-\epsilon, \epsilon)$, we have the equation:

$$\frac{dc}{dt}(t) = \widetilde{X}(c(t))$$

where $\widetilde{X}$ is the left-invariant vector field generated by X. Let $x = c(\frac{\epsilon}{2})$. We first remark that for $s, t \in (\epsilon, \epsilon)$, such that $s + t \in (-\epsilon, \epsilon)$, we have $c(s + t) = c(s)c(t)$. This is because both the curves $c_1(t) := c(s + t)$ and $c_2(t) := c(s)c(t)$ take the value $c(s)$ at $t = 0$, and both are trajectories for the vector field $\widetilde{X}$. Thus by uniqueness of trajectories, they agree wherever they are defined. For $s \in (\frac{\epsilon}{2}, \frac{3\epsilon}{2})$ define $\phi(s) := L_x(c(s - \frac{\epsilon}{2}))$. We claim that $\phi(t) = c(t)$ for $\frac{\epsilon}{2} < t < \epsilon$. This is because for each such t, we have $\frac{dc}{dt}(t) = \widetilde{X}(t)$, and also

$$\begin{aligned} \frac{d\phi}{dt}(t) &= L_{x*}\left(\frac{dc}{dt}\left(t - \frac{\epsilon}{2}\right)\right) \\ &= L_{x*}\left(\widetilde{X}\left(c\left(t - \frac{\epsilon}{2}\right)\right)\right) = \widetilde{X}\left(L_x c\left(t - \frac{\epsilon}{2}\right)\right) \\ &= \widetilde{X}(\phi(t)) \end{aligned}$$

since $\widetilde{X}$ is left invariant. Also, since $x = c(\frac{\epsilon}{2})$ we have $\phi(\frac{3\epsilon}{4}) = L_x c(\frac{\epsilon}{4}) = c(\frac{\epsilon}{2})c(\frac{\epsilon}{4}) = c(\frac{3\epsilon}{4})$ by the remark in the beginning. Thus we have extended the trajectory to $(-\epsilon, \frac{3\epsilon}{2})$. We may proceed in this fashion to extend it to all of $\mathbb{R}$.

The fact that this smooth curve becomes a group homomorphism follows from the remark at the outset (i.e. uniqueness). We denote $c(t)$ by $\exp(tX)$, because again using uniqueness one easily shows the trajectory corresponding to λX takes the same value at time t as the trajectory corresponding to X at time λt. The smoothness of $\exp : \mathbb{R} \times \mathfrak{g} \to G$ follows from the fact that the trajectory of a smooth vector field varies smoothly if the vector field is varied smoothly and the initial condition is varied smoothly (which is a part of Theorem 5.4.2, and was only stated there without proof). □

Exercise 5.5.5.

(i): Verify that the infinite-series definition of the exponential map given in Definition 3.1.7 for the various linear Lie groups of Exercise 3.3.28 agrees with the exponential map defined above. This is equivalent to saying that the one-parameter group $\exp(tX)$ defined earlier, corresponding to $X \in \mathfrak{g}$, is the trajectory of the left invariant vector field $\widetilde{X}$ passing through the identity element $e \in G$ at $t = 0$. All other trajectories for $\widetilde{X}$ are obtained by left translating this one-parameter group.

(ii): Show (by computing the derivative of exp at 0 and using the inverse function theorem) that the map $\exp : \mathfrak{g} \to G$ maps a neighbourhood of 0 in $\mathfrak{g}$ diffeomorphically onto a neighbourhood of e in G. Thus we have a natural chart around e in a Lie group (called the *exponential chart*, or *exponential coordinates*), and by composing with left translations we get charts around each point in G.

(iii): For $G = S^1 = SO(2)$, note that the exponential map $\exp : \mathbb{R} \to S^1$ may also be represented by $t \mapsto e^{it}$. Show that for an irrational number α, the subalgebra :

$$\mathfrak{h} = \{(t, \alpha t) : \ t \in \mathbb{R}\}$$

exponentiates to the subgroup of $T^2 = S^1 \times S^1$ given by the skew line that was defined in 3.1.5.

Remark 5.5.6 (When is the exponential map surjective?)**.** We saw in Exercise 3.1.13, by considering $GL(2, \mathbb{R})$, that exponential map of a Lie group need not be surjective. On the other hand we also noted in Exercise 3.1.11 (using the Jordan Canonical Form) that for $G = GL(n, \mathbb{C})$, it is surjective. It turns out that the exponential map of any compact connected Lie group is surjective.

Exercise 5.5.7. Give an example of a compact manifold M and a smooth vector field X on it such that all trajectories of X are non-compact.

5.6. **Lie Derivative.** A smooth vector field X on a manifold *apriori* only gives us a method of differentiating functions, to produce other functions. But it is possible to differentiate other objects, such as tensors, and in particular differential forms with respect to X.

We first note that even if X is not a compactly supported vector field, we can multiply it with a cut-off function λ_x which is identically $= 1$ in some neighbourhood $W(x)$ of x (see Corollary 3.2.4), and get a compactly supported vector field which agrees with X on $W(x)$. Denote the flow (=1-parameter family of diffeomorphisms) of this vector field (as guaranteed by Theorem 5.4.11) by ϕ_t^X. It is easy to see that this well-defined in a neighbourhood of x and small t, regardless of choice of cut-off function.

Definition 5.6.1 (Lie derivative)**.** For $f \in \Lambda^0(M)$, a smooth function, define:

$$L_X f = X(f)$$

which clearly means, since $\frac{d\phi_t^X(x)}{dt}(0) = X(x)$, that

$$(L_X f)(x) = \lim_{t \to 0} \frac{f(\phi_t^X(x)) - f(x)}{t}$$

so that:

$$L_X f = \lim_{t \to 0} \frac{(\phi_t^X)^* f - f}{t}$$

If Y is another vector field, then ϕ_{-t}^X being a diffeomorphism, there exists the vector field $(\phi_{-t}^X)_* Y$, as explained in (v) of Exercise 5.5.2, whose value at x is precisely $D\phi_{-t}^X(\phi_t^X(x))(Y(\phi_t^X(x))$. Thus we define:

$$L_X Y := \lim_{t \to 0} \frac{(\phi_{-t}^X)_* Y - Y}{t}$$

Similarly, for a 1-form ω, we define:

$$L_X \omega := \lim_{t \to 0} \frac{(\phi_t^X)^* \omega - \omega}{t}$$

Now we extend L_X to tensors of type (k, l) by requiring that it acts as a *derivation*, that is, for two tensors τ, σ, we have:

$$L_X(\tau \otimes \sigma) = L_X \tau \otimes \sigma + \tau \otimes L_X \sigma$$

Since every tensor is locally the sum of decomposable tensors, this definition makes sense.

Remark 5.6.2. We examine the reason for the opposite signs of t in the definitions of L_X for 1-forms and vector fields is that $\omega(Y)$ is a function. For brevity, let us denote ϕ_t^X by ϕ_t. By the definition of pullback of a form ω under a map f, we have $[(f^*\omega)](x)Z = \omega(f(x))(Df(x)Z)$ for a tangent vector $Z \in T_x(M)$. Thus if we let the vector field $Z := (\phi_{-t})_*Y$, then $Z(x) = D\phi_{-t}(\phi_t(x))Y(\phi_t(x))$. Thus the number $(\phi_t^*\omega)((\phi_{-t})_*Y)(x)$ is the same as the number $\omega(\phi_t(x))(D\phi_t\, D\phi_{-t})(\phi_t(x))Y(\phi_t(x))$, which is exactly $\omega(\phi_t(x))(Y(\phi_t x)$. This is, of course, the function $\phi_t^*(\omega(Y))$. In other words, we have the relation of functions:

$$((\phi_t)^*\omega)(\phi_{-t\,*}Y) = (\phi_t)^*(\omega(Y))$$

and the formula for the Lie derivatives of vector fields, 1-forms, and functions match up to give the following:

Proposition 5.6.3. Let ω be a 1-form and X, Y smooth vector fields on M. Then the following formulas hold:

(i):

$$L_X(\omega(Y)) = X(\omega(Y)) = (L_X\omega)(Y) + \omega(L_XY)$$

(ii):

$$L_XY = [X, Y]$$

Proof: As before, let us denote ϕ_t^X by ϕ_t for brevity. From the formula deduced in the remark above:

$$((\phi_t)^*\omega)(\phi_{-t\,*}Y) = (\phi_t)^*(\omega(Y))$$

it follows by taking the derivative $\frac{d}{dt}$ at $t = 0$ that:

$$(L_X\omega)(Y) + \omega(L_XY) = X(\omega(Y)) = L_X(\omega(Y))$$

which implies (i).

For (ii), note that for a smooth function f, and a point $x \in M$, we have:

$$\begin{aligned}[(\phi_{-t\,*}Y)(x)]f &= [D\phi_{-t}(Y(\phi_t(x))]f \\ &= Y(\phi_t(x))(f \circ \phi_{-t})\end{aligned}$$

so that :

$$[(\phi_{-t\,*}Y)(x)](f \circ \phi_t) = Y(\phi_t(x))f = (Yf)(\phi_t(x))$$

which implies that

$$\begin{aligned}(Yf) \circ \phi_t - Y(f \circ \phi_t) &= (\phi_{-t\,*}Y - Y)(f \circ \phi_t) \\ &= t(L_XY)(f \circ \phi_t) + o(t^2) \\ &= t(L_XY)f + o(t^2)\end{aligned}$$

where the second equality results from the definition of Lie derivative, and the third from the fact that $f \circ \phi_t = f + tXf + o(t^2)$. Adding and subtracting Yf to the left hand side of the first equation above, the equation above may be rewritten as:

$$\left(\frac{(Yf) \circ \phi_t - Yf}{t}\right) - \left(\frac{Y(f \circ \phi_t) - Yf}{t}\right) = (L_XY)f + o(t)$$

Taking the limit as $t \to 0$, we get $L_XY = X(Yf) - Y(Xf) = [X, Y]f$. $\square$

Corollary 5.6.4 (The adjoint action of a Lie group)**.** Let G be a Lie group, and let $X \in \mathfrak{g} = T_e(G)$. Then we have the map $\mathrm{Ad}(\exp(tX)) : G \to G$ which takes g to $\exp(tX)g(\exp(-tX)$. Its derivative $\mathrm{Ad}(\exp tX)_* = D(\mathrm{Ad}(\exp tX)$ is an automorphism of $\mathfrak{g}$. We have the equation:

$$\lim_{t \to 0} \frac{\mathrm{Ad}(\exp tX)_* - \mathrm{Id}}{t} = \mathrm{ad}\, X := [X, \_]$$

Proof: Let $Y \in \mathfrak{g}$, and let $\widetilde{Y}$ (resp. $\widetilde{X}$) denote the left-invariant vector fields corresponding to Y (resp. X). Note that $\exp(tX)$ is the trajectory of the vector field $\widetilde{X}$ passing through the identity element e at $t = 0$. This follows easily from the fact that the exponential map is a homomorphism. We need to see what the trajectories passing through other points are. We claim that $R_{\exp tX}(x) = x.\exp(tX)$ is the trajectory of $\widetilde{X}$ passing through x at $t = 0$ (Recall that this is the same as $L_x(\exp(tX))$, as noted in (i) of Exercise 5.5.5). This is because:

$$\frac{d(x \exp tX)}{dt}(0) = L_{x*}\left(\frac{d \exp tX}{dt}(0)\right) = L_{x*}(X) = L_{x*}(\widetilde{X}(0)) = X(x)$$

by the left invariance of $\widetilde{X}$. Thus, the flow of $\widetilde{X}$ is given by $\phi_t^{\widetilde{X}} = R_{\exp tX}$, which we abbreviate as ϕ_t. Consequently:

$$\begin{aligned}
(\phi_{-t,*}\widetilde{Y})(e) &= DR_{\exp(-tX)}\widetilde{Y}(e.\exp tX) = DR_{\exp(-tX)}\widetilde{Y}(\exp tX) \\
&= DR_{\exp(-tX)}DL_{\exp tX}(Y) = D(R_{\exp(-tX)} \circ L_{\exp tX})(Y) \\
&= D(Ad(\exp tX))Y = \mathrm{Ad}(\exp tX)_* Y
\end{aligned}$$

since the left invariance of $\widetilde{Y}$ implies $\widetilde{Y}(\exp tX) = DL_{\exp tX}(Y)$.

From (ii) of the previous proposition it follows that:

$$[X, Y] = [\widetilde{X}, \widetilde{Y}](e) = L_{\widetilde{X}}\widetilde{Y}_{|e} = \lim_{t \to 0} \frac{\mathrm{Ad}(\exp tX)_* Y - Y}{t}$$

which proves our assertion. □

Proposition 5.6.5 (Characterisation of one-parameter subgroups)**.** A smooth homomorphism:

$$\eta : \mathbb{R} \to G$$

into a Lie group G (or sometimes the image of η) is called a 1-*parameter subgroup* of G. Then η is a one-parameter subgroup of G iff $\eta(t) = \exp tX$ for some $X \in \mathfrak{g} = T_e(G)$.

Proof: The "if" part is clear. So suppose $\eta : \mathbb{R} \to G$ is a one parameter subgroup of G. Then let us denote:

$$X := \frac{d\eta}{dt}(0)$$

which is an element of $\mathfrak{g}$. We claim that $\eta(t)$ is a trajectory of the left invariant vector field $\widetilde{X}$. Noting that $t \mapsto s + t$ is a curve in $\mathbb{R}$ starting at s, with velocity vector $\frac{d}{dt}_{|s}$, we have:

$$\begin{aligned}
\frac{d\eta}{dt}(s) &= D\eta(s)\left(\frac{d}{dt}_{|s}\right) \\
&= \frac{d\eta(s+t)}{dt}(0) = \frac{d(\eta(s)\eta(t))}{dt}(0) \\
&= \frac{d(L_{\eta(s)}(\eta(t))}{dt}(0) = (L_{\eta(s)})_*\left(\frac{d\eta(t)}{dt}\right)(0) \\
&= \widetilde{X}(\eta(s))
\end{aligned}$$

Further, $\eta(0) = e$. The curve $\phi(t) := \exp tX$ is also a trajectory of $\widetilde{X}$ which satisfies $\phi(0) = e$. By the uniqueness of trajectories, it follows that $\eta \equiv \phi$. □

Corollary 5.6.6. For a Lie group G:

(i):

$$(\mathrm{Ad} \exp tX)_* = \exp t(\mathrm{ad}\, X)$$

where $\mathrm{ad}\, X \in \mathfrak{gl}(\mathfrak{g})$ is the linear map $Y \mapsto [X, Y]$, so that its exponential is the power series $\sum_{i=0}^{\infty} \frac{(\mathrm{ad}\, X)^i}{i!}$, an element of $GL(\mathfrak{g})$.

(ii): The following formula holds:

$$(\mathrm{Ad}\exp tX)(\exp sY) = (\exp tX)(\exp sY)(\exp tX)^{-1} = \exp(s\exp(t\,\mathrm{ad}\,X)Y)$$

Note that $\exp(t\,\mathrm{ad}\,X)$ is an element of $GL(\mathfrak{g})$, so $\exp(t\,\mathrm{ad}\,X)Y \in \mathfrak{g}$, and its exponential is an element of G.

(iii): Let G be a coonected Lie group. Then G is abelian iff $\mathfrak{g}$ is an abelian Lie algebra. That is $[X,Y] \equiv 0$ for all $X, Y \in \mathfrak{g}$.

Proof: To see (i), we consider the 1-parameter subgroup:

$$\begin{array}{rcl} \eta : \mathbb{R} & \to & GL(\mathfrak{g}) \\ t & \mapsto & (\mathrm{Ad}\exp tX)_* \end{array}$$

Note that

$$\frac{d\eta}{dt}(0) = \lim_{t\to 0}\frac{(\mathrm{Ad}\exp tX)_* - I}{t} = \mathrm{ad}\,X$$

by the Corollary 5.6.4 above. Thus, by the Proposition 5.6.5 above, we have the result.

To see (ii), fix a t and X. For a general $Y \in \mathfrak{g}$ consider the one-parameter subgroup in G:

$$\begin{array}{rcl} \phi : \mathbb{R} & \to & G \\ s & \mapsto & \mathrm{Ad}(\exp tX)(\exp sY) = (\exp tX)(\exp sY)(\exp tX)^{-1} \end{array}$$

Clearly, from (i) it follows that $\frac{d\phi}{ds}(0) = (\mathrm{Ad}\exp tX)_*(Y) = (\exp t\mathrm{ad}\,X)Y$, by the foregoing, so that by the Lemma 5.6.5 above, it follows that:

$$\phi(s) = \exp(s\exp(t\,\mathrm{ad}X)Y)$$

thus proving (ii).

For (iii), note that the only "if" part is trivial, because the adjoint action of G on itself is the identity map, and so its derivative $(\mathrm{Ad}\exp tX)_*$ is the identity map for all t, and so by (i) of 5.6.4, $\mathrm{ad}\,X = 0$ for all X, and thus $\mathfrak{g}$ is abelian.

Conversely, if $\mathfrak{g}$ is abelian, the part (ii) proved above shows that

$$(\mathrm{Ad}\exp tX)(\exp sY) = (\exp tX)(\exp sY)(\exp tX)^{-1} = \exp(sY)$$

for all $s, t \in \mathbb{R}$ and $X, Y \in \mathfrak{g}$.

This shows that all elements in the image of $\exp : \mathfrak{g} \to G$ commute. By the inverse function theorem, (see (ii) of Exercise 5.5.5) every element in a neighbourhood of e is in this image, so all elements in a neighbourhood of the identity commute. Since every connected topological group is generated by any neighbourhood of the identity, G is abelian. $\square$

Exercise 5.6.7 (Derivative of a Lie homomorphism)**.** Let $\phi : G \to H$ be a smooth homomorphism of Lie groups.

(i): Show that there is a unique linear map of Lie algebras, which is denoted $\phi_* := D\phi(e)$ from $\mathfrak{g}$ to $\mathfrak{h}$, which is a homomorphsim of Lie algebras, i.e. $[\phi_* X, \phi_* Y] = \phi_*[X, Y]$.

(ii): Show that if ϕ, ψ are two Lie homomorphisms from G to H, and G is connected, then $\phi_* = \psi_*$ implies $\phi = \psi$. (Hint: Just look at one parameter subgroups of G.)

Exercise 5.6.8. Let $G = SL(2, \mathbb{R})$. Recall that its Lie Algebra is given by:

$$\mathfrak{sl}(2, \mathbb{R}) = \{Y \in \mathfrak{gl}(2, \mathbb{R}) : \operatorname{tr} Y = 0\}$$

Compute the tangent space $T_A(SL(2, \mathbb{R}))$ for $A \in SL(2, \mathbb{R})$. For $Y \in \mathfrak{sl}(2, \mathbb{R})$, compute the left-invariant vector field $\widetilde{Y}$.

Exercise 5.6.9. Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Let X be the smooth vector field on S^2 defined by:

$$X(x, y, z) = (-y, x, 0)$$

Compute the one-parameter family of diffeomorphisms for this vector field. Compute the Lie derivative $L_X \omega$ where ω is the restriction of the 1-form dx to S^2.

Exercise 5.6.10. Let M be a smooth manifold, and X, Y smooth vector fields on it. For a 1-form ω on M, show that:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

(*Hint:* Use local coordinates).

Exercise 5.6.11. Consider the map:

$$\begin{array}{rcl} \theta : G \times G & \to & G \\ (x, y) & \mapsto & xyx^{-1}y^{-1} \end{array}$$

(i): Show that $D\theta(e, e) = 0$. (*Hint:* Note that the curve $(\exp(tX), e)$ has tangent vector $(X, 0)$, and the tangent vector $(e, \exp(sY))$ has tangent vector $(0, Y)$).

(ii): Show that :

$$\frac{\partial}{\partial t}_{|t=0} \left(\frac{\partial}{\partial s} \right)_{|s=0} \theta(\exp(tX)\exp(sY)) = [X, Y]$$

(*Hint:* Use the formula in (ii) of Corollary 5.6.6).

5.7. **Homogeneous manifolds.** Let G be a Lie group of dimension n, and $H \subset G$ be a closed subgroup. It is a theorem due to Cartan-Lie (see [BMST], p.58 Thm 4.5 for the case when G is a linear Lie group) that H is also a Lie group, say of dimension m. Look at the space of left cosets, denoted G/H. This is just the space of orbits of the smooth *right* action of H on G by multiplication. It is given the quotient topology. Since G can be given a right G-invariant Riemannian metric (see example 5.1.14), this metric will also be right H-invariant. Now let d be the distance on G, in accordance with 5.2.3, which will also be right H-invariant by 5.2.5. That is, it will satisfy $d(ah, bh) = d(a, b)$ for all $a, b \in G$, and $h \in H$.

Because H is closed, one can define a distance on G/H as:

$$\delta(aH, bH) := \inf_{h\in H}(d(a, bh)) = d(a, bH) = \inf_{h,h'\in H} d(ah, bh') = \delta(bH, aH)$$

Using the right H-invariance of the distance d, it is an easy exercise to check that δ defines a distance on G/H, which therefore makes it Hausdorff and paracompact. It is second countable (=separable) because the image of a countable dense subset $\Sigma \subset G$ under the natural quotient map $\pi : G \to G/H$ is a countable dense subset of G/H.

Proposition 5.7.1 (Homogeneous manifolds)**.** For a Lie group G and a closed subgroup H, the space G/H, as defined above, is a smooth manifold of dimension $\dim G - \dim H = n - m$.

Proof:
Note that the space $M := G/H$ is homogeneous in the following sense: given any point $x = gH$ in M, there is a homeomorphism, also denoted L_g, taking the identity coset $o = H$ to $x = gH$. Thus if we can prove that a neighbourhood of the identity coset o is homeomorphic to an open subset of $\mathbb{R}^{n-m}$, and also show that the coordinate changes are smooth, we will be done.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H. It is a fact (see [BMST], p.58) that the exponential map $\exp : \mathfrak{g} \to G$ takes $\mathfrak{h}$ to H. Let $\mathfrak{m}$ be a vector space complement to $\mathfrak{h}$ inside $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Define the map:

$$\begin{aligned} \phi : \mathfrak{m} \oplus \mathfrak{h} &\to G \\ (X, Y) &\mapsto \exp X \exp Y \end{aligned}$$

It is easily checked that the derivative $D\phi(0)$ is the identity map $\mathfrak{g} \to \mathfrak{g}$ so that, by the inverse function theorem, there exists a neighbourhood $\widetilde{U}$ of 0 in $\mathfrak{g}$ on which ϕ is a diffeomorphism. Let $\phi(U_1) \subset \phi(\widetilde{U})$ be a smaller neighbourhood such that it is symmetric (i.e. such that $y \in \phi(U_1) \Rightarrow y^{-1} \in \phi(U_1)$, and such that $\phi(U_1)\phi(U_1) \subset \phi(\widetilde{U})$. Thus it follows that if $y_1, y_2 \in \phi(U_1)$, we have $y_1^{-1}y_2 \in \phi(\widetilde{U})$. Now take a neighbourhood of the form $W \times V \subset U_1$, where W and V are neighbourhoods of 0 in $\mathfrak{m}$ and $\mathfrak{h}$ respectively. Let $U := \phi(W \times V) = \exp(W).\exp(V)$. We claim that $\pi(U)$ is an open neighbourhood of the identity coset $o = eH$ in G/H.

Recall that with the quotient topology on G/H, a set is open in G/H iff its inverse image under π is open in G. But $\pi^{-1}(\pi(U))$ is just $\cup_{h\in H} R_h(U)$, which is clearly open. Consider the map $\psi : W \to \pi(U)$ defined by $\psi(X) \mapsto (\exp X)H$. That is, $\psi = \pi \circ \phi_{|\mathfrak{m}}$. Clearly ψ is continuous.

We claim that its inverse is the continuous map:

$$\begin{aligned} \theta : \pi(U) &\to W \\ \pi(y) &\mapsto \mathrm{pr}_{\mathfrak{m}}\phi^{-1}(y) \end{aligned}$$

To see that this map is well-defined, note that if $\pi(y_1) = \pi(y_2)$ with $y_1, y_2 \in U$, then by writing $y_1 = \phi(X_1, Y_1)$ and $y_2 = \phi(X_2, Y_2)$ with $X_i \in W$ and $Y_i \in V$, we have $(\exp X_1)^{-1} \exp X_2 \in \phi(\widetilde{U}) \cap H$, so that $\exp X_2 = \exp X_1 \exp Z$ for some $Z \in \widetilde{U} \cap \mathfrak{h}$. That is $\phi(X_2, 0) = \phi(X_1, Z)$. Since ϕ was injective on $\widetilde{U}$, it follows that $X_1 = X_2 = X$ and $Z = 0$. Thus $y_i = \exp X \exp Y_i$ for $i = 1, 2$. Thus $\phi^{-1}(y_i) = (X, Y_i)$, and $\mathrm{pr}_{\mathfrak{m}}\phi^{-1}(y_i) = X$ is well defined.

Hence (ψ^{-1}, V), where $V := \pi(U)$ is a local chart around o. Local charts around other points are given by composing with left translations. That is, for a $g \in G$, we define $gV = L_g(V) = \pi(gU)$, and ψ_g^{-1} by the diagram:

$$\begin{array}{ccc} V & \xleftarrow{L_g^{-1}} & gV \\ \psi^{-1}\downarrow & & \downarrow\psi_g^{-1} \\ W & \longleftrightarrow & W \end{array}$$

Thus on the overlap $W_g := \psi^{-1}(V \cap gV)$, the coordinate change $\psi_g^{-1} \circ \psi$ is the composite:

$$W_g \xrightarrow{\psi} V \cap gV \xrightarrow{L_g^{-1}} V \cap g^{-1}V \xrightarrow{\psi^{-1}} W_{g^{-1}}$$

which we denote by ρ. Again, since $\pi \circ \phi = \psi$, we conclude from the diagram:

$$\begin{array}{ccc} V \cap gV & \xrightarrow{L_g^{-1}} & V \cap g^{-1}V \\ \pi\uparrow & & \uparrow\pi \\ U \cap gU & \xrightarrow{\widetilde{L}_g^{-1}} & U \cap g^{-1}U \\ \phi\uparrow & & \uparrow\phi \\ W_g & & W_{g^{-1}} \end{array}$$

that the composite ρ is also the composite $\phi^{-1}\widetilde{L}_g^{-1}\phi$ of the above diagram, (where $\widetilde{L}_g$ denotes left translation on the group G. This is clearly smooth.

Thus we have defined a smooth atlas, and $X = G/H$ is a smooth manifold of dimension $n - m$. $\square$

Example 5.7.2 (G-manifolds as homogeneous spaces)**.** Let M be a smooth manifold of dimension m, and let G be a Lie group of dimension n acting smoothly on it. (See definition 5.1.11). We define the *isotropy group* (or *stabiliser*) of x to be the subgroup of G given by:

$$G_x = \{g \in G : \; gx = x\}$$

Recall that the action is said to be *transitive* if M consists of a single orbit, i.e. $M = Gx$ for each $x \in M$. We call the action a *good action* if it is a transitive action, and if for some $x \in M$, the derivative at the identity $e \in G$ of the map $\rho_x : G \to M$ defined by $\rho_x(g) = gx$, namely:

$$D\rho_x(e) : T_e(G) \to T_x(M)$$

is surjective. Since we have a commutative square:

$$\begin{array}{ccc} T_e(G) & \xrightarrow{D\rho_x(e)} & T_x(M) \\ L_{g*}\downarrow & & \downarrow L_{g*} \\ T_g(G) & \xrightarrow{D\rho_x(g)} & T_{gx}(M) \end{array}$$

and $L_g : G \to G$ and $L_g : M \to M$ are diffeomorphisms, it follows from the square above that the lower horizontal arrow is surjective for all $g \in G$, i.e. each point of M is a regular value of the map ρ_x.

Thus $\rho_x^{-1}(x) = G_x$ is a closed submanifold of G, i.e. a closed Lie subgroup of G of dimension $n - m$, by the (submersive) implicit function theorem 1.6.4.

Claim: M is diffeomorphic to the homogeneous manifold G/G_x of 5.7.1.

For, fix a point $x \in M$, and again consider the map ρ_x. Since ρ_x is continuous, and takes gh and g to the same point gx for each $h \in G_x$, it descends to a continuous map $\phi : \; G/G_x \to M$ by the property of the quotient topology. This map ϕ is easily checked to be a bijection. On the other hand, since $D\rho_x$ is surjective at each point $g \in G$, the (submersive form of) the implicit function theorem implies that ρ_x is locally a projection, and hence an open map. Thus ϕ is also open, and we have that ϕ is a homeomorphism.

By the example 3.3.22, the tangent space to G_x at e is the kernel:

$$\ker D\rho_x(e) : T_e(G) \to T_x(M)$$

Thus, if $\mathfrak{m}$ is the vector space complement to $T_e(G_x)$ that was chosen in the proposition 5.7.1 above, it follows that $D\rho_x(e)_{|\mathfrak{m}}$ is an isomorphism. That is, the derivative of the map:

$$\mathfrak{m} \stackrel{\exp}{\to} G \stackrel{\rho_x}{\to} M$$

at $0 \in \mathfrak{m}$, is an isomorphism. Hence, by the inverse function theorem, this composite is a local diffeomorphism. But on a neighbourhood W of 0 in $\mathfrak{m}$, this map is also the composite:

$$W \stackrel{\psi}{\to} V \stackrel{\phi}{\to} M$$

where ψ is the local chart around $o \in G/G_x$ constructed in 5.7.1. Thus ϕ is smooth around $o \in G/G_x$, and in fact a local diffeomorphism of a neighbourhood V of o with a neighbourhood of x in M. By applying left translations on G/G_x and M, we see that ϕ is a local diffeomorphism at all points of G/G_x. Since ϕ is a homeomorphism which is a local diffeomorphism at all points, it is a diffeomorphism. This proves the claim. $\square$

Exercise 5.7.3. Show that the natural left action of G on G/H as defined in 5.7.1 is a smooth and good action.

One can obviously define *a right* G-action in analogous fashion, and carry all of the above over to right actions. Given any left action, one can get a right action by defining $xg := g^{-1}x$, and vice versa, so that everything above can be reformulated for right actions. On a Lie group, there is the left action of left multiplication, and the right action of right multiplication, and these actions are distinct if the group is non-abelian. Indeed, the adjoint action of G on itself (which is a left action) measures the "discrepancy" between the left and right actions, as was discussed in the preceding subsection.

5.8. **Homogeneous metrics.** In the last subsection, we saw two alternative descriptions of a homogeneous manifold, i.e. as a coset space of a Lie group with respect to a closed subgroup, or as a manifold with a smooth "good" left-action by a Lie group G.

Note that the Riemannian metric we got on G/H at the beginning of the last subsection need not be left G-invariant. It is quite natural to ask how one can put a left G-invariant (= G-homogeneous) Riemannian metric on such a manifold.

Proposition 5.8.1 (Homogeneous metrics)**.** Let G and H be as in 5.7.1. Assume there exists a Riemannian metric $\langle\, , \rangle$ which is left G-invariant and right H- invariant. Then there exists a (left) G-invariant (or homogeneous) Riemannian metric on G/H.

Proof: Consider the quotient map, which is a submersion:

$$\begin{array}{rcl} \pi : G & \to & G/H \\ g & \mapsto & gH \end{array}$$

which induces the exact sequence of tangent bundles:

$$\pi_* : TG \to \pi^*T(G/H) \to 0$$

The kernel of this bundle morphism is the bundle $\eta \to G$, whose fibre η_g at $g \in G$ is given as the kernel of $D\pi(g) : T_g(G) \to T_{\pi(g)}(G/H)$. Let ξ_g be the orthogonal complement of η_g in $T_g(G)$, with respect to the given Riemannian metric $\langle\, , \rangle$. It follows from the discussion of 5.7.1 that $\pi_{*,g} : \xi_g \to T_{\pi(g)}(G/H)$ is an isomorphism for each $g \in G$, and so

$$\pi_{*|\xi} : \xi \to \pi^*T(G/H)$$

is an isomorphism of vector bundles. Thus its inverse gives an isomorphism of $\pi^*T(G/H)$ with the subbundle ξ of TG.

Notice that both bundles ξ and $\pi^*T(G/H)$ are mapped to themselves by the derivative of the right translation R_{h*}, for $h \in H$. Further, by right H-invariance of $\langle\,,\,\rangle$, the isomorphism above commutes with R_{h*}. (All this is a rerun of part (ii) of exercise 5.1.5.) Thus, by restricting the left G-invariant Riemannian metric on TG, we have a left G-invariant Riemannian metric (call it $(\,,\,)$) on the bundle $\pi^*T(G/H)$. Because the original metric was invariant under R_{h*} for all $h \in H$, this induced metric $(\,,\,)$ is also right H-invariant. Identifying ξ with $\pi^*T(G/H)$, we have an orthogonal decomposition

$$TG \simeq \eta \oplus \pi^*T(G/H)$$

which is preserved under R_{h*} for all $h \in H$.

Now, the restricted bundle $\pi^*T(G/H)_{|gH}$ is a trivial bundle, and for $h \in H$, (R_{h*}) carries the fibre of this bundle at g isomorphically to the fibre at gh. Since the metric $(\,,\,)$ is invariant under R_{h*}, this metric naturally descends to a Riemannian metric on $T(G/H)$, which is left G-invariant. □

Corollary 5.8.2. If G is a Lie group, and H is compact, then there always exists a left G-invariant Riemannian metric on G/H. In particular, if G is itself compact, then we have such a metric.

Proof: H being compact, a positive definite bilinear form on $\mathfrak{g}$ (which automatically yields a left G-invariant Riemannian metric on G) can be averaged out with respect to the adjoint H-action $\mathrm{Ad}(h)_*$ for $h \in H$, as in the proposition 5.1.17, to yield a left G right H invariant metric on G. □

Exercise 5.8.3. Show that if G is a Lie group, and H a *connected* closed subgroup, and $\langle\,,\,\rangle$ is a positive definite bilinear form on $\mathfrak{g}$ which satisfies:

$$\langle(\mathrm{ad}\,Z)X, Y\rangle + \langle X, (\mathrm{ad}\,Z)Y\rangle = 0$$

for all $Z \in \mathfrak{h}$, and all $X, Y \in \mathfrak{g}$, then the homogeneous manifold G/H acquires a left G-invariant Riemannian metric. (Hint: use the formula for $\mathrm{Ad}(\exp(tZ))_*$ and the condition for an H-bi-invariant metric discussed in example 5.1.14).

Example 5.8.4 (The Sphere)**.** It is easy to realise the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as a homogeneous space as follows. Consider natural action of the $SO(n+1)$ on $\mathbb{R}^{n+1}$ given by left matrix multiplication with a column vector. Since $SO(n+1)$ preserves lengths, this action restricts to an action of $SO(n+1)$ on S^n. If we let $x = (0, 0, .., 1)$ be the unit vector of the x_{n+1}-axis (the "north pole" of S^n), we can always find a matrix $A \in SO(n+1)$ such that $Ax = y$ for any given $y \in S^n$ (why?), so that this action is transitive. It is easy to check that it is smooth. To see that it is a good transitive action, one just has to verify that the map:

$$\rho_x : SO(n+1) \to S^n$$

which maps a matrix $A \in SO(n+1)$ to its last column Ax has derivative of full rank at the identity. In a neighbourhood U of the identity, we write $A(t) = \exp tX$ (by (ii) of 5.5.5) where X is a skew-symmetric matrix of trace 0, and thus:

$$\begin{aligned} D\rho_x(e)(X) &= \lim_{t\to 0}\left(\frac{(\exp tX)x - x}{t}\right) \\ &= Xx \end{aligned}$$

Note that X being skew-symmetric, $\langle Xx, x\rangle = \langle -X^t x, x\rangle = -\langle x, Xx\rangle$ which implies Xx is orthogonal to x, and hence lies in $T_x(S^n)$ (see exercise 4.1.3 above). Since any tangent vector $v \in T_x(S^n)$ is just a vector with last entry zero, one can view it as the last column of the skew-symmetric matrix X which has v as the last column, $-v$ as the last row, and zeros elsewhere. Thus $v = Xx$, and our map is surjective, and the transitive action is good. Clearly, the isotropy subgroup of x is the group $SO(n)$ on the first n coordinates. By the foregoing example 5.7.2, the sphere S^n is diffeomorphic to the homogeneous space $SO(n+1)/SO(n)$.

There is the natural the bi-invariant metric arising from $\langle X, Y\rangle = \mathrm{tr}(X^tY)$ on $SO(n+1)$. With respect to this metric, the Lie algebra $\mathfrak{o}(n+1)$ of $SO(n+1)$ decomposes as an orthogonal direct sum:

$$\mathfrak{o}(n+1) = \mathfrak{o}(n) \oplus^\perp \mathfrak{m}$$

where

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & 0 & 0 & ... & v_1 \\ 0 & 0 & 0 & ... & v_2 \\ .. & .. & .. & .. & .... \\ .. & .. & .. & .. & v_n \\ -v_1 & -v_2 & ... & -v_n & 0 \end{pmatrix} : v_i \in \mathbb{R} \right\}$$

For two elements $X, Y \in \mathfrak{m}$, their inner product $\langle X, Y\rangle = \mathrm{tr} X^t Y$ is just twice the euclidean inner product of their non-vanishing (last) columns, which is exactly twice the inner product on $\mathbb{R}^n = T_o(S^n)$, where o is the identity coset $x = (0, 0, .., 1)$. Thus the natural metric on S^n is (twice) the one coming from the proposition 5.8.1. (Since both metrics are left $SO(n+1)$-invariant, they are completely determined by their values at $x = o$).

Example 5.8.5 (Grassmannians)**.** The orthogonal group $O(n)$ acts on the grassmannian $G_k(\mathbb{R}^n)$, as explained in 5.1.9. That this action is transitive and, in fact, good is easily verified. The isotropy group of the point $o = \mathrm{span}(e_1, .., e_k)$ is the subgroup $H = O(k) \times O(n-k)$. Again, the bi-invariant Riemannian metric coming from the positive definite bilinear form $\langle X, Y\rangle = \mathrm{tr}(X^t Y)$ yields an orthogonal decomposition:

$$\mathfrak{o}(n) = \mathfrak{h} \oplus^{\perp} \mathfrak{m}$$

where

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -X^t \\ X & 0 \end{pmatrix} : X \in \hom(o, o^{\perp}) \right\}$$

The natural identification of $T_o(G_k(\mathbb{R}^n)$ with $\mathfrak{m}$, and the fact that the restriction of the metric $\mathrm{tr}(X^t Y)$ to $\mathfrak{m}$ is exactly twice the Riemannian metric introduced on $T_o(G_k(\mathbb{R}^n))$ in 5.1.9, proves that metric to be equal to twice the one coming from 5.8.1, by left $SO(n)$ invariance of both these metrics.

Example 5.8.6 (Hyperbolic 2-space)**.** We let H^2 denote the *upper-half plane* in $\mathbb{R}^2 = \mathbb{C}$, defined as:

$$H^2 = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$$

On this 2-dimensional manifold we put the Riemannian metric given by:

$$g(z) = g(x+iy) = \frac{dx^2 + dy^2}{y^2}$$

There is an action of the group $G = SL(2, \mathbb{R})$ on H^2 defined as follows:

$$\begin{array}{rcl} \mu : SL(2,\mathbb{R}) \times H^2 & \to & H^2 \\ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

Clearly this makes sense, since it is easily checked that:

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = |cz+d|^{-2} \mathrm{Im}\, z$$

which is > 0 if $\mathrm{Im}\, z > 0$. It is also easily checked to be smooth. It is transitive because the element:

$$\begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{-1}{2}} \\ 0 & y^{\frac{-1}{2}} \end{pmatrix}$$

takes $i = \sqrt{-1}$ to $x + iy \in H^2$. We need to check that the action is a good action. That is, the map:

$$\rho := \rho_i : G \to H^2$$

defined by $\rho(g) = gi$ has surjective derivative at the identity (see example 5.7.2 above). Consider the matrix in $\mathfrak{g} = sl(2, \mathbb{R})$ given by:

$$X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

Clearly

$$\exp tX = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{-t\lambda} \end{pmatrix}$$

so that $(\exp tX)i = e^{2t\lambda}i$, so that

$$(D\rho(e))(X)i = \frac{d}{dt}_{t=0}(e^{2t\lambda i}) = 2\lambda i$$

Similarly, consider the matrix $Y \in sl(2,\mathbb{R})$ defined by:

$$Y = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$$

so that

$$\exp tY = \begin{pmatrix} \cosh t\mu & \sinh t\mu \\ \sinh t\mu & \cosh t\mu \end{pmatrix}$$

so that

$$(D\rho(e))(Y) = \frac{d}{dt}_{t=0}(\exp tY)i = 2\mu$$

Thus $D\phi(e)$ has image $\mathbb{C} = T_i(H^2)$ and is surjective. To identify H^2 as a G-space with $G = SL(2,\mathbb{R})$ we just need to determine the isotropy subgroup of the point i, which is easily checked to be the subgroup:

$$K := SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

which is just the circle group. Thus, as a manifold $H^2 = SL(2,\mathbb{R})/SO(2)$.

We now claim that the group $G = SL(2,\mathbb{R})$ acting on H^2 by left translations is acting by isometries. For this, note that the metric is given by:

$$g(z) = \frac{dzd\bar{z}}{(\operatorname{Im} z)^2}$$

If we let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and denote $gz = \frac{az+b}{cz+d}$ by w, we compute:

$$dw = \frac{dz}{(cz+d)^2}$$

Since

$$\operatorname{Im} w = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = |cz+d|^{-2}\operatorname{Im} z$$

it follows that

$$\frac{dzd\bar{z}}{(\operatorname{Im} z)^2} = \frac{dwd\overline{w}}{(\operatorname{Im} w)^2}$$

so that G is acting by isometries, and the *hyperbolic 2-space* H^2 has a G invariant metric.

The next question is obviously whether there is a left-G and right-K invariant metric on $G = SL(2,\mathbb{R})$ which descends to the hyperbolic metric above, in accordance with proposition 5.8.1.

As usual, define the bilinear form $\langle X, Y\rangle = \operatorname{tr}(XY^t)$. This will lead to a left G-invariant Riemannian metric on G. All we need to do is check that it is right K-invariant, and in fact K-bi-invariant. But this is obvious, because for $g \in K$, $g^{-1} = g^t$, and $\operatorname{Ad}g(X) = gXg^t$ for $g \in K$. Thus

$$\langle X, Y\rangle = \operatorname{tr}(XY^t) = \operatorname{tr}(gXY^tg^{-1}) = \operatorname{tr}(gXg^tgY^tg^t) = \langle \operatorname{Ad}g(X), \operatorname{Ad}g(Y)\rangle$$

proving left-G and right-K invariance of the resulting Riemannian metric on G, and by proposition 5.8.1, we get a left G-invariant (homogeneous) Riemannian metric on $G/K = H^2$.

Now, we just need to verify that this metric coincides with the hyperbolic metric above on H^2. For this we note that $\mathfrak{g}$ has the *polar decomposition*

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\mathfrak{k} = \mathfrak{so}(2) = \left\{X \in \mathfrak{sl}(2,\mathbb{R}) : X = -X^t\right\}, \quad \mathfrak{p} = \left\{X \in \mathfrak{sl}(2,\mathbb{R}) : X = X^t\right\}$$

and by our proposition 5.7.1, the vector space $\mathfrak{p}$ gets identified with the tangent space $T_i(H^2)$. The identification is via $D\rho_i(e)$ which, from the first para above is the mapping

$$\begin{array}{rcl} D\rho_i(e) : \mathfrak{m} & \to & T_i(H^2) \\ X := \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix} & \mapsto & 2\lambda i + 2\mu \end{array}$$

which shows that the $\mathrm{tr}(XX^t) = 2(\lambda^2+\mu^2) = 2\langle(\lambda,\mu),(\lambda,\mu)\rangle$, which is exactly half the square of the hyperbolic length of the vector $D\rho(e)(X) \in T_i(H^2)$. Thus, as before, the G-invariant Riemannian metric from 5.8.1 is the hyperbolic metric upto a scalar multiple.

Remark 5.8.7. We remark that the above time-tested positive definite bilinear form $\langle X, Y\rangle = \mathrm{tr}(XY^t)$ is *not* AdG invariant, as is seen from the example 5.1.14. In fact:

Claim: There does not exist any positive definite AdG invariant bilinear form on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$.

Proof:
Note that there is the decomposition:

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}Y \oplus \mathbb{R}H \oplus \mathbb{R}X$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and it is easily calculated that $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$. We recall that, as in the exercise 5.8.3, an AdG invariant bilinear form $\langle\, ,\,\rangle$ would have to satisfy:

$$0 = \left(\frac{d}{dt}\right)_{t=0} \langle \mathrm{Ad}\exp tX(Y), \mathrm{Ad}\exp tX(Z)\rangle = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle \quad \text{for all } X, Y, Z \in \mathfrak{g}$$

So that we would have:

$$2\langle X, X\rangle = \langle [H,X], X\rangle = -\langle X, [H,X]\rangle = -2\langle X, X\rangle$$

which would imply $\langle X, X\rangle = 0$ and thus $\langle\, ,\,\rangle$ is not positive definite. (In fact, it is an exercise to show that any non-degenerate symmetric bilinear form on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ is a non-zero multiple of the (non-degenerate but indefinite) symmetric bilinear form $\langle A, B\rangle = \mathrm{tr}(AB)$. This last form is indefinite because $\langle I, I\rangle = 2$, and $\langle X-Y, X-Y\rangle = -2$, where $X, Y \in \mathfrak{sl}(2,\mathbb{R})$ are the matrices defined above.) This proves the claim. □

Example 5.8.8 (Disc Model of Hyperbolic 2-space)**.** There is another useful "Poincare Disc Model" of hyperbolic space. We equip the open unit disc:

$$D^2 = \{z \in \mathbb{C} : |z| < 1\}$$

with the "Poincare metric":

$$g(z) = g(x+iy) = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2} = \frac{4dzd\bar{z}}{(1-|z|^2)^2}$$

This Riemannian manifold is isometric to H^2 above via the "Cayley transform" mapping:

$$\begin{array}{rcl} \phi : H^2 & \to & D^2 \\ z & \mapsto & \dfrac{z-i}{z+i} \end{array}$$

The map ϕ^{-1} maps w to $i\left(\frac{1+w}{1-w}\right)$. The imaginary axis in H^2 maps to the real axis in D^2, and the distinguished point (identity coset) $i = \sqrt{-1} \in H^2$ maps to the origin in D^2. It is a straightforward calculation to verify that ϕ is an isometry. The good transitive action of $SL(2,\mathbb{R})$ gets replaced by a good transitive action of $SU(1,1)$ on D^2. The group $SU(1,1)$ is defined by:

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in GL(2,\mathbb{C}) : |a|^2 - |b|^2 = 1 \right\}$$

and this acts on D^2 by $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$. Again, one checks that this action leaves the Poincare metric above invariant. The isotropy of the origin is the circle subgroup $S^1 \subset SU(1,1)$ of elements with $b = 0$ and $|a| = 1$.

Exercise 5.8.9 (Equivalence of both models)**.** Construct the explicit isomorphism between $SL(2,\mathbb{R})$ and $SU(1,1)$.)

Example 5.8.10 (Hyperbolic 3-space)**.** Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$. View $\mathbb{R}^4$ as the non-commutative (so be careful in the calculations below!) algebra of quaternions. Let H^3 be the subspace of $\mathbb{H} = \mathbb{R}^4$ defined by:

$$H^3 = \{x + iy + jw + kt \in \mathbb{H} : t = 0,\ w > 0\}$$

This is the upper-space of $\mathbb{R}^3 \subset \mathbb{R}^4$, and is a Riemannian manifold with the metric given by:

$$g(x + iy + jw) = \frac{dx^2 + dy^2 + dw^2}{w^2}$$

We claim that H^3 is the homogeneous manifold $SL(2,\mathbb{C})/SU(2)$, and the metric above is $G = SL(2,\mathbb{C})$ invariant. First let us define a transitive action of $SL(2,\mathbb{C})$ on H^3 as follows:

$$\begin{array}{rcl} \mu : G \times H^3 & \to & H^3 \\ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, r) & \mapsto & (ar + b)(cr + d)^{-1} \end{array}$$

(Exercise: show that $cr + d \neq 0$ for $r \in H^3$). Here we are regarding $\mathbb{C}$ as the subalgebra of $\mathbb{H}$ consisting of quaternions of the type $x + iy$. Let us see that this is well-defined. Regard a quaternion as $z + jw$ where $z, w \in \mathbb{C}$. Note that the quaternion multiplication rules imply that $jz = \bar{z}j$ for $z \in \mathbb{C}$, and hence that:

$$(z + jw)(z_1 + jw_1) = (zz_1 - \bar{w}w_1) + j(wz_1 + \bar{z}w_1)$$

There is also the *conjugation* involution of $\mathbb{H}$ which takes $r = x + iy + jw + tk$ to $\bar{r} = x - iy - jw - tk$. If we express $r = z + jw$ with $z, w \in \mathbb{C}$, then $\bar{r} = \bar{z} - jw = \bar{z} - \bar{w}j$. It is easily checked that this conjugation is an anti- homomorphism, i.e.:

$$\overline{rs} = \bar{s}\,\bar{r}$$

for all $r, s \in \mathbb{H}$. For any quaternion $r \neq 0$, it is easy to check that its inverse r^{-1} is just $\frac{\bar{r}}{\|r\|^2}$.

To see that $s = (ar + b)(cr + d)^{-1} \in H^3$ for $r \in H^3$, note that for $r = z + jw \in H^3$, we have $2w = j\bar{r} - rj$. So we have to compute the quantity $j\bar{s} - sj$. Since

$$s = \frac{(ar + b)(\bar{r}\,\bar{c} + \bar{d})}{\|cr + d\|^2}$$

so that

$$\bar{s} = \frac{(cr + d)(\bar{r}\,\bar{a} + \bar{b})}{\|cr + d\|^2}$$

from which we compute (using brute force and $ad - bc = 1$) that

$$\begin{aligned} j\bar{s} - sj &= \frac{jcr\bar{b} + jd\bar{r}\,\bar{a} - b\bar{r}\,\bar{c}j - ar\bar{d}j}{\|cr + d\|^2} \\ &= \frac{2w}{\|cr + d\|^2} \end{aligned}$$

which is clearly positive for $w > 0$. If we denote $s = q + ju$, then this formula implies that:

$$u = \frac{w}{\|cr + d\|^2}$$

Since the matrix inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

it follows that:

$$w = \frac{u}{\|a - cs\|^2}$$

and hence that:

$$\frac{u^2}{w^2} = \frac{\|a - sc\|^2}{\|cr + d\|^2} \qquad (*)$$

an equation which we use later.

To see that the action is good, is a simple verification as in the case of H^2 above. As before, let ρ denote the map $G \to H^3$ taking g to $g.j$. Then for

$$X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad \lambda \in \mathbb{R} \;\; \lambda > 0$$

we find that

$$D\rho(e)(X) = \left(\frac{d(\exp tX.j)}{dt}\right)(0) = \left(\frac{d(e^{2t\lambda j})}{dt}\right)(0) = 2\lambda j$$

whereas for

$$Y = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 0 & i\nu \\ -i\nu & 0 \end{pmatrix} \qquad \mu, \nu \in \mathbb{R}$$

we have:

$$\begin{aligned} D\rho(e)(Y) &= \left(\frac{d(\exp tY.j)}{dt}\right)(0) \\ &= \left(\frac{d(\cosh t\mu j + \sinh t\mu)(\bar{j}\sinh t\mu + \cosh t\mu)}{dt}\right)(0) \\ &= \left(\frac{d(\sinh 2\mu t + j)}{dt}\right)(0) = 2\mu \end{aligned}$$

and similarly $D\rho(e)(Z) = 2i\nu$, which clearly shows that $D\rho(e)$ maps the subspace $\mathfrak{p} \subset sl(2, \mathbb{R})$ surjectively onto $T_j(H^3)$. Here $\mathfrak{p}$ denotes the subspace of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ consisting of traceless hermitian matrices.

For $r = z + jw \in H^3$, where $w \in \mathbb{R}$ is strictly positive, the matrix:

$$\begin{pmatrix} \sqrt{w} & \frac{z}{\sqrt{w}} \\ 0 & \frac{1}{\sqrt{w}} \end{pmatrix}$$

takes $j \in H^3$ to $r = z + jw$. Thus the action is transitive, and by the foregoing, a good action.

To compute the isotropy of $j \in H^3$, we have the equation:

$$(aj + b)(cj + d)^{-1} = j$$

which implies that $aj + b = jcj + jd = -\bar{c} + \bar{d}j$ so that $b = -\bar{c}$ and $a = \bar{d}$. But this group:

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : \; a, b \in \mathbb{C} \right\}$$

is precisely the subgroup $SU(2) \subset SL(2, \mathbb{C})$. Thus H^3 is diffeomorphic to the manifold $SL(2, \mathbb{C})/SU(2)$ by 5.7.2.

Finally, to check that the action leaves the metric above invariant, note that on differentiating the equation $s(cr + d) = ar + b)$ we obtain $ds(cr + d) + sc(dr) \;=\; adr$, which implies:

$$(a - sc)dr = ds(cr + d)$$

The conjugate of this equation is

$$d\bar{r}\overline{(a - sc)} = \overline{(cr + d)}d\bar{s}$$

and multiplying the two equations, and noting that real quantities commute with everything, we get:

$$\|a - sc\|^2 \, dr d\bar{r} = \|cr + d\|^2 \, ds d\bar{s}$$

and using $(*)$ above we have that:

$$\frac{dr d\bar{r}}{w^2} = \frac{ds d\bar{s}}{u^2}$$

which implies the invariance of the metric since $drd\bar{r} = dx^2 + dy^2 + dw^2$.

The final issue is whether this metric comes from a left-G and right-K invariant Riemannian metric on $G = SL(2,\mathbb{C})$ in accordance with 5.8.1. As in the example of $SL(2,\mathbb{R})$ in 5.8.6 above, we have a direct sum decomposition of Lie algebras:

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{p}$$

where $\mathfrak{su}(2)$ is the Lie algebra $\mathfrak{k}$ of K, consisting of all traceless skew-hermitian complex matrices (i.e. $X^* = -X$), and $\mathfrak{p}$ those of tracelss hermitian ones. Under $D\rho(e)$, $\mathfrak{p}$ gets identified with $T_j(H^3)$. We leave it as an exercise for the reader to prove that the positive definite bilinear form $\langle X, Y\rangle = \operatorname{tr}(XY^*)$ (where $Y^* := \overline{Y}^t$) on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ is AdK bi-invariant, and the corresponding homogeneous metric resulting from 5.8.1 on $M = H^3$ is a scalar multiple of the above hyperbolic one.

Both the hyperbolic spaces above are examples of *Riemannian symmetric spaces of non-compact type.*

5.9. **Classification of compact surfaces.** The hyperbolic 2-space above is the source of a great deal of beautiful mathematics, unifying Lie theory, Riemannian geometry, complex analysis and number theory. For example, there is the following fundamental:

Theorem 5.9.1 (Riemann mapping theorem)**.** A simply connected complex manifold of (complex) dimension one is biholomorphically (or "conformally") equivalent to $\mathbb{C}$, H^2 or S^2. (Here S^2 is the *Riemann sphere* $\mathbb{C}\cup\infty$, and can be made into a complex manifold of dimension one in a natural way. It can also be viewed as the complex projective space $\mathbb{CP}(1)$.)

The reader may consult Ahlfors' book *Complex Analysis*, or Rudin's *Real and Complex Analysis* for a proof of this deep fact. The original proof given by Riemann was a heuristic one, and it took the better part of a century for a rigorous proof (due to Ahlfors).

As a consequence, one can classify (upto diffeomorphism) all compact connected smooth manifolds of (real) dimension 2. First one classifies the orientable ones. If M is a compact connected orientable manifold, one can give it a smooth Riemannian metric, and then prove the existence of *local isothermal parameters.* That is, in a neighbourhood of every point, there is a coordinate system (u, v), with respect to which the metric g takes on the form $\lambda(u, v)(du^2 + dv^2)$, where λ is a strictly positive function. This remarkable fact is due to Gauss, and a proof maybe found in Springer's *Riemann Surfaces.*

If on an overlapping region, we have isothermal coordinates (x, y), then the reader can easily verify that the relation $\lambda(u, v)(du^2+dv^2) = \mu(x, y)(dx^2+dy^2)$, coupled with the extra fact that the jacobian of the coordinate change $(u, v) \mapsto (x, y)$ is positive (which can be assumed without loss of generality by the orientability of M) shows that this coordinate change obeys the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on the overlap, which makes M a comlex manifold of dimension 1.

By the Riemann mapping theorem above, its simply-connected covering $\widetilde{M}$ is therefore either $\mathbb{CP}(1)$, $\mathbb{C}$ or H^2, and the group of covering transformations Γ must be a subgroup of the group of holomorphic automorphisms $\mathrm{Aut}_h(\widetilde{M})$.

For the reader who is familiar with a bit of one-variable complex analysis, it is not difficult to show that:

$$\begin{aligned}
\mathrm{Aut}_h(\mathbb{CP}(1)) &= PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\{\pm I\} \\
\mathrm{Aut}_h(\mathbb{C}) &= \{z \mapsto \alpha z + \beta : \alpha \in \mathbb{C}^*, \text{ and } \beta \in \mathbb{C}\} \\
\mathrm{Aut}_h(H^2) &= PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}
\end{aligned}$$

where the groups $PGL(2,\mathbb{R})$ and $PSL(2,\mathbb{R})$ act via fractional linear transformations $z \mapsto \left(\frac{az+b}{cz+d}\right)$.

So now the question boils down to finding out which subgroups Γ of the above automorphism groups $\mathrm{Aut}_h(\widetilde{M})$ can act properly discontinuously on the complex manifolds $\widetilde{M}$ above. Per force, such subgroups have to be discrete subgroups. In fact, by covering space theory, it will turn out that Γ is isomoprhic to the fundamental group $\pi_1(M)$. Suitably classifying these discrete subgroups of the various automorphism groups listed above will lead to a classification of compact connected complex manifolds of dimension 1, upto holomorphic or conformal equivalence.

This is a deeper question than what we are interested in. The fact that a compact connected orientable surface becomes a complex manifold of dimension 1 leads to the fact that it is "triangulable", that is, it is a two dimensional simplicial complex. This fact is proved in the book *Riemann Surfaces* by Ahlfors and Sario.

Now, using triangulability, one can give combinatorial arguments to show that a compact connected orientable surface M is homeomorphic to one of the following:

(i): S^2, if M is simply connected.

(ii): $T^2 \# T^2 \# ... \# T^2$ (g-times), if M is not simply connected.

Here $\#$ denotes *connected sum*. That is, for two surfaces X, Y, their connected sum $X \# Y$ is defined as the surface obtained by removing an open disc from each, thus yielding $X_1 = X \setminus D^2$ and $Y_1 = Y \setminus D^2$, and then gluing X_1 and Y_1 along the boundary circles of the holes. One has to check that the homeomorphism type of this object is well-defined. The number g is called the *genus* of M, and is a topological invariant. The surface of genus g above is often called a *g-handle*.

The proof of this topological classification maybe found in W. Massey's book *Introduction to Algebraic Topology*. As it turns out, the smooth (diffeomorphic) classification is the same as the topological classification, because each of the surfaces listed above admits a unique differentiable structure. (See Hirsch's *Differential Topology*, Chapter 9, for a direct classification of smooth orientable surfaces using Morse Theory).

The sphere S^2 is already simply connected. The simply connected cover of the genus-1 surface, i.e. the torus T^2, is $\mathbb{R}^2 = \mathbb{C}$, and in this case Γ is any subgroup of $\mathbb{C}$ of rank 2 (e.g. $\mathbb{Z} + i\mathbb{Z}$), called a lattice. In fact *all* lattices in $\mathbb{C}$ are of the form $\Gamma = \mathbb{Z} + j\mathbb{Z}$ where j is a complex number such that $\operatorname{Im} j > 0$. It turns out that for two lattices $\Gamma = \mathbb{Z} + j\mathbb{Z}$ and $\Gamma' = \mathbb{Z} + j'\mathbb{Z}$ the tori $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Gamma'$ are biholomorphically equivalent iff $j' = \frac{aj+b}{cj+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. The simply connected cover of all compact connected orientable surfaces of genus ≥ 2 is the upper half plane H^2.

Each of the surfaces listed above also has a nice Riemannian metric. We saw earlier that S^2 has a left-$SO(2)$ invariant Riemannian metric, viz. the one induced from $\mathbb{R}^3$. Similarly, since the lattices $\Gamma \subset \mathbb{C}$ act by translations, they are isometries with respect to the euclidean metric on $\mathbb{C}$, and thus $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ acquires a "flat" metric, denoted by $d\theta^2 + d\phi^2$.

Finally for the surfaces above of genus ≥ 2, the universal cover H^2 has the hyperbolic metric, and since the group Γ is a subgroup of $SL(2, \mathbb{R})$, it is a group of hyperbolic isometries, and the quotient naturally acquires the quotient hyperbolic metric.

The homogeneity of the metrics on S^2, $\mathbb{C} = \mathbb{R}^2$ and H^2 will result in their curvature (to be defined later) being constant functions (in fact, 1, 0, and -1 respectively). Thus the quotient surfaces above with these quotient metrics will also have constant curvature.

The compact *non-orientable* surfaces, of course cannot be complex manifolds, since all complex manifolds are orientable. However their orientable double covers will belong to the above list. They are all of the following type:

$$\mathbb{RP}(2) \# \mathbb{RP}(2) \# ... \# \mathbb{RP}(2) \quad k \text{ copies}$$

(which is called the *k-crosscap*). The involutions τ of the orientable surfaces M above, which give rise to the non-orientable surfaces $M/\{1, \tau\}$, are all isometries of M with respect to the metrics defined above. (For example, if $M = S^2$, the τ is the antipodal map, and $M/\{1, \tau\} = \mathbb{RP}(2)$.) Thus the quotients, viz. the crosscaps all acquire nice quotient metrics.

6. Connections

6.1. **Principal G-bundles.** In the sequel G will always a smooth Lie group, and $\mathfrak{g}$ its Lie algebra. L_g and R_g denote left and right translations by $g \in G$, and $\mathrm{Ad}g := L_g \circ R_{g^{-1}}$.

Definition 6.1.1 (Principal G-bundles)**.** A smooth *principal G-bundle* on a smooth manifold M is a smooth manifold P of dimension $\dim G + \dim M$ such that G acts freely and smoothly from the left on it. Further, there is a smooth surjection $\pi : P \to M$ such that each fibre $P_x = \pi^{-1}(x)$ is a free and transitive G-space, and a G-orbit of the action of G on P. Further, there exists an open covering $\{U_i\}_{i\in\Lambda}$ of M, and smooth maps (called *local trivialisations* or *bundle charts*):

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times G$$

making the diagram:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_i} & U_i \times G \\ \pi \searrow & & \swarrow \mathrm{pr}_1 \\ & U_i & \end{array}$$

commute. For each $x \in U_i$, and each i, the restriction $\Phi_{i|P_x}$ is a G-equivariant (=commuting with the left action of G) map to $\{x\} \times G$. P is called the *total space* of the bundle, and M the *base space*. π is called the *bundle projection*. G is called the *structure group* of the principal bundle. (Physicists call it the "*restricted gauge group*")

If there exists such an open covering with just a single element $\{M\}$, then the bundle is said to be a *trivial bundle*. In this case, $P = M \times G$, and the left action of G on P is via left-multipication on the second factor.

It is easy to check that local triviality forces π to be open, and thus π is a quotient map, and $M \simeq P/G$, where P/G is the orbit space of the left G-action on P.

For a principal G-bundle, we can define transition functions, analogous to what we did for vector bundles in 4.3.3. Namely, on the overlap $U_i \cap U_j$, the *left* G-equivariant bundle coordinate change:

$$(U_i \cap U_j) \times G \xrightarrow{\Phi_i^{-1}} \pi^{-1}(U_i \cap U_j) \xrightarrow{\Phi_j} (U_i \cap U_j) \times G$$

must carry (x, g) to $(x, g.g_{ij}(x))$, where $g_{ij}(x) \in G$. Thus, for each i, j, we have a smooth map:

$$g_{ij} : U_i \cap U_j \to G$$

such that the following cocycle conditions hold:

(a): $g_{ii} = \mathrm{Id}$, $g_{ij}(x) = g_{ji}(x)^{-1}$ for all i, j.

(b): $g_{ij}(x)g_{jk}(x)g_{ki}(x) = \mathrm{Id}$ for all $x \in U_i \cap U_j \cap U_k$.

A smooth map $s : M \to P$ is called a *section of* the bundle if $\pi \circ s = \mathrm{Id}_M$. As in the part (iv) of the exercise 4.3.4, this is equivalent to giving smooth functions $s_i : U_i \to G$ for each i, satisfying $s_j(x) = s_i(x)g_{ij}(x)$ for all $x \in U_i \cap U_j$.

Lemma 6.1.2. A principal G-bundle $\pi : P \to M$ is (smoothly) trivial iff it has a smooth section $s : M \to P$.

Proof: If the bundle is trivial, then the section $x \mapsto (x, e)$, (where $e \in G$ is the identity element) is a section of the trivial bundle $M \times G \to M$, and a G-bundle isomorphism $\Phi : M \times G \to P$ of this bundle to a principal G-bundle $\pi : P \to M$ would result in the section $x \to \Phi(x, e)$ of that bundle.

Conversely, if $\pi : P \to M$ admits a section, the isomorphism:

$$\begin{array}{rcl} \Phi : M \times G & \to & P \\ (x, g) & \mapsto & gs(x) \end{array}$$

is a G-bundle isomomorphism of P with the trivial G-bundle. $\square$.

Example 6.1.3. All the examples of homogeneous manifolds:

$$\pi : G \to G/H$$

constructed so far (see the section 5.7) are examples of principal H-bundles. The only difference with the definition above is the fact that in those examples, the closed subgroup H was acting from the right by right multiplication. One can easily convert it into a left action by defining $h * g := gh^{-1}$. To see local triviality, one needs to show local triviality of the bundle only around a neighbourhood of the identity coset $o = e.H \in G/H$. In the notation of proposition 5.7.1, a neighbourhood of the identity coset is $\pi(\phi(W \times V))$, which is diffeomorphic to the neighbourhood W of the origin in $\mathfrak{m}$. The open set $\pi^{-1}(W)$ in G is just $\exp(W)H$. We may therefore find a trivialisation of $\pi^{-1}(W) \to W$ by setting:

$$\begin{aligned} \Phi : \exp(W) \times H &\to \pi^{-1}(W) \\ (\exp X, h) &\mapsto (\exp X)h \end{aligned}$$

which commutes with the left action (=right multiplication) of H. Indeed, on the open set $\exp(W) \times \exp(V)$, we have seen in the proposition above that this Φ is essentially the diffeomorphism ϕ of that proposition. The H-equivariance proves it is a diffeomorphism all over $\pi^{-1}(W)$, and is the trivialisation sought.

Example 6.1.4 (Frame bundle of a vector bundle)**.** Let $\pi : E \to M$ be a smooth vector bundle of rank k. We define the set of all *frames* in the fibre E_x to be set of all ordered tuples $(v_1, .., v_k)$ of k linearly independent vectors in E_x (=ordered bases or *frames* of E_x), and denote it by $F(E)_x$. Now, define the *frame bundle* $F(E)$ to be the disjoint union:

$$F(E) = \coprod_{x \in M} F(E)_x$$

and the projection $p : F(E) \to M$ by mapping an element in $F(E)_x$ to x. There is a left action of $G = GL(k, \mathbb{R})$ on each $F(E_x)$, where $g = [g_{ij}] \in G$ takes the frame $(v_1, .., v_k)$ of E_x to the frame $(w_1, .., w_k)$, where $w_j = \sum g_{ij}^{-1} v_i$. (The funny definition stems from the fact that tuples of basis elements are to be treated as "row vectors", and the right action of G on "row vectors" has to be converted to a left action). We note that if $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ is a smooth bundle atlas for E, then we may define:

$$\begin{aligned} \Psi_i : p^{-1}(U_i) &\to U_i \times GL(k, \mathbb{R}) \\ (v_1, .., v_k) \in F(E)_x &\mapsto (x, \mathrm{pr}_{\mathbb{R}^k}(\Phi_i(v_1), ..., \Phi_i(v_k)) \end{aligned}$$

where we are naturally identifying all frames in $\mathbb{R}^k$ with G (by writing the elements of a frame as columns of a matrix). These Ψ_i then become G-equivariant bijections, and we transport the topology from $U_i \times G$, and check that they are compatible on overlaps. With this topology, it is easy to check that $F(E)$ is a smooth manifold, and $p : F(E) \to M$ is a prinicpal $GL(k, \mathbb{R})$ bundle. The bundle $F(E)$ is also sometimes denoted $GL(E)$.

Example 6.1.5. If $\pi : E \to M$ is a smooth vector bundle with a Riemannian metric $\langle \, , \rangle$, we may define the bundle of *orthonormal frames* $O(E)$ in a fashion analogous to $F(E)$ above. $O(E)_x$ is then the set of all *orthornormal frames* in E_x with respect to the inner product $\langle \, , \rangle_x$ on E_x. The discussion above goes through, *mutatis mutandis*, and exhibits $p : O(E) \to M$ as a principal $O(k)$-bundle.

In analogous fashion, if the vector bundle $\pi : E \to M$ is an oriented bundle (i.e. there is a nowhere vanishing section s of $\wedge^k(E)$), we can construct a principal $SL(k, \mathbb{R})$ bundle $p : SL(E) \to M$ by choosing only those frames $(v_1, .., v_k)$ of E_x which satisfy $v_1 \wedge .. \wedge v_k = s(x)$. If it is an oriented Riemannian vector bundle, then we can form the oriented orthonormal frame bundle $SO(E)$ by choosing only those orthonormal frames $(v_1, .., v_k)$ of E_x which satisfy $v_1 \wedge .. \wedge v_k = s(x)$.

Exercise 6.1.6. Define the set:

$$V_k(\mathbb{R}^n) = \{(v_1, .., v_k) \subset (\mathbb{R}^n)^k : \langle v_i, v_j \rangle = \delta_{ij}\}$$

of orthonormal k-frames in $\mathbb{R}^n$. Show that it is a homogeneous manifold $O(n)/O(n-k)$. It is called the *Stiefel manifold*. For example $V_1(\mathbb{R}^n) = S^{n-1}$. Show that the natural map:

$$\begin{array}{rcl} \pi : V_k(\mathbb{R}^n) & \to & G_k(\mathbb{R}^n) \\ (v_1, .., v_k) & \mapsto & \text{span}(v_1, ..., v_k) \end{array}$$

is a principal $O(k)$ bundle. Prove that it is isomorphic to the orthonormal frame bundle $O(\gamma_n^k)$ of the tautological bundle $\gamma_n^k \to G_k(\mathbb{R}^n)$. In particular, the principal $O(1) = \mathbb{Z}_2$ orthonormal frame bundle of the tautological line bundle $\gamma^1 \to \mathbb{RP}(n-1)$ is the natural quotient mapping:

$$\pi : S^{n-1} \to \mathbb{RP}(n-1)$$

Exercise 6.1.7 (Orthonormal frame bundle of TS^n)**.** Show that the orthonormal (resp. oriented orthonormal) frame bundles of the oreiented Riemannian vector bundle $TS^n \to S^n$ are $\pi : O(n+1) \to S^n$ (resp. $\pi : SO(n+1) \to S^n$), where π maps the orthogonal (resp. special orthogonal) matrix A to its last column Ae_{n+1}.

It is perhaps not yet clear why principal G-bundles were introduced. It is true that we could have continued our discussion by just sticking with vector bundles. However, it is important to note that various different vector bundles may have one *underlying* principal G-bundle. Also, as we saw in the last subsection, different structures on a vector bundle (e.g. an orientation, Riemannian bundle metric) lead to *reduction* of the structure group G (for example, from $GL(k, \mathbb{R})$ to $SL(k, \mathbb{R})$, or $O(k, \mathbb{R})$) of the principal G-bundle. All this is clarified in the sequel.

6.2. **Associated bundles to a principal bundle.** Let G be a Lie group. For the sake of brevity, we denote a set X with a left G action as a *G-space*. If X is a manifold, we require the action to be smooth, if X is a vector space, we require each left translation L_g to be linear and the homomorphism $G \to GL(X)$ taking $g \mapsto L_g$ to be smooth. If X is another Lie group H, we again require the action to be smooth. In all of these situations, we have a homomorphism or *representation*:

$$\rho : G \to \text{Aut}(X)$$

mapping g to L_g, and $Aut(X)$ is to be intepreted according to the context. Just note that when $X = H$, a group, L_g is only usually a diffeomorphism, and *not a* homomorphism. For example, if G is a subgroup of H, then left multiplication by $g \in G$ is not a homomorphism $H \to H$.

Definition 6.2.1 (Associated bundles)**.** Let $\pi : P \to M$ be a principal G-bundle, and let X be a $G-space$ in the sense above. Introduce an equivalence relation on $P \times X$ by coordinatewise left action of G (called the diagonal action). That is $(a, x) \sim (ga, gx)$ for all $g \in G$. Denote the equivalence class of (a, x) as $[p, x]$.

Then the *associated bundle with fibre* X is defined as follows:

$$P \times_\rho X = P \times X/G = \{[a, x] : a \in P, \;\; x \in X\}$$

It has a natural quotient topology. Define the projection map $p : P \times_\rho X \to M$ by $p([a, x]) = \pi(a)$. This is clearly well defined since $\pi(a) = \pi(ga)$ for all $g \in G$.

A typical fibre, say $y \in M$, will be $p^{-1}(y) \simeq a \times X$, where a is some fixed element of $\pi^{-1}(y)$. (Since the action of G on P is free, (a, x) is a well-defined representative of the equivalence class $[a, x]$). We leave it to the reader to show that if P has bundle charts over open sets $U_i \subset M$, then over U_i this fibre bundle will be isomorphic to the trivial bundle $U_i \times X$.

Example 6.2.2. If X is a vector-space of dimension k, and G acts on V by linear automorphisms (i.e. there is a smooth representation $\rho : G \to GL(V)$), then the associate bundle $p : P \times_{rho} V \to M$ is a vector bundle of rank k. If ρ maps into $SL(V)$, then this associated vector bundle becomes an oriented vector bundle. If V has an inner product $\langle\, ,\, \rangle$, and ρ maps into $O(V)$, then this vector bundle acquires a Riemannian metric. If ρ maps into $SO(V)$, it becomes an oriented vector bundle with Riemannian metric. If

$$g_{ij} : U_i \cap U_j \to G$$

are the transition functions of the given principal bundle , show that the transition functions of these associated vector bundles are $\rho \circ g_{ij}$.

Proposition 6.2.3. Let $\pi : E \to M$ be a smooth vector bundle. Let V denote the fibre E_x of some fixed point $x \in M$. We fix a linear isomorphism (= identification) $\tau : \mathbb{R}^k \to V$. Thus get an action of $GL(k\mathbb{R})$ on V, by the (natural) left multiplication on column vectors. That is, we fix an isomorphism $\rho : GL(k, \mathbb{R}) \to GL(V)$. Let $F(E)$ denote the frame bundle of E, constructed in 6.1.4. Then the associated vector bundle $F(E) \times_\rho V \to M$ is isomorphic to the original vector bundle $E \to M$.

Proof: A point $(a, v) \in F(E) \times V$ gives a frame $a = (v_1, .., v_k)$ in $y = \pi(a) \in M$, and a vector $\tau(\sum_i \lambda_i e_i)$ where $\{e_i\}$ is the standard basis of $\mathbb{R}^k$. Define the map:

$$\begin{array}{rcl} \Phi : F(E) \times V & \to & E \\ (a, v) \in F(E)_y \times V & \mapsto & \displaystyle\sum_i \lambda_i v_i \in E_y \end{array}$$

We note that if $g = [g_{ij}] \in GL(k, \mathbb{R})$, then ga is the frame of E_y given by $w_j = \sum_i g_{ij}^{-1} v_i$. Also $g.v = \tau(\sum_j \mu_j e_j)$, where $\mu_j = \sum_j g_{jl} \lambda_l$. Then

$$\begin{aligned} \Phi(ga, gv) &= \sum_j \mu_j w_j = \sum_{i,l,j} g_{jl} \lambda_l g_{ij}^{-1} v_i \\ &= \sum_{i,l} \delta_{il} \lambda_l v_i = \sum_i \lambda_i v_i \end{aligned}$$

This proves that Φ descends to a map $\phi : F(E) \times_\rho V \to E$. The verification that this is an isomorphism of vector bundles is left to the reader. $\square$

Exercise 6.2.4. Prove the analogous facts for an oriented (resp. Riemannian vector bundle) $E \to M$, and the principal bundles $SL(E)$ (resp. $O(E)$) constructed in 6.1.5.

Exercise 6.2.5. In the notation of the preceding exercise, show how the various associated bundles $\wedge^k(E)$, E^*, $(\otimes^k(E)) \otimes (\otimes^l(E^*))$ may be constructed as associated bundles to the prinicpal bundle $F(E) \to M$ and suitable representations of $GL(k, \mathbb{R})$ on vector spaces (such as $\wedge^k(V)$ etc.) associated with V.

Definition 6.2.6 (Reduction of structure group)**.** Let $\pi : P \to M$ be a principal G-bundle, and let H be a subgroup of G. Thus there is the natural left action of H on G by left multiplication, and this makes G into a left H-space, and as usual we denote the structure map by ρ. Note that the associated bundle $Q \times_\rho G$ acquires a left action of G by $g.[x, b] := [x, b.g^1]$. In fact, it becomes a principal G-bundle with this action of G (Prove). We say that the principal G-bundle $P \to M$ admits *reduction of structure group to* H if there exists a principal H-bundle $Q \to H$ such that the associated bundle $Q \times_\rho G$ is isomorphic to P as a principal G-bundle.

For example, in the situation of an oriented (resp. Riemannian) vector bundle, the frame bundle $F(E)$ admitted a reduction of structure group from $GL(k, \mathbb{R})$ to $SL(k, \mathbb{R})$ (resp. $O(k)$). The reader should verify that $F(E) \simeq SL(E) \times_\rho GL(k, \mathbb{R})$ (resp $\simeq O(E) \times_\rho GL(k, \mathbb{R})$.)

For further details about prinicpal bundles and associated fibre bundles, the reader may consult D. Husemoller's book *Fibre Bundles* or Kobayashi and Nomizu's *Foundations of Differential Geometry*, Vol 1.

6.3. **The Maurer-Cartan Form.** Let G be a Lie group (of dimension say m), and $\mathfrak{g}$ its Lie algebra. We have already seen that a trivialisation of $T(G)$ can be obtained by choosing a basis $\{X_i\}_{i=1}^m$ of $\mathfrak{g}$, and generating the m everywhere linearly independent sections $\widetilde{X}_i$, the left-invariant vector fields corresponding to X_i. Similarly, if we take an element ω of $\mathfrak{g}^*$, we can generate a left-invariant 1-form $\widetilde{\omega}$ by the prescription:

$$(L_g)^*\widetilde{\omega}(g) = \widetilde{\omega}(e) = \omega$$

That is, $\widetilde{\omega} \circ L_g = L^*_{g^{-1}}\widetilde{\omega}$. Again, starting with a basis $\{\omega_i\}$ of $\mathfrak{g}^*$ will lead to left invariant 1-forms $\{\widetilde{\omega}_i\}$ trivialising T^*G, the cotangent bundle of G. If we denote by $\Omega^1(G)$ the space of 1-forms on G, then we have the natural identification:

$$\begin{array}{rcl} \theta : \Omega^0(G) \otimes \mathfrak{g}^* & \to & \Omega^1(G) \\ f \otimes \omega & \mapsto & f\widetilde{\omega} \end{array}$$

which preserves the left Ω^0 module structure (multiplication with smooth functions) of both sides.

Now, tensor product $\Omega^1(G) \otimes \mathfrak{g}$ (=the space of $\mathfrak{g}$ valued 1-forms), which we denote $\Omega^1(G, \mathfrak{g})$, therefore becomes isomorphic to $\Omega^0(G) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Now, for any vector space, there V is a unique element $\omega_V \in V^* \otimes V$, which can be denoted by $\sum_i e_i^* \otimes e_i$, where $\{e_i\}$ is *any* basis of V, and $\{e_i^*\}$ the corresponding dual basis. This element is well- defined *independent of choice of basis*, and corresponds to the identity map Id_V under the *canonical isomorphism* $V^* \otimes V \to \hom_{\mathbb{R}}(V, V)$. Similarly, there is the constant function 1 on G that is a distinguished element of $\Omega^0(G)$.

Definition 6.3.1. Under the above identification:

$$\Omega^1(G, \mathfrak{g}) \simeq \Omega^0(G) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$$

the $\mathfrak{g}$-valued 1-form in $\Omega^1(G, \mathfrak{g})$ corresponding to the distinguished element $1 \otimes \omega_{\mathfrak{g}}$ in the right hand side is called the *Maurer-Cartan* 1-*form* of G. If we choose a basis $\{X_i\}$ of $\mathfrak{g}$, and denote by ω_i its dual basis, then $\omega_{\mathfrak{g}} = \sum_i \omega_i \otimes X_i$. This corresponds under θ above to the left-invariant $\mathfrak{g}$ valued 1-form:

$$\omega = \sum_i X_i \otimes \widetilde{\omega}_i$$

We call it *the* Maurer-Cartan form of G, because it is the unique left invariant 1-form on G whose value at $e \in G$ is the (also unique) form $\omega_{\mathfrak{g}}$.

We can similarly define the *right invariant* Maurer-Cartan form of a Lie group G to be the unique $\mathfrak{g}$ valued one form on G which is invariant under right translations, and agrees with $\omega_{\mathfrak{g}}$ at the identity. This right invariant Maurer-Cartan 1-form will not agree with the left invariant one unless the group G is abelian.

We can compute the Maurer-Cartan form of $GL(n, \mathbb{R})$, for example.

Example 6.3.2 (Maurer-Cartan form of $GL(n, \mathbb{R})$)**.** Note that since $G = GL(n, \mathbb{R})$ is an open subset of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^n$, there are global coordinates g_{ij} available on G. We let g denote the $\mathfrak{g}$-valued function given by restricting the identity map of $\mathfrak{g}$ to the open subset G. Thus there are also the globally defined 1-forms dg_{ij}, and indeed the $\mathfrak{g}$-valued 1-form $dg := [dg_{ij}]$ on G.

If $h = [h_{ij}]$ is a fixed matrix, then for a $B \in GL(k, \mathbb{R})$ we have:

$$g_{ij}(L_h B) = (h.B)_{ij} = \sum_l h_{il} g_{lj}(B)$$

which shows that $L_h^* g = g \circ L_h = h.g$ and similarly $R_h^* g = g \circ R_h = g.h$

Hence

$$L_h^* dg = d(L_h^* g) = d(h.g) = h.dg$$

and similarly $R_h^* dg = dg.h$ So we see that dg is neither left nor right invariant. But since the function g on G also undergoes the same effect under left and right translations,

the 1-form $\omega^R := dg.g^{-1}$ on G (the dot "." again means matrix multiplication) will be right-invariant, and similarly the 1-form $\omega^R = g^{-1}dg$ will be left-invariant.

Now we just have to examine their values at $I \in G$. But at the identity we have the basis $\{E_{ij}\}$ of $T_e(G) = \mathfrak{gl}(n, \mathbb{R})$. Our one form $dg.g^{-1}$ at the identity is just the matrix $[dg_{ij}]$, which maybe written as

$\sum_{i,j} E_{ij} \otimes dg_{ij}$. Note that, in Euclidean space $\mathfrak{g}$, $E_{ij} = \frac{\partial}{\partial g_{ij}}$ are the natural basis of coordinate partials, and dg_{ij} are the dual basis. Thus ω^L and ω^R are, by uniqueness, the left and right invariant Maurer-Cartan forms of G respectively. .

Note that $L_h^*\omega^R = hdg.g^{-1}h^{-1} = \mathrm{Ad}(h)_*\omega^R$, and similarly $R_h^*\omega^L = \mathrm{Ad}(h^{-1})_*(\omega^L)$, where the adjoint automorphism $\mathrm{Ad}(h)_* : \mathfrak{g} \to \mathfrak{g}$ is being applied pointwise. This motivates the:

Exercise 6.3.3. Prove the analogous fact for the left and right Maurer- Cartan forms of any Lie group G. (Just examine the effect of L_h^* (resp. R_h^*) on right-invariant (resp. left invariant) 1- forms.)

Remark 6.3.4 (Maurer-Cartan forms of linear lie groups)**.** It is clear that the restriction (=pullback under inclusion) of the Maurer-Cartan form of G to a Lie subgroup H will be the Maurer-Cartan form of H. Thus we can restrict the left (resp. right) Maurer-Cartan form of $GL(n,\mathbb{R})$ to any of the classical subgroups $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$ and get their left (resp. right) Maurer-Cartan forms. For example, for $H = SO(2) \subset GL(2,\mathbb{R})$, we have:

$$\begin{aligned} dg.g^{-1} &= \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} d\theta . \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\theta \end{aligned}$$

When one identifies the Lie-algebra $\mathbb{R}$ of $SO(2)$ as the subspace spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this becomes the well-known volume form $d\theta$ of the circle group, which is right-invariant. The left Maurer-Cartan form for $SO(2)$, viz. $g^{-1}dg$ also comes out to be the same, since $SO(2)$ is abelian!

6.4. **Connections on principal bundles.**

Definition 6.4.1. Let $\pi : P \to M$ be a smooth principal G-bundle. The *vertical tangent bundle*, denoted $T^vP \to P$ is defined by the short exact sequence of vector bundles:

$$0 \to T^vP \to TP \stackrel{\pi_*}{\to} \pi^*TM \to 0$$

That is, it is the kernel bundle of the surjective bundle morphism $\pi_* = D\pi : TP \to \pi^*TM$ (that π_* is surjective follows from local triviality).

Lemma 6.4.2. Let $\pi : P \to M$ be a principal G-bundle, and L_g denote the left-translation map $x \mapsto gx$ of P. Then:

(i): The vertical tangent bundle is mapped isomorphically to itself by $L_{g*} = DL_g$ for each $g \in G$. That is $L_{g*}(T_x^vP) = T_{gx}^vP = (L_g^*T^vP)_x$.

(ii): For each $x \in P$, $T_x^v(P) = T_x(Gx)$. That is, the vertical tangent space at each point $x \in P$ is the tangent space to the orbit Gx.

(iii): T^vP is a trivial vector bundle with fibre $\mathfrak{g}$.

Proof:

(i) is clear since $\pi \circ L_g = \pi$, which implies that $L_g^*\pi^*TM = \pi^*TM$ and $\pi_* = \pi_* \circ L_{g*}$.

For (ii), note that for $x \in P$, $\pi^{-1}(\pi(x)) = Gx$, and since $\pi(x)$ is a regular value of π, the kernel of $D\pi(x)$ is precisely the tangent space $\pi^{-1}(\pi(x))$ by 3.3.22.

For (iii), we recall that the left-action of G on P is given by a smooth map:

$$\begin{aligned} \mu : G \times P &\to P \\ x &\mapsto \mu(g,x) := g.x \end{aligned}$$

Thus it has a derivative $D\mu(e,x) : \mathfrak{g} \times T_x(P) \to T_x(P)$. Note that for a fixed $x \in P$, $\mu_x := \mu(-,x) : G \to Gx$ is a left G-equivariant diffeomorphism, and therefore maps $\mathfrak{g}$ isomorphically onto $T_x(Gx) = T_x^v P \subset T_x P$.

In particular, given an $X \in \mathfrak{g}$, we get a vector field on P, which we also denote as $\widetilde{X}$, defined by $\widetilde{X}(x) := \mu_{x*}(X)$. Now, for an $h \in G$, we have

$$\mu_{hx}(\mathrm{Ad}(h)g) = hgh^{-1}.hx = hg.x = L_h \circ \mu_x(g)$$

Thus

$$L_{h*}\widetilde{X} = L_{h*}(D\mu_x(e)X) = D\mu_{hx}(e)(\mathrm{Ad}(h)_*X) = (\widetilde{\mathrm{Ad}(h)_*X})$$

Thus $\widetilde{X}$ *is not* a left-invariant vector field, but obeys the Ad-invariance property:

$$L_{h*}\widetilde{X} = \widetilde{\mathrm{Ad}(h)_*X})$$

Definition 6.4.3 (Maurer-Cartan section of the vertical bundle)**.** By the last proposition, there is a smooth and canonical isomorphism of vector bundles:

$$\hom(T^vP, T^vP) \simeq \hom(T^vP, \mathfrak{g}) \simeq (T^vP)^* \otimes \mathfrak{g}$$

Denote by ω_v the section of $(T^vP)^* \otimes \mathfrak{g}$ which corresponds to the identity section of $\hom(T^vP, T^vP)$ under this identification. If we choose a basis $\{X_i\}$ of $\mathfrak{g}$, and denote by $\widetilde{e}_i$ the Ad-invariant sections of $T^{v*}P$ which satisfy $\widetilde{e}_i(\widetilde{X}_j) = \delta_{ij}$, then ω_v is given by:

$$\omega_v = \sum_i \widetilde{e}_i \otimes X_i$$

Again, as in the case of the Maurer-Cartan form of a Lie group, the reader may verify that this expression for ω_v holds for all choices of basis for $\mathfrak{g}$. Clearly, ω_v is Ad-invariant under left translations L_g^*. That is, for a fixed $g \in G$,

$$L_g^*\omega_v = \mathrm{Ad}(g)_*\omega_v$$

where the right side denotes pointwise multiplication. It will be called the *Maurer-Cartan section of* $(T^vP)^* \otimes \mathfrak{g}$.

Remark 6.4.4. It is natural to ask what the Maurer-Cartan section of $(T^vP)^* \otimes \mathfrak{g}$ has to do with the left and right Maurer-Cartan forms of G introduced in 6.3.1. The exercise 6.3.3 suggests that the transformation property of the Maurer-Cartan section ω_v above, under left translations, is the same as that of the right Maurer-Cartan form ω^R of the group G. Indeed, the reader can convince herself that if we identify the orbit Gx with G, via the G-equivariant map μ_x, then the pullback (under μ_x) of the Maurer-Cartan section ω_v is the same as the right Maurer-Cartan section ω^R of G.

Definition 6.4.5. A *connection* on a principal G-bundle $\pi : P \to M$ is a vector bundle morphism:

$$\eta : TP \to T^vP$$

such that:

(i): $\eta_{|T^vP} = \mathrm{Id}_{T^vP}$

(ii): $\eta \circ L_{g*} = L_{g*} \circ \eta$

(We recall from (i) of the lemma above that L_{g*} preserves the vertical tangent bundle, so this condition makes sense).

Notation 6.4.6. In the sequel, for $g \in G$, we will denote the automorphism $\mathrm{Ad}(g)_* : \mathfrak{g} \to \mathfrak{g}$ as $\mathrm{Ad}(g)$ for notational convenience.

Proposition 6.4.7. Let $\pi : P \to M$ be a principal G-bundle. The following are equivalent:

(i): A connection on the principal G-bundle $P \to M$.

(ii): A smooth sub-bundle $T^hP \subset TP$ (called the *horizontal bundle*) such that (a) $L_{g*}(T^h_x) = T^h_{gx}P$ for all $x \in P$, $g \in G$, and (b) $T^hP \oplus T^vP = TP$.

(iii): A smooth section ω of $T^*P \otimes \mathfrak{g}$ (=smooth Lie algebra-valued 1-form) satisfying: (a) $L^*_g\omega = \mathrm{Ad}(g)\omega$ and (b) $\omega(X) = \omega_v(X)$ for every vertical tangent vector $X \in T^vP$. (Note that in (a) $\mathrm{Ad}(g)\omega$ denotes the 1-form whose value on a tangent vector v is $\mathrm{Ad}(g)(\omega(v))$).

Proof:

(i) $\Rightarrow$ (ii) Clearly, if η is a smooth connection, then for each $x \in P$, the kernel bundle:

$$T^hP := \ker \eta : TP \to T^vP$$

will satisfy $T^hP \oplus T^vP = TP$, because, by definition, η is a bundle-splitting map for the inclusion $T^vP \to TP$, and hence (b) of (ii) follows. Also, for $v \in T^h_xP = \ker \eta_x : T_xP \to T^v_xP$, $\eta(L_{g*}v) = L^*_g(\eta(v)) = 0$, by the property (ii) in 6.4.5. Thus $L_{g*}(v) \in \ker \eta_{gx}$, which implies (a) of (ii)

(ii) $\Rightarrow$ (iii) Since $TP = T^vP \oplus T^hP$, we have an isomorphism

$$T^*P \otimes \mathfrak{g} = (T^vP)^* \otimes \mathfrak{g} \oplus (T^hP)^* \otimes \mathfrak{g}$$

We already have the smooth Maurer-Cartan section ω_v of $(T^vP)^* \otimes \mathfrak{g}$, by 6.4.3. Define the section ω of the left hand bundle by $\omega = (\omega_v, 0)$. Then (b) of (iii) is trivial. The fact that $L^*_g\omega_v = \mathrm{Ad}(g)\omega_v$, and the fact that L_{g*} preserves the bundles T^hP and T^vP, and hence the direct sum decomposition $TP = T^vP \oplus T^hP$, implies that $L^*_g\omega = \mathrm{Ad}(g)\omega$.

(iii) $\Rightarrow$ (i) Let ω be a smooth form satisfying (a) and (b), and for a $Y \in T_xP$, define $\eta(Y) = \widetilde{\omega(x)(Y)}(x) \in T^v_xP$. Note that $\omega(x)(Y) \in \mathfrak{g}$, and it makes sense to define the Ad-invariant vector field from it, as explained in (iii) of 6.4.2.

Then

$$\begin{aligned} \eta(L_{g*}Y) &= \widetilde{\omega(gx)(L_{g*}Y)}(gx) = \widetilde{L^*_g\omega(x)(Y)}(gx) \\ &= \mathrm{Ad}(g)\widetilde{(\omega(x)(Y))}](gx) \\ &= L_{g*}[\widetilde{\omega(x)Y}(x)] \\ &= L_{g*}(\eta(Y)) \end{aligned}$$

which proves (ii) of the definition of a connection. For some basis X_i of $\mathfrak{g}$, the vector fields $\widetilde{X}_i$ span T^vP at each point, and furthermore $\omega_v = \sum_i \widetilde{e}_i \otimes X_i$. Thus:

$$\omega(x)(\widetilde{X}_j) = \omega_v(\widetilde{X}_j) = \sum_i \widetilde{e}_i(\widetilde{X}_j) \otimes X_i = \sum_i \delta_{ij} X_i = X_j$$

. Hence $\eta(\widetilde{X}_j) = \widetilde{\omega(\widetilde{X}_j)} = \widetilde{X}_j$ which proves (i) in the definition of a connection. □

Corollary 6.4.8. The connection $\eta : TP \to T^vP$ and the connection form ω are related by the formula:

$$\omega(Z) == \omega_v(\eta(Z)) \quad \text{for } \; Z \in TP$$

Proof: This is clear from the step (ii) $\Rightarrow$ (iii) in the proof of the above proposition, for if $Z \in TP$, then $Z = (\eta(Z), Z - \eta(Z))$ in the decomposition $TP = T^vP \oplus T^hP$, and so $\omega(Z) = \omega_v(\eta(Z))$. □.

Definition 6.4.9. Let $\pi : P \to M$ be a principal G-bundle. A G- equivariant automorphism $\phi : P \to P$ such that $\pi \circ \phi = \phi$ (i.e. a bundle isomorphism of the principal bundle) is called a *gauge transformation.* The set of gauge transformations is a group, called the *full gauge group*, and denoted by $\mathcal{G}$.

Remark 6.4.10. If we let $\{(\Phi_i, U_i)\}$ be a G-bundle atlas for $\pi : P \to M$, we can define smooth local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$ of the bundle $P_{|U_i}$ by setting $\sigma_i(x) = \Phi_i^{-1}(x, e)$. We note that these σ_i will satisfy the following patching condition on $U_i \cap U_j$:

$$\sigma_i(x) = (\Phi_j^{-1}\Phi_j)\Phi_i^{-1}(x, e) = \Phi_j^{-1}(x, g_{ij}(x)) = g_{ij}(x)\Phi_j^{-1}(x, e) = g_{ij}(x)\sigma_j(x)$$

where $g_{ij} : U_i \cap U_j \to G$ are the bundle transition functions introduced in the first subsection 6.1.

Now, given a gauge transformation ϕ, we note that for $x \in U_i$, $\phi(g.\sigma_i(x)) = g\phi(\sigma_i(x))$ by G- equivariance of ϕ, and hence ϕ is completely determined by $\phi(\sigma_i)$. Also, since $\phi(\sigma_i(x))$ is in the same fibre as $\sigma_i(x)$, there is a unique $\phi_i(x) \in G$ such that $\phi(\sigma_i(x)) = \phi_i\sigma_i(x)$, and again ϕ is uniquely determined by these maps $\phi_i : U_i \to G$. Finally, for $x \in U_i \cap U_j$:

$$\begin{aligned}
\phi_i(x)\sigma_i(x) &= \phi(\sigma_i(x)) = \phi(g_{ij}(x)\sigma_j(x)) \\
&= g_{ij}(x)\phi(\sigma_j(x)) = g_{ij}(x)\phi_j(x)\sigma_j(x) \\
&= (\mathrm{Ad}g_{ij}(x)(\phi_j(x)))\,\sigma_i(x)
\end{aligned}$$

which shows that the maps ϕ_i patch up to give a section of the associated *adjoint bundle* $P \times_{\mathrm{Ad}} G$. Hence a gauge transformation is equivalent to a section of the adjoint G-bundle of G.

Definition 6.4.11. Let $\eta : TP \to T^vP$ be a smooth connection, and let $\phi : P \to P$ be a gauge transformation. Since ϕ preserves the G-orbit of a point, its derivative ϕ_* maps the vertical tangent space T^v_xP to the vertical tangent space $T^v_{\phi(x)}P$. Also the We define the *pullback connection* η^ϕ on $\pi : P \to M$ to be the composite:

$$TP \xrightarrow{\phi_*^{-1}} TP \xrightarrow{\eta} T^vP \xrightarrow{\phi_*} T^vP$$

Note that $\eta_{T^vP} = \mathrm{Id}_{T^vP}$ implies the same fact for η^ϕ, thus making η^ϕ also a connection.

This gives an action of the full gauge group $\mathcal{G}$ on the space of all connections on P. Before we proceed further, here is another fruitful way of describing a gauge transformation.

Remark 6.4.12. There is a $1-1$-correspondence between the full gauge group $\mathcal{G}$ and the set:

$$\{g : P \to G \text{ smooth} : \ g(hx) = \mathrm{Ad}h.g(x) = hg(x)h^{-1} \ \text{ for all } \ x \in P\}$$

which is a group via pointwise multiplication. For, if $\phi \in \mathcal{G}$, then since ϕ preserves fibres, and G acts freely on the fibres, for each $x \in P$:

$$\phi(x) = g_\phi^{-1}(x).x$$

for a unique $g_\phi(x) \in G$. From the G-equivariance of ϕ, it follows that $g_\phi^{-1}(hx).hx = \phi(hx) = h\phi(x) = hg_\phi^{-1}(x)h^{-1}.hx$, which implies that $g_\phi(hx) = Ad(h)g_\phi(x)$ for $h \in G$ and $x \in P$. Similarly, one easily checks that the map $\phi \mapsto g_\phi$ of $\mathcal{G}$ to the set above is a group homomorphism for:

$$\phi \circ \psi(x) = \phi(g_\psi^{-1}(x)x) = g_\psi^{-1}(x)\phi(x) = g_\psi^{-1}(x)g_\phi^{-1}(x).x = (g_\phi(x)g_\psi(x))^{-1}\,x$$

Since it is clearly a bijection, the map $\phi \mapsto g_\phi$ is an isomorphism of groups. The reader may easily check from the definitions that $g(\sigma_i(y))^{-1} = \phi_i(y)$ for $y \in U_i$, where $\phi_i : U_i \to G$ are defined in 6.4.10. From this relation, and the fact that $g(hx) = \mathrm{Ad}h.g(x)$, the reader may also check that $g : P \to G$ is smooth.

Lemma 6.4.13. We recall that we have trivialised the vertical tangent bundle in 6.4.2, via the map $X \to \widetilde{X}$ for $X \in \mathfrak{g}$. If $\phi : P \to P$ is a gauge transformation, then

$$\phi_*(\widetilde{X}(x)) = \widetilde{X}(\phi(x))$$

Proof: By definition $\mu_x(g) = gx$, and thus by the G-equivariance $\phi \circ \mu_x(g) = g\phi(x) = \mu_{\phi(x)}$. Thus:

$$\phi_*(\widetilde{X}(x)) = D\phi(x)(D\mu_x(e)(X)) = D\mu_{\phi(x)}(e)(X) = \widetilde{X}(\phi(x))$$

Corollary 6.4.14. Let ω_v be the Maurer-Cartan section of the vertical cotangent bundle $(T^vP)^*\otimes\mathfrak{g}$ defined in 6.4.3. If $\phi\in\mathcal{G}$, we have the isomorphism of bundles $\phi_*:T^vP\to\phi^*T^vP$, and hence the dual isomorphism $\phi^*:\phi^*(T^vP)^*\to(T^vP)^*$, and the induced isomorphism, which we also denote as $\phi^*:\phi^*(T^vP)^*\otimes\mathfrak{g}\to(T^vP)^*\otimes\mathfrak{g}$. Then, we claim:

$$(\phi^*\omega_v)=\omega_v$$

for all $x\in P$.

Proof: We let X_i be a basis of $\mathfrak{g}$, and $\widetilde{e}_i$ be the dual basis to the fields $\widetilde{X}_i$, retaining the notation of 6.4.3. The lemma above implies that $\phi_*(\widetilde{X}_i)=\widetilde{X}_i\circ\phi$.

By definition:

$$(\phi^*\widetilde{e}_i)(x)(\widetilde{X}_j(x))=\widetilde{e}_i(\phi(x))(\phi_*(X_i(x)))=\widetilde{e}_i(\phi(x))(\widetilde{X}_j(\phi(x))=\delta_{ij}$$

which shows that $(\phi^*\widetilde{e}_i)(x)=\widetilde{e}_i(x)$. Then, using the explicit expression $\omega_v=\sum_i\widetilde{e}_i\otimes X_i$ for the Maurer-Cartan section (see 6.4.3), the result follows. $\square$.

Proposition 6.4.15 (Effect of $\mathcal{G}$ on the connection form)**.** Let ω be the connection form of a connection η, and ω^ϕ that of η^ϕ, where $\phi\in\mathcal{G}$. Also let $g:=g_\phi:P\to G$ be the smooth map associated with ϕ as described in 6.4.12 above. Then:

$$\omega^\phi=(\mathrm{Ad}\,g)\omega+g^*\omega^R$$

where ω^L is the left-invariant Maurer-Cartan 1-form on G.

Proof: We recall from the corollary 6.4.8 above that $\omega(Z)=\omega_v(\eta(Z))$, for $Z\in T_xP$. If $\phi\in\mathcal{G}$ is a gauge transformation, then:

$$\omega^\phi(Z)=\omega_v(\eta^\phi(Z))=\omega_v(\phi_*\eta\phi_*^{-1}(Z))$$

Since η annihilates horizontal vectors, we need the vertical component of $\phi_*^{-1}(Z)$ in the connection η. Note that ϕ^{-1} is the composite:

$$P\overset{(g_\phi,1)}{\to}G\times P\overset{\mu}{\to}P \tag{9}$$

so we need to compute the (vertical component of) the derivative $D\mu(g(x),x)$ where, for notational convenience, we drop the subscript ϕ on g_ϕ. For any $g\in G$, there is the commutative square:

$$\begin{array}{ccccc} G & \times & P & \xrightarrow{\mu} & P\\ L_g\downarrow & & \downarrow\mathrm{Id} & & \downarrow L_g\\ G & \times & P & \xrightarrow{\mu} & P\end{array}$$

which yields the relation:

$$L_{g*}\circ D\mu(1,x)=D\mu(g,x)\circ(L_{g*},\mathrm{Id})$$

Hence if $(Y,Z)\in T_{(g,x)}(G\times P)=T_gG\oplus T_xP$ is a tangent vector at (g,x), we have:

$$D\mu(g,x)(Y,Z)=L_{g*}D\mu(1,x)(L_{g*}^{-1}Y,Z)$$

Now we recall the right-invariant Maurer-Cartan form ω^R of G, as defined in 6.3.1. Note that $\mathrm{Ad}g^{-1}\omega^R(Y):=\omega^R(L_{g*}^{-1}Y)=L_{g*}^{-1}(Y)$, because $L_{g*}^{-1}Y\in T_1G=\mathfrak{g}$ and ω^R acts on it as identity. Thus we have the formula:

$$D\mu(g,x)(Y,Z)=L_{g*}D\mu(1,x)(\mathrm{Ad}g^{-1}\omega^R(Y),Z) \tag{10}$$

If we just want the vertical component, we apply η to the formula above, and using the fact that η commutes with L_{g*} we have:

$$\begin{aligned}\eta(D\mu(g,x)(Y,Z)) &= \eta(L_{g*}D\mu(1,x)(\mathrm{Ad}g^{-1}\omega^R(Y),Z)\\ &= L_{g*}\left(\eta(\mu_{x*}(\mathrm{Ad}g^{-1}\omega^R(Y))+\eta(D\mu(1,x)(0,Z)\right)\\ &= L_{g*}(\eta(\widetilde{\mathrm{Ad}g^{-1}\omega^R}(Y))+L_{g*}\eta(Z)\\ &= \eta(L_{g*}(\widetilde{\mathrm{Ad}g^{-1}\omega^R}(Y)(x))+L_{g*}\eta(Z)\\ &= \eta(gx)(\widetilde{\omega^R(Y)})+L_{g*}\eta(Z)\\ &= \widetilde{\omega^R(Y)}(gx)+L_{g*}\eta(Z)\end{aligned} \tag{11}$$

since $D\mu(1,x)(0,Z) = Z$, $L_{g*}\widetilde{W} = \widetilde{\mathrm{Ad}g.W}$ for any $W \in \mathfrak{g}$ (by proof of (iii) in Lemma 6.4.2), and $\widetilde{\omega^R(Y)}$ is a vertical vector field, and so η maps it to itself. Thus, from (9) above, we have:

$$\begin{aligned} \eta(\phi_*^{-1}(Z)) &= \eta(D\mu(g,x)(Dg(x)Z, Z)) \\ &= (\omega^R\widetilde{(Dg(x)}Z))(g(x)x) + L_{g(x)*}\eta(Z) \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} \omega^\phi(x)(Z) &= \omega_v(\phi_*\eta\phi_*^{-1}(Z)) \\ &= (\phi^*\omega_v)(x)(\eta\phi_*^{-1}(Z)) \\ &= \omega_v(\phi(x))(\eta\phi_*^{-1}(Z)) \\ &= \omega_v(g(x)x)\omega^R\widetilde{(Dg(x)}Z)(g(x)x) + \omega_v L_{g(x)*}\eta(Z) \\ &= \omega^R(Dg(x)Z)) + \omega_v(L_{g(x)*}\eta(Z)) \\ &= g^*\omega^R(Z) + (L_{g(x)}^*\omega_v)(\eta(Z)) \\ &= g^*\omega^R(Z) + (\mathrm{Ad}g(x)\omega_v)(\eta(Z)) \\ &= g^*\omega^R(Z) + \mathrm{Ad}g(x)(\omega(Z)) \end{aligned} \tag{13}$$

where we used the corollary 6.4.14 in the third line, equation (12) in the fourth line, the fact that $\omega_v(y)(\widetilde{W}(y)) = W$ for all $W \in \mathfrak{g}$ in the fifth line, and and the fact that $L_g^*\omega_v = \mathrm{Ad}g.\omega_v$ from 6.4.3. in the sixth line. This proves the proposition. $\square$

Remark 6.4.16. From the equation (13) in the proof above, it follows that:

$$\omega^\phi(Z) = \omega_v(\eta\phi_*^{-1}(Z)) = \omega(\phi_*^{-1}(Z)) = (\phi^{-1})^*\omega(Z)$$

That is,

$$\omega^\phi = (\phi^{-1})^*\omega$$

Definition 6.4.17 (Local description of a connection)**.** Let $\{\Phi_i, U_i\}$ be a bundle atlas for the principal G-bundle $\pi : P \to M$, and let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be the smooth local sections defined in the remark 6.4.10 above. Let ω be the Lie algebra-valued 1-form of a connection on $\pi : P \to M$ as in (iii) of 6.4.7. We denote the pullback 1-forms $\sigma_i^*\omega$, which are sections of $T^*U_i \otimes \mathfrak{g}$, as ω_i. Since the inverse of the trivialisation map $\Phi_i^{-1} : U_i \times G \to P_{|U_i}$ is defined by $(x,g) \mapsto L_g \circ \sigma_i(x)$, and that $L_g^*\omega = \mathrm{Ad}g(\omega)$, it is clear that all the ω_i completely determine ω.

It is clear that the local forms ω_i defined above will have some compatibility formula on the overlaps $U_i \cap U_j$. This follows from the proposition 6.4.15 above. More precisely:

Proposition 6.4.18 (Compatibility condition for the local forms ω_i)**.** Let ω_i denote the local connection 1-forms on U_i for the connection η as in 6.4.17 above. Then:

$$\omega_i = (\mathrm{Ad}g_{ij})\omega_j + (g_{ij}^*\omega^R) \quad \text{for all } x \in U_i \cap U_j$$

In particular, if $G = GL(n,\mathbb{R})$, we recall that $\omega^R = dg.g^{-1}$, and hence $g_{ij}^*\omega^R = (dg_{ij}.g_{ij}^{-1})$, which yields:

$$\omega_i = (\mathrm{Ad}g_{ij})\omega_j + dg_{ij}.g_{ij}^{-1} \quad \text{for all } x \in U_i \cap U_j$$

Proof: Consider the principal G-bundle $P_{ij} := P_{|U_i \cap U_j} = \pi^{-1}(U_i \cap U_j)$ on $U_{ij} := U_i \cap U_j$. We can define a gauge transformation of this bundle by:

$$\phi(h\sigma_j(x)) = h\sigma_i(x) = hg_{ij}(x)\sigma_j(x)$$

This would mean that if $g_\phi : P_{ij} \to G$ is the corresponding map, then we have $g_\phi(\sigma_j(x)) = g_{ij}(x)$. Now since

$$\sigma_i = \phi^{-1}\sigma_j$$

we have:

$$\sigma_i^*\omega = \sigma_j^*((\phi^{-1})^*\omega) = \sigma_j^*(\omega^\phi)$$

using the remark 6.4.16 above. Now from the proposition 6.4.15 above, we substitute for ω^ϕ to get

$$\begin{aligned} \omega_i(x) &= \left(\sigma_j^*(\mathrm{Ad}g_\phi\omega + g_\phi^*\omega^R)\right)(x) \\ &= \mathrm{Ad}g_\phi(\sigma_j(x))(\sigma_j^*\omega)(x) + \left((g_\phi \circ \sigma_j)^*\omega^R\right)(x) \\ &= \mathrm{Ad}g_{ij}(x)\omega_j(x) + (g_{ij}^*\omega^R)(x) \end{aligned}$$

which proves the proposition. □

Exercise 6.4.19. Show that a collection of smooth $\mathfrak{g}$-valued 1-forms ω_i on U_i which satisfy:

$$\omega_i = Ad(g_{ij})\omega_j + g_{ij}^*\omega^R$$

on the overlaps $U_i \cap U_j$ defines a connection on the bundle $\pi : P \to M$.

Proposition 6.4.20 (Existence of a connection)**.** Let $\pi : P \to M$ be a principal G-bundle. Then there exists a connection on it.

Proof: Let us first note that a connection certainly exists on a trivial bundle $P = M \times G$. Just the pullback $\omega := p^*\omega^R$ of the right-invariant Maurer-Cartan form ω^R on G under the second projection:

$$p : M \times G \to G$$

will do the job. For any $x = (m, g) \in P$, we have that $\mu_x : G \to P$ is the map $h \mapsto (m, h.g) = (m, R_g h)$, so that for $X \in \mathfrak{g}$, the vector field $\widetilde{X}$ is defined by

$$p_*(\widetilde{X}(m, g)) = p_*(0, R_{g*}X) = \widehat{X}(g)$$

where $\widehat{X}$ is the *right*-invariant vector field generated on G by X. Thus:

$$X = \omega^R(\widehat{X}) = \omega^R(p_*\widetilde{X}) = \omega(\widetilde{X})$$

which shows that ω and ω_v agree on vertical vectors. Further, the transformation property of ω under L_g^* follows from the corresponding fact for ω^R and the fact that p preserves the left G-action.

Now let $\pi : P \to M$ be any principal G-bundle, with $\Phi_i : P_{|U_i} \to U_i \times G$ the G-equivariant trivialisations. Clearly, we get connection 1-forms ω_i on $P_{|U_i} = \pi^{-1}(U_i)$ by setting

$$\omega_i := \Phi_i^*(\omega)$$

where ω is a connection 1-form on the trivial bundle $U_i \times G$ as constructed above.

Now let λ_i be a partition of unity subordinate to U_i. Define the 1-form ω by:

$$\omega(x) = \sum_i \lambda_i(\pi(x))\omega_i(x) \quad \text{for } x \in P$$

where $\omega_i(x)$ is taken to be $\equiv 0$ outside $\pi^{-1}(U_i)$. The fact that $L_g^*\omega = \mathrm{Ad}g.\omega$ follows from the corresponding property of the ω_i. Also, for a vertical vector X, we have $\omega(X) = \sum_i \lambda_i\omega_i(X) = \sum_i \lambda_i\omega_v(X) = \omega_v(X)$, since $\omega_i(X) = \omega_v(X)$ for all i and X a vertical vector. □

Example 6.4.21. A simple situation where a natural connection arises on a principal G-bundle $\pi : P \to M$ is when P has a Riemannian metric $\langle\, ,\, \rangle$ that is invariant under left translations. One can simple define the horizontal subspace $T_x^h P$ as the orthogonal complement $(T_x^v P)^\perp$ for each $x \in P$. The fact that $\langle\, ,\, \rangle$ is left-invariant shows that $L_{g*}(T_x^h P) = T_{gx}^h P$ for each $x \in P$. We can easily compute the connection form ω for this connection. Let $\{X_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$, and let $\widetilde{X}_i$ be the corresponding basis of vertical vector fields, and $\widetilde{e}_j$ the basis of 1-forms dual to this basis. Then let Y_i be the basis of vertical vector fields obtained by applying the (smooth) Gram-Schmidt process to $\widetilde{X}_i$. Clearly $\eta(Z) = \sum_i \langle Z, Y_i\rangle\, Y_i$. Thus

$$\omega(Z) = \omega_v(\eta(Z)) = \sum_j (\widetilde{e}_j(Z) \otimes X_j) = \sum_{i,j} A_{ij}(x)\, \langle Z, Y_i\rangle\, X_j$$

for $Z \in T_x P$, where $A_{ij}(x) := \widetilde{e}_j(Y_i(x))$ is the matrix of the Gram-Scmidt operation.

Example 6.4.22 (Connections on homogeneous manifolds)**.** We have seen already in the example 6.1.3 that the projection $\pi : P = G \to G/H = M$ onto a homogeneous manifold is a principal H-bundle. Any right H-invariant Riemannian metric on G will result in a natural connection on this principal bundle in accordance with the example 6.4.21 above. If the Riemannian metric on G is left-G and right H-invariant, and arising from an inner product $\langle\, ,\, \rangle$ on $\mathfrak{g}$, we can explicitly compute the connection form as follows. Let $\{X_i\}_{i=1}^n$ be an orthonormal basis for $\mathfrak{g}$ with respect to $\langle\, ,\, \rangle$ such that $\{X_i\}_{i=1}^m$ form an orthonormal basis for $\mathfrak{h}$. Then, since the left action of H on G is given by R_h^{-1}, the map $\mu_g : H \to G$ is the map $h \mapsto gh^{-1}$, i.e. the map $L_g \circ I$, where I is the inversion map of H. Thus for $X \in \mathfrak{h}$, we get that $\widetilde{X}^P(g) = L_{g*}(-X) = -\widetilde{X}(g)$, where $\widetilde{X}$ on the right hand side is the left invariant vector field on G generated by X. Clearly, then

$$\delta_{ij} = \widetilde{e}_i(\widetilde{X}_j^P) = \left\langle \widetilde{X}_i, \widetilde{X}_j \right\rangle = -\left\langle \widetilde{X}_i, \widetilde{X}_j^P \right\rangle$$

For a $Z \in T_g G$, we have $\eta(Z) = \sum_{i=1}^m \left\langle Z, \widetilde{X}_i^P \right\rangle \widetilde{X}_i^P$, which implies that

$$\omega(Z) = \omega_v(\eta(Z)) = -\sum_{i=1}^m \left\langle Z, \widetilde{X}_i \right\rangle X_i$$

If ω^L denotes the left G-invariant Maurer-Cartan form on G, the reader may easily verify that this connection form is nothing but $(-p \circ \omega^L)$, where $p : \mathfrak{g} \to \mathfrak{h}$ is the orthogonal projection with respect to $\langle\, ,\, \rangle$.

Exercise 6.4.23 (Stiefel bundles)**.** Consider the principal $O(k)$ bundle:

$$\pi : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$$

introduced in the exercise 6.1.6. The bi-invariant metric on $O(n)$ generated by the inner product $\langle X, Y\rangle = \operatorname{tr}(X^* Y)$ on the Lie algebra $\mathfrak{o}(n)$ descends to a left-$O(n)$ right-$O(k)$-invariant metric on $V_k(\mathbb{R}^n) = O(n)/O(n-k)$. (Note that for an orthonormal k-frame $(v_1, ..., v_k)$, where each v_i is an n-column vector, there is a left action of $O(n)$ by $(v_1, .., v_n) \mapsto (Av_1, .., Av_k)$, and also the right action by $O(k)$ given by $(v_1, .., v_k) \mapsto (\sum_j B_{j1} v_j, ..., \sum_j B_{jk} v_j)$ for $B \in O(k)$. It is the right action of $O(k)$ which yields the principal bundle above). Compute the connection form of this bundle for the connection given in the example 6.4.21 above, and also the sections σ_i and local connection forms ω_i, after noting that a system of trivialising charts can be given on the open sets U_P introduced in 2.2.6.

Exercise 6.4.24. Let $\pi : P \to M$ be a principal G-bundle with a connection η, and let $f : N \to M$ be a smooth map of manifolds. Note that the pullback G-bundle f^*P is a bundle which completes the commutative square:

$$\begin{array}{ccc} f^*P & \xrightarrow{\theta} & P \\ \pi\downarrow & & \downarrow\pi \\ N & \xrightarrow{f} & M \end{array}$$

where θ is G-equivariant. Show that there is a natural *pullback connection* denoted $f^*\eta$ on f^*P which satisfies the following property:

$$\theta_*(T_x^h(f^*P)) = T_{\theta(x)}^h P$$

Show that the local connection forms of f^*P are given by $f^*\omega_i : f^{-1}(U_i) \to \mathfrak{g}$, where $\omega_i : U_i \to \mathfrak{g}$ are the local connection forms of η (as in 6.4.17).

Example 6.4.25 (Canonical Cartan Connection on a Lie Group)**.** Let G be a Lie group. We consider the trivial principal G-bundle:

$$\begin{aligned} \pi_1 : G \times G &\to G \\ (g,h) &\mapsto g \end{aligned}$$

with the G-action being $g.(h,k) = \mu(g,(h,k)) = (h, gk)$. This means that for $x = (h,k) \in P$, the map $\mu_x : G \to P$ is given by $g \mapsto (h, gk)$, i.e. $\mu_x = (h, R_k)$, and hence for $X \in \mathfrak{g}$,

$$\mu_{x*}(X) = (0, R_{k*}X) = (0, \hat{X}(k)) \in T_h(G) \times T_k(G)$$

where $\hat{X}$ denotes the right invariant vector field generated by X. If we denote by $\pi_2 : P \to G$ the second projection $(h,k) \mapsto k$, the vertical tangent bundle of P is given by $T^vP = \pi_2^*TG$. Thus the Maurer-Cartan section ω_v of the vertical bundle is the pullback section $\pi_2^*\omega^R$ of the right-invariant Maurer-Cartan 1-form ω^R of G. Since $P = G \times G$, $TP = \pi_1^*(TG) \oplus \pi_2^*(TG)$ in a canonical manner, and by setting the natural connection $\eta : TP \to T^vP$ to be $(X,Y) \mapsto Y$, we get the connection form

$$\omega(X,Y) = \omega_v(Y) = \pi_2^*\omega^R(X,Y)$$

ω is therefore constant along horizontal slices $G \times k$, and satisfies the correct transformation law under L_{g*} by the corresponding fact for ω^R as noted in the exercise 6.3.3.

Incidentally, one gets the tangent bundle TG of G as the associated bundle to this trivial prnicipal bundle by taking the adjoint representation:

$$\begin{aligned} \rho : G &\to GL(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}(g)(:= \mathrm{Ad}(g)_*) \end{aligned}$$

For, we just map the element $[(h,k),X] \in (P \times_\rho \mathfrak{g})_h$ to the element $L_{h*}(\mathrm{Ad}(k^{-1})X) = (\widetilde{\mathrm{Ad}(k^{-1})X})(h) \in T_hG$. Then $[g.(h,k), \rho(g)X] = [(h,gk), \mathrm{Ad}(g)X]$ will get mapped to

$$L_{h*}(\mathrm{Ad}(gk)^{-1}\mathrm{Ad}g\, X) = L_{h*}(\mathrm{Ad}(k^{-1})X)$$

and this map is well defined independent of representatives. We leave it as an exercise for the reader to define the inverse isomorphism.

6.5. **Horizontal lifting of vector fields and paths.**

Definition 6.5.1 (Horizontal fields and lifts)**.** Let U be an open subset of P, where $\pi : P \to M$ is a principal G-bundle with a connection η. We say a vector field Y on U is *horizontal* if $Y(x) \in T_x^hP$ for each $x \in U$. Equivalently, if $\eta(Y) \equiv 0$ at all points of U, which is the same as requiring $\omega(Y) \equiv 0$ on U (since $\omega_v \circ \eta = \omega$).

Note that for a fixed $g \in G$, $L_{g*}Y$ is a horizontal field iff Y is a horizontal field.

If X is a smooth vector field on an open set $U \subset M$, we say a smooth horizontal vector field $\widetilde{X}$ on $\pi^{-1}(U)$ is a *horizontal lift* of X if $\pi_*(\widetilde{X}) = X$. Note that in view of the remark above, we will have $\widetilde{X}(hx) = L_{h*}X(x)$, since both sides are horizontal vectors at hx mapping to $X(\pi(x))$ under π_*, and the only things in the kernel of π_* are vertical vectors.

Remark 6.5.2. Every vector field X on an open subset U of M has a horizontal lift. For, this vector field leads to the pullback section π^*X of π^*TM over $\pi^{-1}(U)$, defined by $\pi^*(X)(x) = X(\pi(x))$. Since $\ker \pi_* = T^vP$, $\pi_* : T^hP \to \pi^*TM$ is an isomorphism of vector bundles. Thus the smooth vector field $\widetilde{X} := \pi_*^{-1}(\pi^*X)$ is the required horizontal lift.

Proposition 6.5.3 (Horizontal lifts of smooth paths)**.** Let $\pi : P \to M$ be as above, with a connection η. Let $c : [0,1] \to M$ be a smooth path in M. Let $x \in P$ be a point in the fibre $\pi^{-1}(c(0))$. Then there exists a unique path $\widetilde{c} : [0,1] \to P$ such that:

(i): $\widetilde{c}(0) = x$.

(ii): $\frac{d\widetilde{c}}{dt}(t)$ is horizontal for all t.

This path $\widetilde{c}$ will be called the *horizontal lift of* c *beginning at* x. For each $g \in G$, the path $L_g \circ \widetilde{c}$ will be the horizontal lift of c beginning at gx, i.e. all the horizontal lifts of c are left translates of each other.

Proof: Clearly, we can assume that c extends to a smooth map $c : (-\delta, 1+\delta) \to M$. Let us denote $(-\delta, 1+\delta)$ as I. We consider the pullback G-bundle $Q := c^*P$ on I. We therefore have a commutative pullback square:

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & P \\ \pi\downarrow & & \downarrow\pi \\ I & \xrightarrow{c} & M \end{array}$$

where θ is a smooth G-equivariant isomorphism of the fibre $Q_t = \pi^{-1}(t)$ with the fibre $P_{c(t)} = \pi^{-1}(c(t))$. Let η' denote the pullback connection on Q, as in exercise 6.4.24.

On I we have the natural coordinate field $Y = \frac{d}{dt}$ such that $c_*(\frac{d}{dt}) = \frac{dc}{dt}$. By the remark 6.5.2 above, there is a horizontal lift $\widetilde{Y}$ of this field Y to Q. Also, as remarked at the end of 6.5.1, the horizontal vector field $\widetilde{Y}$ is invariant under L_{g*} for all $g \in G$.

Claim: For each $x \in Q_0 = \pi^{-1}(0)$, there is a smooth path $a_x : [0,1] \to Q$ such that :

(i): $a_x(0) = x$

(ii): $\pi \circ a_x(t) = t$ for all $t \in [0,1]$, and

(iii): $a_x(t)$ is a trajectory of $\widetilde{Y}$.

(iv): $a_{gx}(t) = L_g a_x(t)$ for all $t \in [0,1]$.

Proof of claim: Fix some $x_0 \in \pi^{-1}(0)$. Then, by the existence theorem for ODE's (5.4.2), there is the germ of a smooth path $a_{x_0} : (-\epsilon, \epsilon) \to Q$ which satisfies (i), (ii) and (iii) for all $t \in [0, \epsilon)$. Also note that for all $g \in G$, we can define $a_{gx_0} : (-\epsilon, \epsilon) \to Q$ as $L_g \circ a_{x_0}$. Since $\widetilde{Y}$ is invariant under L_{g*}, and every $x \in Q_0$ is of the form gx_0 by the uniqueness part of 5.4.2, it is clear that a_{gx_0} will also satisfy the properties (i) through (iv), for $x = gx_0$.

Say a subinterval $[0, \epsilon]$ or $[0, \epsilon)$ with $\epsilon \leq 1$ is *good*, if there exists $a_x : [0, \epsilon] \to Q$ satisfying (i) through (iv) above for all $t \in [0, \epsilon]$. The remarks above show that the collection of good (right closed or right open) subintervals is non-empty. Order it by inclusion, and observe that the union of a chain provides an upper bound for that chain. Hence, by Zorn's lemma, there is a maximal element $[0, b]$ or $[0, b)$. In either case $[0, b)$ is good. If $b < 1$, one can take a point $y \in \pi^{-1}(b)$, and construct a path $\alpha : (-2\epsilon, 2\epsilon) \to Q$ by 5.4.2 which will be a trajectory of $\widetilde{Y}$, and satisfy $\alpha(0) = y$. Since $[0, b)$ is good, there is a path $a_x : [0, b) \to Q$ satisfying (i) through (iv) above all over $[0, b]$. Since $a_x(b - \epsilon) \in \pi^{-1}(b - \epsilon)$, there is a group element g such that $g(\alpha(-\epsilon)) = a_x(b - \epsilon)$. Then the path defined on $[0, b + 2\epsilon)$ by t aking it to be a_x on $[0, b)$, and $L_g \circ \alpha(t - b)$ on $[b, b + 2\epsilon)$ is a smooth path (by uniqueness in 5.4.2, because a_x and $L_g \circ \alpha(t - b)$ must agree for the overlap $[b - 2\epsilon, b]$). This path satisfies (i) through (iv) on $[0, b+2\epsilon)$, and shows that $[0, b+2\epsilon)$ is also good. This contradicts the maximality of $[0, b]$ (or $[0, b)$), and hence $b = 1$. Thus the claim follows. □

Now to prove the proposition, let $x \in P$, and set $y = \theta^{-1}(x) \in Q$. The smooth path $\theta \circ a_y$, where a_y is the path constructed in the above claim, is easily checked to be the required horizontal lift of c. □.

It is useful to write down the differential equation that the horizontal lift of a smooth path satisfies. We do this in the next lemma.

Lemma 6.5.4 (ODE for horizontal lifts)**.** Let $\{(\Phi_i, U_i)\}$ be a bundle atlas for the principal G-bundle $\pi : P \to M$. Let $\sigma_i(x) = \Phi_i^{-1}(x, 1)$ be the smooth local sections over U_i, and $\omega_i = \sigma_i^* \omega$ the local connection forms. Let $\sigma_i(t)$ denote $\sigma_i(c(t))$. Then, for a path $c : [a, b] \to U_i$, the horizontal lift $\widetilde{c}(t)$ beginning at z is given by:

$$\widetilde{c}(t) = g(t)\sigma_i(t)$$

where $z = g(0)\sigma_i(0)$ and

$$\omega^R\left(\frac{dg}{dt}\right) + \mathrm{Ad}g(t)\omega_i(c(t))\left(\frac{dc}{dt}\right) = 0$$

In the particular case when G is a linear group, we get the equation:

$$\frac{dg}{dt} + L_{g(t)*}\omega_i(c(t))\left(\frac{dc}{dt}\right) = 0$$

Proof: Since $\widetilde{c}(t)$ is a path in $\pi^{-1}(U_i)$, we can certainly write

$$\widetilde{c}(t) = g(t)\sigma_i(t)$$

for a uniquely defined smooth function $g : [a, b] \to G$. Now, we need to express the condition $\omega(\widetilde{\frac{dc}{dt}}) = 0$. Letting dots denote t-derivative $\frac{d}{dt}$, we have:

$$\begin{aligned}
\omega(\dot{\widetilde{c}}(t)) &= \omega_v \eta\,(\dot{\widetilde{c}}(t)) \\
&= \omega_v \eta\ (D\mu(g(t), \sigma_i(t))(\dot{g}(t), \sigma_{i*}(c'(t))) \\
&= \omega_v\left(\widetilde{\omega^R(\dot{g}(t))} + L_{g(t)*}\sigma_{i*}(c'(t))\right) \\
&= \omega^R(\dot{g}(t)) + \mathrm{Ad}g(t)\omega(\sigma_{i*}(c'(t))) \\
&= \omega^R(\dot{g}(t)) + \mathrm{Ad}g(t)\omega_i(c'(t))
\end{aligned}$$

where we have used the equation (11) from the proposition 6.4.15 in the third line, and put dots to denote $\frac{d}{dt}$. This proves the lemma. The second part is clear since by 6.3.2, $\omega^R = dg.g^{-1}$, and by definition $dg(\dot{g}(t)) = \dot{g}(t)$. □

6.6. **Connections on vector bundles, parallel transport.** Connections provide a method of differentiating sections of vector bundles. To do this, one needs the notion of what sections are "constant", i.e. have zero derivative. For this one introduces parallel transport or parallel translation. So let $\pi : P \to M$ be a principal G-bundle, and let $\rho : G \to GL(V)$ be a representation of G on a vector space V. For notational simplicity we will write $\rho(g)v$ as $g.v$ or even gv, where $g \in G$ and $v \in V$. Let η be a connection on P, and let ω, (resp. ω_i) denote the $\mathfrak{g}$-valued connection 1-forms on P (resp. U_i) as in the previous subsections. Let $p : E(= P \times_\rho V) \to M$ be the associated vector bundle, as in example 6.2.2.

Let $y \in M$ and $v \in E_y = p^{-1}(y)$. That is $v = [x, w]$ where $x \in P_y = \pi^{-1}(y)$. Let $c : [0, 1] \to M$ be a smooth path with $c(a) = y$, and $\widetilde{c} : [a, b] \to P$ be a horizontal lift of c starting at x, as in 6.5.3. Note that $[\widetilde{c}(a), w] = v$.

Definition 6.6.1. The vector defined by $P_t^c v := [\widetilde{c}(t), w]$ is called the *parallel translate of* v by t along the path c. Note that $p(P_t^c(v)) = \pi(\widetilde{c}(t)) = c(t)$, so that $P_t^c v \in E_{c(t)}$. It is easily checked that this mapping:

$$P_t^c : E_{c(a)} \to E_{c(t)}$$

is linear. In fact, as an exercise the reader can check that:

(i): $P_0^c = Id$, and $P_t^c \circ P_s^c = P_{t+s}^c$, whenever both sides are defined.

(ii): P_t^c is an isomorphism of $E_{c(0)}$ onto $E_{c(t)}$. (Its inverse is "P_{-t}^c" which just denotes P_t^d, where $d : [0,t] \to M$ is the path $d(s) = c(a+t-s)$).

For notational convenience we shall write P_t instead of P_t^c. We remark that for two distinct paths joining the same two points y, y', the two corresponding parallel transport isomorphisms from E_y to $E_{y'}$ will be distinct.

Let c be a smooth path as above. Clearly, any frame $(v_1, .., v_k)$ of a fibre $p^{-1}(y) = E_y$ can be expressed as $v_i = [x, e_i]$, where $(e_1, .., e_k)$ is a frame for V (why?). Denote $\widetilde{v}_i(t) := P_t v_i = [\widetilde{c}(t), e_i]$. This will be a frame for $E_{c(t)}$. If $v = \sum_{i=1}^k a_i v_i$, then we have $P_t v = \sum_{i=1}^k a_i \widetilde{v}_i(t)$. The frame $(\widetilde{v}_1(t), ..., \widetilde{v}_k(t))$ is called the *parallel frame along* c with initial value $(v_1, .., v_k)$.

Definition 6.6.2 (Covariant derivative)**.** Let $p : E \to M$ be a vector bundle, with $E = P \times_\rho V$, and $\pi : P \to M$ a principal G-bundle with a connection η. Let $s : U \to E$ be a smooth section of E over the open set $U \subset M$, and $X \in T_y M$ be a tangent vector at $y \in U$. We define the *covariant derivative* $\nabla_X s$ to be the vector in E_y given by the formula:

$$\nabla_X s = \lim_{t \to 0} \frac{P_{-t}^c(s(c(t)) - s(c(0))}{t}$$

where $c : (-\epsilon, \epsilon) \to U$ is *any* smooth curve with $c(0) = y$ and $c'(0) = X$.

We need to verify that the definition above makes sense, i.e., is independent of the curve c that was chosen. This will follow as soon as we write down equations. Meanwhile, let us just note that if $(\widetilde{v}_j(t) = P_t^c v_j)$ is the parallel transport of v_j where $(v_1, ..., v_k)$ is a frame for E_y, then we may express $s(c(t)) = \sum_j s_j(t)\widetilde{v}_j(t)$, from which it follows that $P_{-t}^c s(c(t)) = \sum_j s_j(t) v_j$, so that:

$$\nabla_X s = \sum_j \frac{ds_j}{dt}(0) v_j$$

From this it easily follows that if $f : U \to \mathbb{R}$ is a smooth function, then for the smooth section fs we have the *Leibniz Formula*:

$$\nabla_X(fs) = f(c(0))\nabla_X s + X(f)s \tag{14}$$

Now, to show that ∇_X does not depend on the choice of path germ c satisfying $c(0) = y$ and $c'(0) = X$, we need only check this for a suitable basis of smooth sections over U, in the light of the Leibniz formula above.

For this purpose, without loss of generality, we assume that U is a coordinate chart, and $\omega : U \to \mathfrak{g}$ is the local connection form on U. Let $\{v_i = [x, e_i]\}_{i=1}^k$ be a frame of E_y, where e_i is a frame of V, and $x \in P_y$. Let c be a path germs at y with $c(0) = y$, and $c'(0) = X$. Let $\widetilde{c}$ denote its horizontal lift beginning at x. Let $\sigma : U \to \pi^{-1}(U)$ be the natural section coming from the bundle trivialisation, and w.l.o.g. take $x = \sigma(0) := \sigma(c(0))$. Then, we have a natural frame (basis of smooth sections) of E all over U coming from σ, namely

$$w_i(z) = [\sigma(z), e_i] \quad \text{for } z \in U$$

Proposition 6.6.3 (Covariant derivative in local coordinates)**.** With the setting above, the covariant derivative $\nabla_X w_j$ is given by:

$$\nabla_X w_j = \sum_i \dot{\rho}(\omega(X))_{ji} w_i(c(0))$$

where $\dot{\rho} := D\rho(1) : \mathfrak{g} \to gl(V)$ is the derived representation. (see the exercise 5.6.7).

Proof: Note that $s_j(c(0)) = v_j$. The parallel translated frame is:

$$\widetilde{v}_i(t) = [\widetilde{c}(t), e_i]$$

We write $\widetilde{c}(t) = g(t)\sigma(c(t))$, as before. Then:

$$\begin{aligned}
w_i(t) &= [\sigma(c(t)), e_i] = [g(t)\sigma(c(t)), \rho(g(t))e_i] \\
&= [\widetilde{c}(t), \sum_j \rho(g(t))_{ji} e_j] \\
&= \sum_j \rho(g(t))_{ji} [\widetilde{c}(t), e_j] \\
&= \sum_j \rho(g(t))_{ji} \widetilde{v}_j(t)
\end{aligned}$$

where $[\rho(g)_{ij}]$ is the matrix of $\rho(g)$ with respect to the basis $\{e_i\}$ of V.

Since $P_{-t}\widetilde{v}_j(t) = v_j$ by definition, we have $P_{-t}w_i(c(t)) = \sum_j \rho(g(t))_{ij} v_j$. Thus can compute:

$$\begin{aligned}
\nabla_X w_j &= \sum_i \left(\frac{\rho(g(t))_{ji}(t) - \rho(g(0))_{ji}}{t} \right) v_i \\
&= \sum_i D\rho_{ji}(1)(\frac{dg}{dt}(0)) v_i
\end{aligned}$$

since $g(0) = 1$. Now, by the ODE of 6.5.4,

$$\frac{dg}{dt} + L_{g(t)*}\omega(c(t)) \left(\frac{dc}{dt} \right) = 0$$

where $g(0) = 1$. Thus $(\frac{dg}{dt}(0)) = -\omega(c(0))(c'(0)) = -\omega(y)(X)$. If we denote the derivative of $\rho : G \to GL(V)$ at the identity by $\dot{\rho}$, we then have the formula:

$$\nabla_X w_j = -\sum_i \dot{\rho}(\omega(X))_{ji} v_i$$

Since $v_i = w_i(c(0))$, we may as well write this as:

$$(\nabla_X w_j) = -\sum_i \dot{\rho}(\omega(X))_{ji} w_j(c(0))$$

This formula clearly shows that $\nabla_X w_j$ depends only on $c(0)$ and $X = c'(0)$. □

Remark 6.6.4. From the local formula of the above proposition, it is clear that

$$\nabla_{cX} s = c \nabla_X s$$

for all $c \in \mathbb{R}$. In particular, if X is a tangent vector field on U, we may define, for a smooth section $s : U \to E$, another section $\nabla_X s$ by the formula $(\nabla_X s)(y) := \nabla_{X(y)} s$. The formula above shows that this is also a smooth section of E on U. In other words, we have a linear operator:

$$\begin{aligned}
\nabla : \Gamma(U, TM) \otimes_{\mathbb{R}} \Gamma(U, E) &\to \Gamma(U, E) \\
X \otimes s &\mapsto \nabla_X s
\end{aligned}$$

where $\Gamma(U, E)$ (resp. $\Gamma(U, TM)$) denotes the $C^\infty(U)$ module of smooth sections of E over U (resp. vector fields over U). This operator is linear with respect to multiplication by a smooth function $f \in C^\infty(U)$ in the first (=X) variable, but obeys the Leibniz rule with respect to the such a multiplication in the second (=s) variable.

Corollary 6.6.5. A smooth connection η on $\pi : P \to M$ gives rise, for every open subset U of M, the *covariant derivative* operator:

$$\begin{array}{rcl} \nabla : \Gamma(U,E) & \to & \bigwedge^1(U) \otimes \Gamma(U,E) \\ s & \mapsto & \nabla s \end{array}$$

where $\Gamma(U,E)$ denotes the $C^\infty(U)$-module of smooth sections of E at $y \in M$. It satisfies the following *Leibniz Formula*):

$$\nabla(fs) = f\nabla s + df \otimes s$$

Proof: Clear from the remark above, by defining $(\nabla\, s)(X) = \nabla_X s$ for a tangent vector field X, and noting that $df(X) = X(f)$. □

Conversely, given a covariant derivative operator on a vector bundle E, one can create a connection on the associated frame bundle. This is the content of the:

Exercise 6.6.6. Let E be a smooth vector bundle of rank k. Fix a vector bundle atlas $\{(\Phi_i, U_i)\}$ for E, and let (for example) $\{s_i(x)\}$ denote the frame $\{\Phi_i^{-1}(x,e_1),..,\Phi_i^{-1}(x,e_k)\}$. Note that on $U_i \cap U_j$, we have the formula $s_i(x) = g_{ij}(x)s_j(x)$ where $g_{ij}(x) \in GL(k,\mathbb{R})$. Suppose there is given a covariant differentiation operator ∇ satisfying the Leibniz formula in 6.6.5 above. Set

$$\nabla s_i = -\omega_i.s_i$$

where ω_i is a matrix of 1-forms on U, and if we denote its component 1-forms by ω_i^{kl}, and $s_i = (s_i^1,..,s_i^k)$ then $(\omega_i.s_i)^k := \sum_l \omega_i^{kl} \otimes s_i^l$. Using the Leibniz formula, show that the collection of $\mathfrak{gl}(k)$ valued 1-forms ω_i satisfy the transformation property of 6.4.17, and hence define a connection on the principal $GL(k)$ frame bundle of E.

Exercise 6.6.7 (Covariant derivative w.r.t. Cartan connection)**.** Let G be a Lie group, and let η be the Cartan connection introduced in the example 6.4.25. For two left invariant vector fields $\widetilde{X}, \widetilde{Y}$, compute the covariant derivative $\nabla_{\widetilde{X}}\widetilde{Y}$ (on the tangent bundle TG).

6.7. **Metric connections.** Let $p : E \to M$ be a smooth vector bundle of rank k, and a Riemannian bundle metric $\langle\, , \rangle$. We will denote the inner product on the fibre E_y by $\langle\, , \rangle$. Thus we may form the bundle of *orthonormal frames* $\pi : P \to M$, which will be a principal $O(k)$ bundle. Its fibre over y will be the set of all orthonormal frames in E_y. (That is E will admit a reduction of structure group from $GL(k,\mathbb{R})$ to $O(k)$ in the sense of 6.2.6.

Definition 6.7.1. A connection on the principal $O(k)$-bundle of a Riemannian vector bundle E will be called a *metric connection* or *Riemannian connection.*

There is a formula that connects covariant differentiation of sections on E with respect to a metric connection, and the metric itself. That is:

Proposition 6.7.2. Let E be as above, and let $X \in T_y(M)$, and σ and τ two smooth sections of E over U, an open neighbourhood of y in M. Then

$$\langle \nabla_X \sigma, \tau\rangle + \langle \sigma, \nabla_X \tau\rangle = X\,\langle \sigma, \tau\rangle \tag{15}$$

for the covariant derivative ∇_X with respect to any metric connection.

Proof:

If c is a path germ at $y \in M$, the horizontal lift $\widetilde{c(t)}$ beginning at any point $x \in P_y$ will be a path of orthonormal frames. If $v_i = [x, e_i]$, for some orthonormal frame $e_i \in V$, then the parallel translated frame along c is $\widetilde{v}_i(t) = [\widetilde{c}(t), e_i]$.

The isomorphism $P \times_\rho V \to E$ is the map $[(w_1, .., w_k), \sum_j a_j e_j] \mapsto \sum_j a_j w_j$, as noted in 6.2.3. Thus $\widetilde{v}_i(t) = \widetilde{c}_i(t)$. Consequently

$$\langle \widetilde{v}_i(t), \widetilde{v}_j(t) \rangle_{c(t)} = \langle \widetilde{c}_i(t), \widetilde{c}_j(t) \rangle_{c(t)} = \delta_{ij}$$

since $\widetilde{c}(t)$ is an orthonormal frame for all t. That is,

$$\langle P^c_t v_i, P^c_t v_j \rangle_{c(t)} = \langle v_i, v_j \rangle$$

and parallel translation along each curve is an isometry with respect to $\langle\ ,\ \rangle$. In particular, for any sections σ, τ of E, we have:

$$\left\langle P^c_{-t}\sigma(c(t)), P^c_{-t}\tau(c(t)) \right\rangle_{c(0)} = \langle \sigma(c(t)), \tau(c(t)) \rangle_{c(t)}$$

Taking $\frac{d}{dt}_{|t=0}$, and setting $c'(0) = X$ shows that:

$$\langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle = X \langle \sigma, \tau \rangle$$

which proves the proposition. □

Remark 6.7.3. If $e = (e_1, .., e_k)$ is a local orthonormal frame of E over some open set U, then we recall that

$$\nabla_X e_i = -\sum_j \omega_{il}(X) e_l$$

where $\omega : U \to \mathfrak{o}(k)$ is the local connection form over U. Since $\langle e_i, e_j \rangle$ is the constant matrix δ_{ij}, the relation (15) above implies that:

$$\omega_{ij}(X) + \omega_{ji}(X) = 0$$

which is precisely the statement that ω is $\mathfrak{o}(k)$ valued, and hence a skew-symmetric matrix.

Clearly, since every principal bundle admits many connections, there is nothing unique about a metric connection. However, on the tangent bundle of a Riemannian manifold, there is a unique connection if one imposes an extra (*torsion free*) condition, which is the content of the following subsections.

6.8. **Covariant differentiation on Vector Bundles.** If one wishes to avoid considering principal G-bundles altogether, it is possible to directly define connections on vector bundles.

We recall some notation first. If $\pi : E \to M$ is a vector bundle over a smooth manifold M, and $U \subset M$ is an open set, the $\mathbb{R}$ vector space of smooth sections of this bundle over U will be denoted $\Gamma(U, E)$, as usual. Multiplication by smooth functions on U makes this vector space a module over $C^\infty(U)$. The space of differential 1-forms on U is denoted by $\Lambda^1(U)$, which is precisely $\Gamma(U, T^*(M))$, and therefore also a module over $C^\infty(U)$.

Definition 6.8.1 (Connection on a vector bundle)**.** Let $\pi : E \to M$ be a vector bundle. A *connection* or *covariant derivative* on this vector bundle is an $\mathbb{R}$-linear operator:

$$\begin{aligned} \nabla : \Gamma(U, E) &\to \Lambda^1(U) \otimes \Gamma(U, E) \\ s &\mapsto \nabla s \end{aligned}$$

for each open set $U \subset M$. This operator should further satisfy the following *Leibniz formula*:

$$\nabla(fs) = f\nabla s + df \otimes s \tag{16}$$

for all $f \in C^\infty(U)$, all $s \in \Gamma(U, E)$, and all $U \subset M$ open. A connection on the tangent bundle TM of a manifold M is called a *connection on M*. Note that this definition is just the axiomatisation of what we defined in Corollary 6.6.5 of the previous subsection.

Let $X \in T_x(M)$ be a tangent vector. We note that there is a natural operation of *contraction* defined by:

$$\begin{aligned} T_x(M) \otimes T_x^*(M) &\rightarrow \mathbb{R} \\ (X \otimes \omega) &\mapsto i_X\omega := \omega(X) \end{aligned}$$

which leads to a natural (pointwise) operator:

$$\begin{aligned} T_x(M) \otimes \Lambda^1(U) \otimes \Gamma(U,E) &\rightarrow E_x \\ (X \otimes \omega \otimes s) &\mapsto (i_X\omega)(x)s(x) = \omega(x)(X)s(x) \end{aligned}$$

which we also denote $i_X(\omega \otimes s)$, for notational convenience.

If $U \subset M$ is an open set, then the above operators extend as natural (local) operators over U:

$$\begin{aligned} \Gamma(U,TM) \otimes \Lambda^1(U) &\rightarrow C^\infty(U) \\ (X \otimes \omega) &\mapsto i_X\omega := \omega(X) \end{aligned}$$

which satisfies $i_{fX}\omega = fi_X\omega = i_Xf\omega$ for all $f \in C^\infty(U)$, by definition.

Hence we have the corresponding (local) operator:

$$\begin{aligned} \Gamma(U,TM) \otimes \Lambda^1(U) \otimes \Gamma(U,E) &\rightarrow \Gamma(U,E) \\ (X \otimes \omega \otimes s) &\mapsto (i_X\omega)s \end{aligned}$$

where the smooth function $i_X\omega$ on U has the obvious pointwise definition $(i_X\omega)(x) = i_{X(x)}(\omega(x))$. Also note that we have

$$i_{fX}(\omega \otimes s) = fi_X(\omega \otimes s) = i_X(\omega \otimes fs)$$

for all smooth functions $f \in C^\infty(U)$, all $\omega \in \Lambda^1(U)$ and all $s \in \Gamma(U,E)$.

Definition 6.8.2 (Covariant derivative with respect to a tangent vector)**.** Let ∇ be a connection on a smooth vector bundle $\pi : E \rightarrow M$. Let $x \in U$, where $U \subset M$ is an open set. For a tangent vector $X \in T_x(M)$, define:

$$\begin{aligned} \nabla_X : \Gamma(U,E) &\rightarrow E_x \\ s &\mapsto \nabla_X s = i_X(\nabla s) \end{aligned}$$

and similarly define for a vector field $X \in \Gamma(U,TM)$, the operator:

$$\begin{aligned} \nabla_X : \Gamma(U,E) &\rightarrow \Gamma(U,E) \\ s &\mapsto \nabla_X s = i_X(\nabla s) \end{aligned}$$

where i_X are the two operators defined above. It is trivial to check the following, using the facts about i_X mentioned above, and the Leibniz rule, that:

(i): $\nabla_{fX}s = f\nabla_X s$

(ii): (*Leibniz formula*) $\nabla_X(fs) = f\nabla_X s + X(f)s$

for all for all smooth functions $f \in C^\infty(U)$, and all $s \in \Gamma(U,E)$. This is again exactly what we defined in 6.6.2 of the last subsection.

Remark 6.8.3. A connection on the bundle $\pi : E \rightarrow M$ can also be defined just as a global $\mathbb{R}$-linear operator operator

$$\nabla : \Gamma(M,E) \rightarrow \Lambda^1(M) \otimes \Gamma(M,E)$$

which satisfies the Leibniz rule $\nabla(fs) = f\nabla s + df \otimes s$ for all $f \in C^\infty(M)$ and all $s \in \Gamma(M,E)$. This is because one can use cut-off functions etc. to define the operator on all open subsets $U \subset M$. The Leibniz formula above tells us how to deal with multiplication by a function, and the details are left to the reader.

Remark 6.8.4 (Local description of a connection on a vector bundle)**.** If $\{s_j\}_{j=1}^k$ is a frame (= basis of sections) of the rank-k vector bundle $\pi : E \to M$ on some open set $U \subset M$, then we can define a $k \times k$-matrix of 1-forms $[\omega_{ij}]$ on U by:

$$\nabla s_i = \sum_j \omega_{ij} \otimes s_j$$

which is sometimes written in more compact notation simply as

$$\nabla s = \omega.s$$

where s denotes the frame $\{s_j\}_{j=1}^k$, and ω denotes the matrix $[\omega_{ij}]$, and the dot denotes matrix tensor product.

These 1-forms will change by a formula, if we change the frame to another frame. More precisely, if g is an invertible matrix of smooth functions, and we have another frame $t := gs$ (matrix multiplication!), with corresponding matrix $\widetilde{\omega}$, then:

$$\widetilde{\omega}.t = \widetilde{\omega}.gs = \nabla t = \nabla(gs) = g\nabla s + dg.s = g(\omega.s) + dg.s$$

by Leibnitz rule, so that:

$$\widetilde{\omega}g = g\omega + dg$$

and hence

$$\widetilde{\omega} = g\omega g^{-1} + (dg)g^{-1}$$

where dg is the $k \times k$-matrix whose (ij)-th entry is dg_{ij}, and all multiplications are matrix multiplications. This is the same formula we had encountered in 6.4.15 (see also the Exercise 6.6.6).

From the transformation formula above, it also follows that the matrix of connection 1-forms ω^α defined from some local frames s_α on trivialising open sets $\{U_\alpha\}$ coming from some bundle atlas *will not patch up* and give a global matrix of 1-forms on M. If we write $s_\alpha = g^{\alpha\beta} s_\beta$, the term $dg^{\alpha\beta}.(g^{\alpha\beta})^{-1}$ in the formula shows that $\omega^\alpha(x)$ does not depend just on $g^{\alpha\beta}(x)$ and $\omega^\beta(x)$, but also on $dg^{\alpha\beta}$.

Remark 6.8.5 (Compact supports)**.** Let s be a smooth section of E with compact support $C \subset U$, with U open. Then, for any connection ∇ on E, we claim ∇s also has compact support in U. For, letting λ be a smooth bump function which is $\equiv 1$ on C and with compact support in U, we can write $s = \lambda s$, so that $\nabla s = \nabla(\lambda s) = \lambda \nabla s + d\lambda \otimes s$, by Leibnitz formula. Since both terms on the right have compact support within U, our claim follows.

Proposition 6.8.6 (Existence of smooth connections)**.** Let $\pi : E \to M$ be a smooth vector bundle of rank k on a smooth manifold M. Then there exists a connection on E.

Proof: We first note that if $\pi : E \to M$ is a trivial bundle, then there is always a connection on it. One could choose a global frame $\{s_i\}_{i=1}^k$ for E, and choose any matrix $[\omega_{ij}]$ of smooth 1-forms defined on M to define ∇ (as indicated in Remark 6.8.4 above) by the formula $\nabla s_i = \sum_i \omega_{ij} \otimes s_j$. For example, we can define the *flat* connection with respect to this frame by setting $\omega_{ij} \equiv 0$ for all $1 \leq i, j \leq k$.

If $\pi : E \to M$ is a general vector bundle, we can find a *locally-finite* open covering $\{U_\alpha\}$ such that $E_\alpha := E_{|U_\alpha}$ is trivial for each α. Let $\widetilde{\nabla}^\alpha$ be a connection on E_α, which exists by the previous paragraph. In view of Remark 6.8.5 above, we can view $\widetilde{\nabla}^\alpha$ as defining an operator:

$$\nabla^\alpha : \Gamma_c(U_\alpha, E) \to \Lambda^1(M) \otimes \Gamma_c(U_\alpha, E)$$

where $\Gamma_c(U_\alpha, E)$ denotes compactly supported sections of E over U_α. Note that if $s \in \Gamma_c(U_\alpha, E)$, then for $f \in C^\infty(M)$, fs is also in $\Gamma_c(U_\alpha, E)$. Thus we have that ∇^α is $\mathbb{R}$-linear as an operator on $\Gamma_c(U_\alpha, E)$, and obeys Leibnitz formula for sections in $\Gamma_c(U_\alpha, E)$, $f \in C^\infty(M)$, from the corresponding property of $\widetilde{\nabla}_\alpha$.

Now let λ_α be a smooth partition of unity subordinate to the covering $\{U_\alpha\}$, with $\operatorname{supp}\lambda_\alpha$ a compact subset of U_α for each α. Define

$$\begin{aligned} \nabla : \Gamma(M,E) &\to \Lambda^1(M)\otimes\Gamma(M,E) \\ s &\mapsto \sum_\alpha \nabla^\alpha(\lambda_\alpha s) \end{aligned}$$

Note that $\nabla^\alpha(\lambda_\alpha s)$ has compact support in U_α since λ_α has compact support in U_α, by Remark 6.8.5 above. Since the open covering $\{U_\alpha\}$ is locally-finite, it follows that the sum over α is a finite one at each point of M, so makes sense. ∇ is clearly $\mathbb{R}$-linear, since all the ∇^α are $\mathbb{R}$-linear. To verify Leibnitz formula, we note :

$$\begin{aligned} \nabla(fs) &= \sum_\alpha \nabla^\alpha(\lambda_\alpha fs) = \sum_\alpha \nabla^\alpha(f.\lambda_\alpha s) \\ &= \sum_\alpha \left(f.\nabla^\alpha(\lambda_\alpha s) + \sum_\alpha df\otimes(\lambda_\alpha s)\right) \\ &= f\sum_\alpha \nabla^\alpha(\lambda_\alpha s) + df\otimes\sum_\alpha(\lambda_\alpha s) = f\nabla s + df\otimes s \end{aligned}$$

where the second line follows from the Leibnitz formula for ∇^α acting on compactly supported sections as noted in the last paragraph. Hence ∇ is a connection on E. □

6.9. **The Levi-Civita Connection on a Riemannian Manifold.**

Proposition 6.9.1 (Levi-Civita connection)**.** Let M be a Riemannian manifold with Riemannian metric g. For tangent vectors $X, Y \in T_x(M)$, denote $g(x)(X,Y) = \langle X, Y\rangle$. Then there is a unique connection on M (i.e. on the tangent bundle $\pi : TM \to M$ which satisfies:

(i): $\nabla_X Y - \nabla_Y X = [X,Y]$ for all smooth vector fields X, Y on M (torsion free condition).

(ii): $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (compatible with metric).

It is called the *Levi-Civita connection* on M.

Proof: Let (ϕ, U) be a local chart on M, and let $\frac{\partial}{\partial x_j}$ denote the corresponding coordinate vector fields on U. For notational convenience, we will denote $\frac{\partial}{\partial x_j}$ by ∂_j. Recall that these coordinate fields commute, i.e. $[\partial_i, \partial_j] = 0$ for $1 \le i, j \le n$.

Unfortunately, these coordinate fields ∂_i will *not* constitute an orthonormal frame. However any connection on the principal $O(n)$ frame bundle of TM will result in a covariant derivative operator on TM, which in turn will result in a connection on the principal $GL(n)$-bundle of *all* frames in TM, call it $\pi : P \to M$. Using the section $\sigma : U \to \pi^{-1}(U)$ defined by $\sigma(x) = (\partial_{1,x}, .., \partial_{n,x})$, we can pullback the connection form ω on P to the local connection form $\sigma^*\omega$ on U. For notational convenience, we call it ω. Clearly the connection is completely determined on U once we specify $\omega(\partial_i) \in \mathfrak{gl}(n,\mathbb{R})$.

If ∇ is a covariant derivative operator on M, it is completely determined on U by ∇_{∂_i} for $i = 1, .., n$. These in turn are completely determined on U by the vector fields $\nabla_{\partial_i}\partial_j$. We may expand this last field as:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

where we use the Einstein summation convention, i.e. repeated indices are summed over 1 through n. These Γ^k_{ij} are smooth functions on U, and are called the *"Christoffel symbols"* or *"connection coefficients"* of the connection. They are related to the matrix of connection 1-forms as follows:

$$\Gamma^k_{ij} = \omega_{jk}(\partial_i)$$

where ω is the local connection form on U defined above in remark 6.8.4. This is clear from the relation $\nabla\partial_j = (\omega.\sigma)_j = \sum_k \omega_{jk}\partial_k$, as in the Remark 6.8.4 (and also Proposition 6.6.3).

Let $g = \langle\, ,\, \rangle$ denote the Riemannian metric. On U, this metric is completely determined by the symmetric matrix of smooth functions $[g_{ij}(x)]$ where $g_{ij}(x) = \langle \partial_i, \partial_j \rangle_x$. Since (ii) above is to hold, we get

$$\begin{aligned} \partial_k g_{ij} &= \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle \\ &= \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} \end{aligned}$$

The above equation is called the *First Christoffel Identity*. By cyclically permuting i, j, k we get the equations and:

$$\begin{aligned} \partial_k g_{ij} &= \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} \\ \partial_i g_{jk} &= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} \\ \partial_j g_{ki} &= \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl} \end{aligned}$$

Now, since the connection is to be torsionless, we have:

$$\nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i = [\partial_i, \partial_j] = 0$$

so that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Using this, and the symmetry of g_{ij}, we subtract the second of the above equations from the sum of the first and third to obtain:

$$\frac{1}{2} \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk} \right) = \Gamma^l_{jk} g_{li}$$

Multiplying the equation above by g^{im}, where $[g^{im}]$ is the inverse matrix of g_{ij}, and summing over i, we obtain:

$$\Gamma^m_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk} \right) \tag{17}$$

This determines the Christoffel symbols[2], and hence the connection form ω in terms of the metric g_{ij} on any coordinate patch U and is called the *second Christoffel Identity.* Hence the connection is uniquely determined by the conditions (i) and (ii), and is *the* Levi-Civita connection. □

Remark 6.9.2 (Torsion of a connection)**.** Let ∇ be *any* connection on a manifold, viz., a connection on its tangent bundle. Then, for vector fields X, Y on M, define the *torsion* of the connection ∇ by:

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

A priori, the definition above just tells us that τ is an $\mathbb{R}$- bilinear map $\Gamma(M, TM) \times \Gamma(M, TM) \to \Gamma(M, TM)$. However we first make the:

Claim: The tangent vector $[\tau(X, Y)](x) \in T_x(M)$ depends only on $X(x)$ and $Y(x)$. In particular, τ gives a well defined bundle morphism $TM \times TM \to TM$, that is, it is a tensor of type $(1, 2)$, and because of skew symmetry, a section of $TM \otimes \Lambda^2(T^*(M))$ (=a "TM-valued 2-form" on M).

To see the claim, we first note that for smooth functions f, g on M, we have:

$$\tau(fX, gY) = fg\tau(X, Y)$$

Because $\tau(X, Y) = -\tau(Y, X)$ by definition, it is enough to see that $\tau(X, fY) = f\tau(X, Y)$. This follows because

$$\begin{aligned} \tau(X, fY) &= \nabla_X(fY) - \nabla_{fY}(X) - [X, fY] \\ &= (f\nabla_X Y + X(f)Y) - f\nabla_Y X - (X(f)Y + f[X, Y]) \\ &= f\,(\nabla_X Y - \nabla_Y X - f[X, Y]) = f\tau(X, Y) \end{aligned}$$

by using (i) and (ii) of 6.8.2, and (ii) of 5.5.2. In fact, this same formula holds for vector fields X, Y on an open set U, and smooth functions $f, g \in C^\infty(U)$.

Now let X be a vector field that vanishes at a point x, where $x \in U_\alpha$ for some coordinate patch U_α. Letting ∂_i be the coordinate fields on U_α, we may write X on U_α as:

$$X = \sum_{i=1}^{n} f_i \partial_i$$

[2]The way to remember the confusing array of indices is to note that the subscripts jk on Γ^m_{jk} are the subscripts on the term containing g_{jk} on the right hand side, which is the term with the minus sign. The other terms are obtained by cyclic permutation and carry a plus sign.

where f_i are smooth functions on U_α, and $f_i(x) = 0$. Then

$$\tau(X, Y) = \sum_i \tau(f_i \partial_i, Y) = \sum_i f_i \tau(\partial_i, Y)$$

by the observations above. Thus

$$[\tau(X, Y)](x) = \sum_i f_i(x)(\partial_{i,x}, Y(x)) = 0$$

By skew-symmetry, it similarly follows that $[\tau(X, Y)](x) = 0$ if $Y(x) = 0$. Hence, if $X(x) = \widetilde{X}(x)$, and $Y(x) = \widetilde{Y}(x)$ for some smooth vector fields X, $\widetilde{X}$, Y, $\widetilde{Y}$, then we have $[\tau(X, Y)](x) = [\tau(\widetilde{X}, \widetilde{Y})](x) =: \tau(X(x), Y(x))$, and hence the claim.

Example 6.9.3 (Cartan connection on a Lie group)**.** We recall the (left invariant) Cartan connection on a Lie group G (see Example 6.4.25 and Exercise 6.6.7). It is the unique connection on TG satisfying:

$$\nabla_X \widetilde{Y} = \nabla_{\widetilde{X}} \widetilde{Y} = [\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]} = [X, Y]$$

for $X, Y \in \mathfrak{g} = T_e(G)$, and $\widetilde{X}, \widetilde{Y}$ the corresponding left invariant vector fields. Then let $\{X_i\}$ be an $\mathbb{R}$-basis of the Lie algebra $\mathfrak{g} = T_e(G)$, which in turn determines a global smooth frame $\{\widetilde{X}_i\}$ of the corresponding left invariant vector fields for the (trivial) tangent bundle TG. So the quantities $\nabla_{\widetilde{X}} \widetilde{X}_j$ completely determine $\nabla_{X(g)} Y$ for *all* tangent vectors $X(g) \in T_g(G)$ and *all* smooth vector fields Y on G, by forcing Leibnitz's Rule.

However, this natural Cartan connection on a Lie group is *not* torsionless. In fact, from the definition above it follows that for left invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ we have:

$$\nabla_{\widetilde{X}} \widetilde{Y} - \nabla_{\widetilde{Y}} \widetilde{X} - [\widetilde{X}, \widetilde{Y}] = [\widetilde{X}, \widetilde{Y}]$$

There is an easy way to remedy this, of course. If we define $\nabla_X \widetilde{Y} = \frac{1}{2}[\widetilde{X}, \widetilde{Y}] = \frac{1}{2}[X, Y]$, then we find that this new connection is torsionless.

The Cartan connection does however satisfy a formula analogous to that of a metric connection. We first make the following :

Definition 6.9.4 (Cartan-Killing Form)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and (as usual) for $X \in \mathfrak{g}$, let $\operatorname{ad} X$ denote the endomorphism $[X, -]$ of $\mathfrak{g}$. The symmetric bilinear form defined on $\mathfrak{g}$ by

$$B(X, Y) = \mathrm{tr}(\mathrm{ad} X \, \mathrm{adY})$$

is called the *Cartan-Killing form on* $\mathfrak{g}$.

Exercise 6.9.5. Verify that the Cartan-Killing form has the following *ad-invariance property*:

$$B((\mathrm{ad}X)Y, Z) + B(Y, (\mathrm{ad}X)Z) = B([X, Y], Z) + B(Y, [X, Z]) = 0$$

By the above Exercise 6.9.5, the Cartan-Killing form B leads to a bi-invariant symmetric bilinear form on the tangent bundle defined by setting $\left\langle \widetilde{X}, \widetilde{Y} \right\rangle := B(X, Y)$. Then, for *left invariant* vector fields, since $\left\langle \widetilde{Y}, \widetilde{Z} \right\rangle$ is a constant function on G, we will have:

$$\left\langle \nabla_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \right\rangle + \left\langle \widetilde{Y}, \nabla_{\widetilde{X}} \widetilde{Z} \right\rangle = 0 = \widetilde{X} \left\langle \widetilde{Y}, \widetilde{Z} \right\rangle \tag{18}$$

Unfortunately, in general, this symmetric form could be degenerate, or even identically 0 (e.g. if the group is abelian, or even nilpotent). It is a fact from the elementary theory of Lie algebras that B is a non-degenerate for a semisimple Lie algebra. For a *compact Lie group* whose Lie algebra is semisimple, (e.g. $SO(n), U(n), SU(n)$), it further turns out that this Cartan-Killing form on $\mathfrak{g} = \mathrm{Lie}(G)$ is *negative definite*. Hence, changing its sign yields a *bi-invariant Riemannian metric* [3]. In that case, setting $\nabla_{\widetilde{X}}\widetilde{Y} = \frac{1}{2}[\widetilde{X}, \widetilde{Y}]$ would yield a torsionless connection, and in view of the formula (18) above, yield the unique Levi-Civita connection on G.

However, in the general case, it is useful to write down the Levi-Civita connection with respect to a *left-invariant* metric $\langle\, , \rangle$ on *any* general Lie group G, and work the above (compact) case out as a special case. We first need to make the following:

Definition 6.9.6 (Structure constants of a Lie algebra)**.** Let $\{X_i\}_{i=1}^n$ be a basis of $\mathfrak{g}$. The *structure constants* c_{ij}^k of $\mathfrak{g}$ (with respect to this basis) are defined by the formula:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k$$

They satisfy certain obvious relations, i.e. $c_{ij}^k = -c_{ji}^k$, and another quadratic identity, viz.

$$c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l = 0 \qquad \text{for all } \ 1 \le i, j, k, m \le n$$

which follows from from the Jacobi identity. (Repeated indices are summed over, as per the Einstein summation convention).

Proposition 6.9.7 (Levi-Civita connection on a Lie group)**.** Let G be a Lie group, and let $\langle .,. \rangle$ be a *left invariant* metric on G. With respect to this metric, let $\{X_i\}$ be an *orthonormal basis* of $\mathfrak{g} = \mathrm{Lie}(G)$, so that $\{\widetilde{X}_i\}$ is a global smooth left invariant orthonormal frame field of TG. Let c_{ij}^k be the structure constants of $\mathfrak{g}$ with respect to the basis $\{X_i\}$. Then the connection coefficients of the Levi-Civita connection on G are given by:

$$\Theta_{jk}^i = \frac{1}{2}\left(c_{jk}^i - c_{ki}^j + c_{ij}^k\right)$$

(We note that Θ_{jk}^i are *not* Christoffel symbols, because the vector fields $\widetilde{X}_i$ will not be coordinate fields in general, since they don't commute unless all the structure constants vanish!).

In particular, if the left invariant metric happens to be *bi-invariant*, the the Levi-Civita connection coincides with the Cartan connection of Example 6.9.3 above.

Proof: Since X_i are orthonormal, it follows from the Definition 6.9.6 that the structure constants are given by the relation $c_{ij}^k = \langle [X_i, X_j], X_k \rangle$.

Let us express

$$\nabla_{\widetilde{X}_i}\widetilde{X}_j = \sum_k \Theta_{ij}^k \widetilde{X}_k$$

where Θ_{ij}^k are the smooth connection coefficient functions G that we seek. It will turn out presently that they are constants. The torsion free condition immediately implies that:

$$\Theta_{jk}^i - \Theta_{kj}^i = c_{jk}^i \tag{19}$$

The metric connection condition implies that:

$$\Theta_{ji}^k + \Theta_{jk}^i = \left\langle \nabla_{\widetilde{X}_j}\widetilde{X}_i, \widetilde{X}_k \right\rangle + \left\langle \widetilde{X}_i, \nabla_{\widetilde{X}_j}\widetilde{X}_k \right\rangle = \widetilde{X}_j \left\langle \widetilde{X}_i, \widetilde{X}_k \right\rangle = \widetilde{X}_j \left\langle X_i, X_k \right\rangle = 0 \tag{20}$$

where the last-but-one equality follows from left invariance of the metric.

Permuting i, j, k cyclically, we get the equations:

$$\begin{aligned} \Theta_{ji}^k + \Theta_{jk}^i &= 0 \\ \Theta_{kj}^i + \Theta_{ki}^j &= 0 \\ \Theta_{ik}^j + \Theta_{ij}^k &= 0 \end{aligned}$$

[3]We had given a prescription for constructing a bi-invariant Riemannian metric on a compact Lie group in Proposition 5.1.17 on G. It turns out that a bi-invariant Riemannian metric on a connected compact semisimple Lie Group is unique upto scaling by a positive constant

By subtracting the third from the sum of the first two, and using (19), we have:

$$\Theta^i_{jk} + \Theta^i_{kj} = -c^j_{ki} + c^k_{ij}$$

which, together with the equation (19) implies that

$$\Theta^i_{jk} = \frac{1}{2}\left(c^i_{jk} - c^j_{ki} + c^k_{ij}\right)$$

In particular, the Θ^i_{jk}'s are all constants, and we get the formula claimed.

To see the second part of the proposition, the definition of Θ^i_{jk}'s and our computation above gives the formula:

$$\nabla_{\widetilde{X}_j}\widetilde{X}_k = \frac{1}{2}\sum_i \left(c^i_{jk} - c^j_{ki} + c^k_{ij}\right)\widetilde{X}_i \tag{21}$$

If the inner product is *bi-invariant*, then $c^k_{ij} = \langle [X_i, X_j], X_k\rangle = \langle X_j, [X_k, X_i]\rangle = c^j_{ki}$, and in this case our formula (21) reads as:

$$\nabla_{\widetilde{X}_j}\widetilde{X}_k = \frac{1}{2}\sum_i c^i_{jk}\widetilde{X}_i = \frac{1}{2}[\widetilde{X}_j, \widetilde{X}_k] = \frac{1}{2}\widetilde{[X_j, X_k]}$$

which is the formula we had in Example 6.9.3 for the (torsionless) Cartan connection. In fact, as we remarked earlier, for a compact semisimple group, every bi-invariant metric is a positive scalar multiple of $-B$ above, and the formula above is therefore just the statement of ad-invariance of the Cartan-Killing metric $-B$. We thus see that on a compact semisimple group, there is a *unique, upto scaling* bi-invariant Riemannian metric, and the torsionless Cartan connection is the *unique* connection which is compatible with any bi-invariant metric on G. □

Remark 6.9.8 (Abelian Lie Groups)**.** If the Lie group G is abelian, then with respect to the Levi-Civita connection with respect to a left invariant metric on G, the left-invariant vector fields will have zero covariant derivatives (ie. will be covariantly constant) with respect to any tangent vector at any point. This follows from the fact that all the c^i_{jk} and hence all Θ^i_{jk} vanish from the previous proposition.

Example 6.9.9 (Hyperbolic space H^2)**.** We recall the hyperbolic space $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian (hyperbolic) metric:

$$\frac{dx^2 + dy^2}{y^2}$$

so that $\langle \partial_i, \partial_j\rangle = y^{-2}\delta_{ij}$ for $1 \leq i, j \leq 2$. We calculate the Christoffel symbols.

Recalling that

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

and differentiating:

$$0 = \partial_1 \langle \partial_1, \partial_1\rangle = \frac{\partial}{\partial x}(y^{-2}) = 2\langle \nabla_{\partial_1}\partial_1, \partial_1\rangle \Rightarrow \Gamma^1_{11} = 0$$

Similarly the relation $\partial_2 \langle \partial_1, \partial_1\rangle = -2y^{-3}$ yields $\Gamma^1_{21} = -y^{-1}$. Differentiating all the inner products $\langle \partial_i, \partial_j\rangle$ partially with respect to x and y yields all the Christoffel symbols:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0$$
$$\Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^1_{21} = -\Gamma^2_{22} = y^{-1}$$

Exercise 6.9.10. Compute the Christoffel symbols of the Levi-Civita connection on the sphere S^2.

6.10. **Parallel transport and geodesics.** Let M be a Riemannian manifold, and let η be a connection on the principal $GL(n,\mathbb{R})$-frame bundle P of TM, leading to a covariant differentiation operator ∇ on the vector bundle TM. If $c:(a,b)\to M$ is a smooth curve in M, then it is clear that the pullback of the tangent bundle c^*TM will have the corresponding pulled back principal bundle c^*P on (a,b) as its principal frame bundle. It will also have the pullback connection $c^*\eta$ introduced in the Exercise 6.4.24. Note that $\frac{d}{dt}$ is a smooth vector field on (a,b). We can also see this connection more directly as follows.

Definition 6.10.1 (Vector field along a curve)**.** A *vector field X along the smooth curve $c:(a,b)\to M$* is defined to be a smooth section of c^*TM. In particular $X(t)\in T_{c(t)}M$ for all t. However note that X is *not* a function of the point $c(t)$ but of t. For example if c crosses itself, with $c(t_1)=c(t_2)$ and $t_1\neq t_2$, then $X(t_1)$ can be different from $X(t_2)$.

Example 6.10.2 (Velocity field of a curve)**.** A prime example of a vector field along a curve is its velocity field c', defined by:

$$c'(t):=c_*\left(\frac{d}{dt}\right)=\frac{dc}{dt}(t)\in T_{c(t)}(M)$$

Lemma 6.10.3 (Covariant derivative along a curve)**.** Let M be a Riemannian manifold M and $c:I\to M$ be a smooth curve on it, where $I=(a,b)$. Then there exists an $\mathbb{R}$-linear operator:

$$\frac{D}{dt}:\Gamma(I,c^*TM)\to\Gamma(I,c^*TM)$$

which satisfies:

(i): Let $X\in\Gamma(I,c^*TM)$ be a vector field along c and $f\in C^\infty(I)$ be a smooth function. Then

$$\frac{D(fX)}{dt}=f\left(\frac{DX}{dt}\right)+\frac{df}{dt}X\qquad\text{(Leibnitz's formula)}$$

(ii): For $X,Y\in\Gamma(I,c^*TM)$, we denote their inner product at $t\in I$ under the bundle metric on c^*TM obtained by pulling back the Riemannian metric $\langle -,-\rangle$ on TM by $\langle X,Y\rangle_t$. That is, $\langle X,Y\rangle_t:=\langle X(t),Y(t)\rangle$ where the right hand side is the inner product on $T_{c(t)}(M)$. Then

$$\frac{d}{dt}\langle X,Y\rangle_t=\left\langle\frac{DX}{dt},Y\right\rangle_t+\left\langle X,\frac{DY}{dt}\right\rangle_t\qquad\text{(Compatibility with the metric)}$$

Proof: We first let c be a smooth curve whose image lies entirely in one coordinate chart U, with coordinate functions x_i and coordinate fields $\partial_i:=\frac{\partial}{\partial x_i}$. From Remark 6.8.4 we recall the matrix ω_{ij} of connection 1-forms on U defined by:

$$\nabla\partial_i=\sum_i\omega_{ij}\otimes\partial_j$$

which are related to the Christoffel symbols of the Levi-Civita connection by $\omega_{ij}(\partial_k)=\Gamma^j_{ki}$ (see proof of Proposition 6.9.1). Clearly these 1-forms ω_{ij} on U can be pulled back via c, and used to define the pullback connection operator $\widetilde{\nabla}$ via the formula:

$$\widetilde{\nabla}(\widetilde{\partial}_i)=\sum_i c^*\omega_{ij}\otimes\widetilde{\partial}_j$$

where the vector fields along c given by $\widetilde{\partial}_i:=c^*\partial_i\in\Gamma(I,c^*TM)$ constitute a smooth frame of c^*TM, obtained by pulling back the smooth coordinate sections $\partial_i\in\Gamma(U,TM)$ via c. Then the covariant derivatives with respect to the velocity field $\frac{d}{dt}$ is

$$\frac{D(\widetilde{\partial}_i)}{dt}:=\widetilde{\nabla}_{\frac{d}{dt}}(\widetilde{\partial}_i)=\sum_j(c^*\omega_{ij})\left(\frac{d}{dt}\right)\widetilde{\partial}_j=\sum_j\omega_{ij}(c(t))\left(\frac{dc}{dt}\right)\widetilde{\partial}_j=\sum_{j,k}\Gamma^j_{ki}(c(t))\frac{dc_k}{dt}\widetilde{\partial}_j$$

where we have used the expansion $\frac{dc}{dt} = \sum_k \left(\frac{dc_k}{dt}\right) \partial_k$ to get the last equality above.

Now, if $X = \sum_i X_i(t)\widetilde{\partial}_i \in \Gamma(I, c^*TM)$ is a vector field along the curve c, we define the *covariant derivative along the curve* c by forcing Leibnitz's rule and using the previous formula, i.e.:

$$\begin{aligned} \frac{DX}{dt} := \widetilde{\nabla}_{\frac{d}{dt}} X &= \sum_i \widetilde{\nabla}_{\frac{d}{dt}} (X_i(t)\widetilde{\partial}_i) = \sum_i \left(\frac{dX_i}{dt}\widetilde{\partial}_i + \sum_{k,j} \Gamma^j_{ki}(c(t)) \frac{dc_k}{dt} X_i \widetilde{\partial}_j \right) \\ &= \sum_j \left(\frac{dX_j}{dt} + \sum_{k,i} \Gamma^j_{ki}(c(t)) \frac{dc_k}{dt} X_i \right) \widetilde{\partial}_j \end{aligned} \tag{22}$$

This is again be a smooth section of c^*TM on I, viz. another vector field along c. This takes care of the case when the image of c lies in one coordinate chart.

In the general situation, we note that if (ϕ_α, U_α) and (ϕ_β, U_β) are two coordinate charts on M, and ω^α and ω^β respectively, denote the matrix of connection 1-forms on these two charts, then they are related on the overlap $U_\alpha \cap U_\beta$ by:

$$\omega^\alpha = g\omega^\beta g^{-1} + dg.g^{-1}$$

where $g := g_{\alpha\beta} = D(\phi_\alpha \phi_\beta^{-1})$ is the transition cocycle of the tangent bundle TM (see Remark 6.8.4). Since pullback commutes with matrix multiplication and exterior differentiation, this formula of 1-forms on $U_\alpha \cap U_\beta$ is preserved under pullback c^*, yielding the exact same formula on the overlap $c^{-1}(U_\alpha) \cap c^{-1}(U_\beta)$. Hence on the naturally defined trivialising open covering $\{c^{-1}(U_\alpha)\}$ of c^*TM, these pullback matrices $c^*(\omega^\alpha)$ patch up to define a global pullback connection $\widetilde{\nabla}$ on the vector bundle c^*TM. The local formula on each $c^{-1}(U_\alpha)$ is exactly the one calculated in equation (22) of the last paragraph. This defines $\frac{D}{dt}$ in the general case.

Finally, the Leibnitz formula and compatibility with the pullback metric follow from the definition of $\widetilde{\nabla}$, and the corresponding properties of ∇. This establishes the lemma. $\square$

Definition 6.10.4 (Parallel vector field, geodesic)**.** We will say that the vector field X along c is *parallel along* c if $\frac{DX}{dt} \equiv 0$. If $X = c'(t) = \frac{dc}{dt}$ is the velocity field of c, the vector field along c defined by $\frac{Dc'}{dt}$ is called the *covariant acceleration* of c. If the velocity field c' is parallel along c (i.e. the covariant acceleration of c is identically zero), then we say c is a *geodesic.*

Example 6.10.5. If we look at the canonical Cartan connection on a Lie group G, (see Example 6.9.3), then we have $\nabla_{\widetilde{X}}\widetilde{X} = [\widetilde{X}, \widetilde{X}] = 0$. We recall from the Theorem 5.5.4, and the proof of Corollary 5.6.4 that the integral curves c to the left invariant vector field $\widetilde{X}$ are the one parameter groups $(\exp(tX))$, and all their left translates $g.\exp(tX)$, for $X \in \mathfrak{g}$, $g \in G$. Thus for such integral curves c, the velocity fields $c'(t)$ are precisely $\widetilde{X}(t)$, which are thus parallel along c. Thus, for the Cartan connection, all one-parameter groups $\exp(tX)$ and their left translates are geodesics.

From Example 6.9.3 it also follows that these are all the geodesics for the Levi-Civita connection of a *bi-invariant* Riemannian metric on G. In general, however, they will *not* be geodesics for the Levi-Civita connection with respect to a *left invariant* metric. In fact, there is the following :

Example 6.10.6. Using the standard basis $X_1 = H, X_2 = X, X_3 = Y$ of the group $SL(2, \mathbb{R})$ introduced in Remark 5.8.7, namely:

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

we have the commutators:

$$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1$$

This leads to the structure constants

$$c^2_{12} = -c^2_{21} = c^3_{31} = -c^3_{13} = 2, \;\; c^1_{23} = -c^1_{32} = 1$$

and all other $c^i_{jk} = 0$. When we use the connection coefficients Θ^k_{ij} (from the computation in Proposition 6.9.3) for the Levi-Civita connection of the left-invariant metric coming from the metric $\langle X_i, X_j \rangle = \delta_{ij}$ on $\mathfrak{sl}(2,\mathbb{R})$, we find, for example, that

$$\Theta^1_{22} = \frac{1}{2}\left(c^1_{22} - c^2_{21} + c^2_{12}\right) = c^2_{12} = 2$$

Similarly, one finds that $\Theta^2_{22} = \Theta^3_{22} = 0$. This implies that:

$$\nabla_{\widetilde{X}_2} \widetilde{X}_2 = \sum_{i=1}^{3} \Theta^i_{22} \widetilde{X}_i = \Theta^1_{22} \widetilde{X}_1 = 2\widetilde{X}_1$$

This shows that the left-invariant vector field $\widetilde{X}_2$ is not parallel along the one-parameter group $\exp(tX_2)$, and hence this one parameter group is *not* a geodesic for the Levi-Civita connection with respect to the left-invariant metric defined by $\left\langle \widetilde{X}_i, \widetilde{X}_j \right\rangle = \delta_{ij}$.

Proposition 6.10.7 (Existence of parallel transport)**.** Let M be a Riemannian manifold, and let ∇ be the Levi-Civita connection on it. For a smooth curve $c : [0,1] \to M$ in M, there exists an $\mathbb{R}$-linear operator $P_t : T_{c(0)}(M) \to T_{c(t)}(M)$ satisfying:

(i): For each $X \in T_{c(0)}(M)$, the vector field along the curve c defined by $X(t) := P_t X$ is parallel along c.

(ii): P_t is an isometry (with respect to the inner products on $T_{c(0)}(M)$ and $T_{c(t)}(M)$ coming from the Riemannian metric on M).

(iii): $P_t \circ P_s = P_{t+s}$, whenever both sides make sense.

The operator P_t is called *parallel transport* or *parallel translation*. (Compare the definition in Definition 6.6.1 for principal G-bundles)

Proof: If the image of c lies in one coordinate chart, then we find the parallel vector field $X(t) = P_t X$ along c by solving the system of n-first order linear ODEs:

$$\frac{dX_j}{dt} + \sum_{k,i} \Gamma^j_{ki}(c(t)) \frac{dc_k}{dt} X_i = 0$$

arising from setting all the components of $\frac{DX(t)}{dt}$ to be zero. This follows from the formula (22) in the proof of Lemma 6.10.3. Since this is a linear system, it has a solution for $t \in [0,1]$, and the solution $X(t) = (X_1(t), .., X_n(t))$ is uniquely determined by the initial value $X_i(0) = X_i$ (n-initial conditions), as a linear function of them.

For the general situation, we can use the Lebesgue covering lemma to find a partition of $[0,1]$ into subintervals $I_j := [\frac{j}{N}, \frac{j+1}{N}]$, $0 \leq j \leq N-1$ such that each $c(I_j)$ is contained in some coordinate chart. Then if $t \in I_j$, we may define P_t as the composite:

$$P_{t-\frac{j}{N}} \circ P_{\frac{j}{N}} \circ ... \circ P_{\frac{1}{N}}$$

This proves (i). For the second assertion, we note that $P_t X = X(t)$ and $P_t Y = Y(t)$ are parallel along c by definition, so the formula (ii) of Lemma 6.10.3 shows that:

$$\frac{d}{dt} \langle X(t), Y(t) \rangle = \left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle = 0$$

which shows that $\langle P_t X, P_t Y \rangle = \langle X, Y \rangle$.

The last assertion is also clear from the semigroup property of the solution to a linear system of ODE's. $\square$

Corollary 6.10.8 (Moving frame or repere mobile)**.** Let $c : [0,1] \to M$ be a smooth curve on a Riemannian manifold M. Let $F_0 := \{e_i\}_{i=1}^n$ be an orthonormal frame for $T_{c(0)}(M)$. Let $e_i(t) := P_t e_i$, the parallel transports of e_i along c. Then:

(i): The set of vectors $F_t := \{e_i(t)\}_{i=1}^n$ is an orthonormal frame for $T_{c(t)}(M)$, and is called the *moving frame* (or Cartan's *repere mobile*) along c (Compare Definition 6.6.1).

(ii): If X is a vector field along c, then the covariant derivative:

$$\frac{DX}{dt} = \sum_{i=1}^{n} \frac{dX_i(t)}{dt} e_i(t)$$

where $X_i(t) = \langle X(t), e_i(t)\rangle$. Thus covariant derivative along c becomes ordinary derivative with respect to t when the vector field X along c is expressed in terms of the moving frame as $X(t) = \sum_i X_i(t)e_i(t)$.

Proof: That $F_t = \{e_i(t)\}_{i=1}^n$ is an orthonormal frame at $c(t)$ follows from (ii) of the previous Proposition 6.10.7. Thus we clearly have $X(t) = \sum_{i=1}^n X_i(t)e_i(t)$, where $X_i(t) = \langle X(t), e_i(t)\rangle$. Then the second assertion follows from the fact that $\frac{De_i}{dt} = 0$, by (i) of the previous Proposition 6.10.7 and the Leibnitz formula. $\square$

Example 6.10.9 (Hyperbolic space)**.** Let $c : [0,1] \to H^2$ be the curve $c(t) = (0, 1+t)$, the linear line segment joining $(0,1)$ to $(0,2)$ in the hyperbolic plane H^2. Using the Christoffel symbols computed in Example 6.9.9, and that $c_1'(t) = 0,\ c_2'(t) = 1$, we find that a vector field $X(t) = X_1(t)\partial_x + X_2(t)\partial_y$ a along c is parallel along c iff

$$\frac{dX_1}{dt} - \frac{1}{(1+t)}X_1(t) = \frac{dX_2}{dt} - \frac{1}{(1+t)}X_2(t) = 0$$

This leads to $X(t) = (1+t)X(0)$. Note that

$$\|X(t)\|^2 = \frac{1}{(1+t)^2}\sum_{i=1}^{2} X_i(t)^2 = \|X(0)\|^2$$

as it should be, since P_t is an isometry and $X(t) = P_t X(0)$. If we, for example, choose the frame $e_1 = \partial_x,\ e_2 = \partial_y$ at $T_{c(0)}(H^2)$, then the corresponding moving frame along c is $e_i(t) = (1+t)e_i$, for $i = 1, 2$.

Exercise 6.10.10. Let $M = S^2$ be the unit sphere centred at the origin in $\mathbb{R}^3$, and let $c(t) = (\cos\frac{\pi t}{2}, 0, \sin\frac{\pi t}{2})$ for $t \in [0,1]$. This is the arc in S^2 joining $e_1 = (1,0,0)$ to $e_3 = (0,0,1)$ in the xz-plane. Let $F = \{e_2, e_3\}$, which is an orthonormal frame for $T_{e_1}(S^2)$.

(i): Determine the moving frame $F_t = \{P_t e_2, P_t e_3\}$ along c in S^2.

(ii): Determine the parallel transports $P_t e_2$, and $P_t e_3$ along c in $\mathbb{R}^3$. This shows that even though the Riemannian metric on S^2 is induced from the euclidean metric on $\mathbb{R}^3$, parallel transport on S^2 differs from that on $\mathbb{R}^3$.

Remark 6.10.11. Note that $P_t : T_{c(0)}(M) \to T_{c(t)}(M)$ being an isometry for all t does not quite completely pin down P_t in general. For example, if we take $\mathbb{R}^3$ with its euclidean metric and choose the curve $c(t) = (t, 0, 0)$ in $\mathbb{R}^3$, and choose the frame $e_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$ at $T_{c(0)}(\mathbb{R}^3)$, then the frame:

$$e_1(t) = e_1,\ e_2(t) = (\cos t)e_2 + (\sin t)e_3,\ e_3(t) = (-\sin t)e_2 + (\cos t)e_3$$

also makes $\{e_i(t)\}_{i=1}^3$ an orthonormal frame in $T_{c(t)}(\mathbb{R}^3)$ for all t. However this frame is *not* parallel along c. The parallel (moving) frame along c is $e_i(t) = e_i$.

Proposition 6.10.12 (Existence of geodesics)**.** Let M be a smooth manifold, and let us equip the tangent bundle TM with *any* connection. Let $x \in M$ and let $v \in T_x(M)$ be a tangent vector. Then there exists an $\epsilon > 0$ and a smooth curve:

$$c : [-\epsilon, \epsilon] \to M$$

such that:

(i): c is a geodesic.

(ii): $c(0) = x$

(iii): $c'(0) = v$

Proof: Let x lie in the coordinate patch U, and let x_j denote the coordinate functions with respect to this chart. The conditions (i) and (ii) above lead to the system of $2n$ first order ordinary differential equations (for $Z_j(t)$ and $c(t)$:

$$\begin{aligned} \frac{dc_j}{dt}(t) &= Z_j(t) \\ \frac{dZ_k}{dt}(t) &+ \Gamma^k_{ij}\frac{dc_i}{dt}Z_j = 0 \end{aligned} \tag{23}$$

from the local formula in the Lemma 6.6.3, using the fact that the velocity field of a geodesic is parallel along it. The initial conditions are $c_j(0) = x_j,\ Z_j(0) = v_j$. This can also be recast as the system of n second order ODE's for $c(t)$:

$$\frac{d^2c_k}{dt^2}(t) + \Gamma^k_{ij}\frac{dc_i}{dt}\frac{dc_j}{dt} = 0 \tag{24}$$

with the initial conditions $c(0) = x,\ c'(0) = v$. By the existence and uniqueness theorem for ODE's, there is a unique solution for this system for $t \in [-\epsilon, \epsilon]$ and some $\epsilon > 0$, thus proving the proposition. □

Example 6.10.13 (Euclidean Space)**.** Since the metric is δ_{ij} in the standard coordinates on $\mathbb{R}^n$, all Christoffel symbols $\Gamma^k_{ij} = 0$. Thus the geodesic equations say that the second derivative of $c(t)$ is zero, so that $c(t) = ta+b$ for some $a, b \in \mathbb{R}^n$, i.e. a straight line.

Remark 6.10.14. Consider $M = (0, 1)$ and $x \in M$, with the tangent vector $v = \frac{d}{dt}$ and the Levi-Civita connection from the euclidean metric on M. As in the previous example, the geodesic starting at x with initial velocity v is given by $c_v(t) = x + t$, whose domain of definition is $(-\epsilon, \epsilon)$ with $\epsilon = \min\{x, 1 - x\}$. Thus the ϵ guaranteed by the previous proposition will shrink as x gets closer to the end-points of the interval M.

Definition 6.10.15 (Exponential map on a manifold)**.** Let M be a smooth manifold with *any* smooth connection ∇. We denote the curve $c(t)$ of the Proposition 6.10.12 above by $\mathrm{Exp}_x(t, v)$. Now we make the following:

Claim: $\mathrm{Exp}_x(t, v) = \mathrm{Exp}_x(Rt, \frac{v}{R})$.

To see this we need to show that the geodesic curve $c_v(t)$ defined on $[-\epsilon, \epsilon]$ with initial data $c(0) = x,\ c'_v(0) = v$ is the same as the geodesic curve $\widetilde{c}(t) := c_{v/R}(Rt)$ defined on $[-\delta, \delta]$ satisfying the initial data $\widetilde{c}(0) = x$ and $\widetilde{c}'(0) = \frac{v}{R}$ on their common domain of definition. Since $c_{v/R}$ satisfies the system of ODE's (24), and $\widetilde{c}'(t) = Rc'_{v/R}(t)$, $\widetilde{c}''(t) = R^2c''_{v/R}(t)$, it follows that $\widetilde{c}$ satisfies the system of equations:

$$R^{-2}\left(\frac{d^2\widetilde{c}_k}{dt^2}(t) + \Gamma^k_{ij}\frac{d\widetilde{c}_i}{dt}\frac{d\widetilde{c}_j}{dt}\right) = 0$$

which is exactly the same as the geodesic ODE's (24). Thus the curve $\widetilde{c}$ is a geodesic curve. Further, $\widetilde{c}(0) = x = c(0)$. Also $\widetilde{c}'(0) = R.c'_{v/R}(0) = v$. Thus $\widetilde{c} \equiv c$ on their common domain of definition, by the uniqueness part of the Theorem 5.4.2. So we see that $\mathrm{Exp}_x(t, v)$ is the same as $\mathrm{Exp}_x(Rt, \frac{v}{R})$ and we may thus denote $\mathrm{Exp}_x(t, v)$ as $\mathrm{Exp}_x(tv)$.

In fact we have the following stronger statement, which we don't prove here, but which essentially follows from the fact the solution to the above system of geodesic ODE's varies smoothly in the initial conditions, as stated in the Proposition 5.4.2:

Proposition 6.10.16. Let M be a smooth manifold with $x \in M$ a point on it. Let ∇ denote the Levi-Civita connection M. Then there exists an open neighbourhood U of the origin in $T_x(M)$, and a neighbourhood V of x in M such that the map:

$$\begin{aligned} \mathrm{Exp}_x : U &\to V \\ v &\mapsto \mathrm{Exp}_x(v) \end{aligned}$$

is a smooth diffeomorphism of U onto V. Thus one gets a natural chart (V, Exp_x^{-1}) around each point x in X. These are known as *exponential* or *geodesic coordinates.*

Proof: If we grant the fact that Exp_x is defined on some open neighbourhood of 0 in $T_x(M)$, then the rest of the proposition follows easily. This is because, by definition:

$$\frac{d\mathrm{Exp}_x(tv)}{dt}(0) = v$$

which shows that $D\mathrm{Exp}_x(0)v = v$, since $c(t) = tv$ is a curve passing through $0 \in T_x(M)$ with $c'(0) = v$. Thus $D\mathrm{Exp}_x = \mathrm{Id}$, and by the inverse function theorem, it follows that Exp_x maps a neighbourhood U of 0 in $T_x(M)$ diffeomorphically to its image V in M. $\square$

Corollary 6.10.17. If V is the neighbourhood x in the proposition above, then each point of V can be joined to x by a *unique geodesic*. There is a stronger theorem of J.H.C. Whitehead which asserts that for the Levi-Civita connection on a Riemannian manifold, each point $x \in X$ has a *geodesically convex neighbourhood*, i.e. a neighbourhood V such that every pair of points in V can be joined by a unique minimal length (among all piecewise smooth curves joining these points in X) geodesic which lies entirely in V. For a proof, see the book *Notes on Differential Geometry* by Hicks, p. 134.

We note here that if G is a Lie group with a left invariant metric $\langle\, , \rangle$, the corresponding map Exp defined above may not tally with the exponential map exp defined earlier. In fact:

Proposition 6.10.18 (Exp and exp for a Lie group)**.** Let $\langle -, - \rangle$ be a left invariant Riemannian metric on a Lie group G, corresponding to a positive definite inner product $\langle\, , \rangle$ on the Lie algebra $\mathfrak{g}$. Then the exponential map Exp_e at the identity $e \in G$ coincides with exp precisely when $\langle\, , \rangle$ on $\mathfrak{g}$ satisfies ad-invariance, viz.:

$$\langle Z, [Y, X] \rangle + \langle [Y, Z], X \rangle = 0 \quad \forall \;\; X, Y, Z \in \mathfrak{g}$$

(For example, this will happen if the left-invariant metric on G is bi-invariant (=Ad-invariant). Compare Proposition 6.9.7).

Proof: We let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ denote the left invariant vector fields corresponding to X, Y, Z respectively. Then we have $\frac{d}{dt}(\exp tX) = \widetilde{X}$. Thus, for the curve $\exp tX$ to be a geodesic, it is necessary and sufficient that:

$$\left\langle \nabla_{\widetilde{X}} \widetilde{X}, \widetilde{Y} \right\rangle = 0$$

for all Y. By compatibility with the metric, this means:

$$\widetilde{X} \left\langle \widetilde{X}, \widetilde{Y} \right\rangle - \left\langle \widetilde{X}, \nabla_{\widetilde{X}} \widetilde{Y} \right\rangle = - \left\langle \widetilde{X}, \nabla_{\widetilde{X}} \widetilde{Y} \right\rangle = 0$$

since the first term is $\widetilde{X} \langle X, Y \rangle = 0$ by the left invariance of $\langle -, - \rangle$. Since the connection is torsionless the last expression above is just

$$- \left\langle \widetilde{X}, \nabla_{\widetilde{Y}} \widetilde{X} \right\rangle + \left\langle \widetilde{X}, [\widetilde{Y}, \widetilde{X}] \right\rangle = \left\langle \widetilde{X}, [\widetilde{Y}, \widetilde{X}] \right\rangle$$

again because $\widetilde{Y}\left\langle \widetilde{X},\widetilde{X}\right\rangle = \widetilde{Y}\langle X,Y\rangle = 0$ by the left invariance of $\langle -,-\rangle$. Thus, for all left-invariant vector fields $\widetilde{X},\widetilde{Y}$, we have $\left\langle \widetilde{X},[\widetilde{X},\widetilde{Y}]\right\rangle = 0$, which implies (because of left-invariance of the metric and the definition of $[-,-]$ on $\mathfrak{g}$) that we may remove all the tildes, and get:

$$\langle X,[X,Y]\rangle = 0 \qquad \text{for all } \ X,Y\in\mathfrak{g} \tag{25}$$

This means that for all $X,Y,Z\in\mathfrak{g}$ we have by repeated use of equation (25) above:

$$\begin{aligned} 0=\langle X+Z,[X+Z,Y]\rangle &= \langle X,[X,Y]\rangle+\langle X,[Z,Y]\rangle+\langle Z,[X,Y]\rangle+\langle Z,[Z,Y]\rangle \\ =\langle X,[Z,Y]\rangle+\langle Z,[X,Y]\rangle &= -\langle X,\mathrm{ad}Y(Z)\rangle-\langle Z,\mathrm{ad}Y(X)\rangle \end{aligned}$$

which is precisely the ad-invariance of $\langle -,-\rangle$ we asserted. □

Corollary 6.10.19. On $SL(2,\mathbb{R})$, the maps Exp_e and exp are distinct. This is because every non-degenerate form satisfying the condition of the proposition above is a multiple of the Cartan-Killing form on $sl(2,\mathbb{R})$, which is non-degenerate, but neither positive definite nor negative definite. See the Example 6.10.6 where we explicitly showed that $\exp(tX_2)$ was not a geodesic for the Levi-Civita connection, viz., not the same as $\mathrm{Exp}_e(tX_2)$.

We now concentrate on geodesics in the Levi-Civita connection on a Riemannian manifold M.

Definition 6.10.20 (Piecewise smooth variation)**.** Let $c:[0,1]\to M$ be a piecewise-smooth curve in a smooth manifold M, viz. c is continuous and there exist $0=a_0<a_1<...<a_N=1$ with c smooth on each $[a_i,a_{i+1}]$. A *one parameter piecewise-smooth variation of the path* c with fixed end points x and y is a continuous map:

$$\begin{aligned} \phi:(-\epsilon,\epsilon)\times[0,1] &\to M \\ (s,t) &\mapsto \phi(s,t) \end{aligned}$$

further satisfying:

(i): $\phi(s,0)=x, \quad \phi(s,1)=y$ for all s; $\quad \phi(0,t)=c(t)$ for all t

(ii): ϕ is smooth on $(-\epsilon,\epsilon)\times[a_j,a_{j+1}]$ for all $j=1,..,N-1$

(iii): From (ii) above, it follows that for a fixed $t\in[0,1]$, the map $s\mapsto\phi(s,t)$ is smooth for all $s\in(-\epsilon,\epsilon)$.

By assumption (ii) above, ϕ is smooth on $V_i:=(-\epsilon,\epsilon)\times[a_i,a_{i+1}]$ for all $0\le i\le N-1$. So for $(s,t)\in V_i$, we define the tangent vectors $\partial_s:=\phi_*\left(\frac{\partial}{\partial s}\right)$ and $\partial_t:=\phi_*\left(\frac{\partial}{\partial t}\right)$ in $T_{\phi(s,t)}M$. Indeed, ∂_s is a vector field along the smooth curve $\phi(t,-)$ for each $t\in[0,1]$ (by (iii) above). Also for each $s\in(-\epsilon,\epsilon)$, ∂_t is a vector field along the curve $c_s:=\phi(s,-)$ defined for $t\in(a_i,a_{i+1})$.

Proposition 6.10.21 (The First Variation Formula)**.** Suppose M is a Riemannian manifold (with metric $\langle -,-\rangle$), and ∇ is the unique torsionless Levi-Civita connection on it as guaranteed by Proposition 6.9.1. For a piecewise-smooth curve $c:[0,1]\to M$, consider the *Energy Functional* which is defined as:

$$E(c)=\sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}}\langle c'(t),c'(t)\rangle\,dt$$

where c is smooth on each $[a_i,a_{i+1}]$, $c(0)=x$ and $c(1)=y$. Let ϕ be a one-parameter variation of c as defined in Definition 6.10.20 above. Then letting $c_s:=\phi(s,-)$, we have:

$$\frac{1}{2}\left(\frac{dE(c_s)}{ds}(0)\right)=\sum_{i=1}^{N-1}\Delta_i\langle\partial_s,c'(t)\rangle-\int_0^1\left\langle\partial_s,\frac{Dc'(t)}{dt}\right\rangle dt$$

where the jump discontinuity:

$$\Delta_i\langle\partial_s,c'(t)\rangle:=\lim_{t\to a_i-}\langle\partial_s,c'(t)\rangle_{c(t)}-\lim_{t\to a_i+}\langle\partial_s,c'(t)\rangle_{c(t)}$$

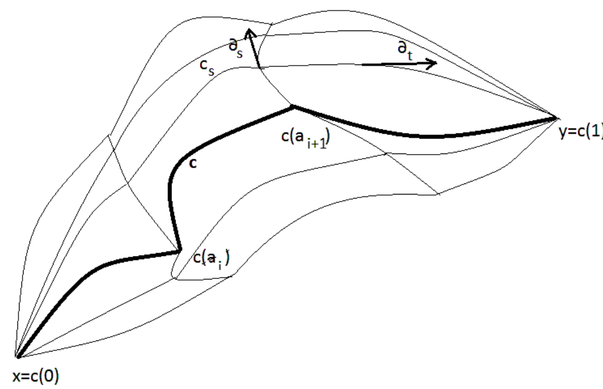

FIGURE 8. Piecewise smooth variation of c

and the integral on the right hand side is interpreted as a sum of integrals over the subintervals $[a_i, a_{i+1}]$ on which c is smooth.

Proof: This formula is essentially a version of the Euler-Lagrange formula that occurs in the calculus of variations. For $(s,t) \in V_i := (-\epsilon, \epsilon) \times [a_i, a_{i+1}] \subset \mathbb{R}^2$, the compatibility of the Riemannian metric with the Levi-Civita connection on TM implies that :

$$\partial_s \langle X, Y \rangle = \left\langle \frac{DX}{ds}, Y \right\rangle + \left\langle X, \frac{DY}{ds} \right\rangle \tag{26}$$

holds for smooth sections X, Y of $\phi^* TM_{|V_i}$ (or "vector fields along ϕ"). We compute:

$$\begin{aligned}
\frac{1}{2}\left(\frac{dE(c_s)}{ds}(0)\right) &= \frac{1}{2}\sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}} \partial_s \langle \partial_t, \partial_t\rangle \, dt_{|s=0} \\
&= \sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}} \left\langle \frac{D}{ds}\partial_t, \partial_t \right\rangle dt_{|s=0} \quad \text{(by equation (26) above)} \\
&= \sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}} \left\langle \frac{D}{dt}\partial_s, \partial_t \right\rangle dt_{|s=0} \quad \text{(by torsionlessness of connection)} \\
&= \sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}} \left(\partial_t \langle \partial_s, \partial_t\rangle \, dt - \left\langle \partial_s, \frac{D}{dt}\partial_t \right\rangle dt \right)_{|s=0} \quad \text{(by equation (26) above)}
\end{aligned}$$

By the fundamental theorem of calculus, the first integral above is just the sum

$$\sum_{i=0}^{N-1}\left(\lim_{t\to a_{i+1}-} \langle \partial_s, \partial_t\rangle_{c(t)} - \lim_{t\to a_i+} \langle \partial_s, \partial_t\rangle_{c(t)} \right)_{|s=0}$$

Since $\phi(s,0) \equiv x$ and $\phi(s,1) \equiv y$, we have $\partial_s = 0$ at $t = a_0$ and $t = a_N$. So $\lim_{t\to a_N-} \langle \partial_s, \partial_t\rangle_{c(t)} = 0 = \lim_{t\to a_0+} \langle \partial_s, \partial_t\rangle_{c(t)}$. Thus, using the fact that $\partial_{t|s=0} = c'(t)$, the above sum rearranges into the sum $\sum_{i=1}^{N-1} \Delta_i \langle \partial_s, c'(t)\rangle$, which is exactly the first term of our asserted formula. The second term is as defined above. The proposition follows. $\square$

Lemma 6.10.22 (Localised variations)**.** Let $c : [0,1] \to M$ be a continuous piecewise-smooth curve as in the last proposition, with $0 = a_0 < a_1 < ... < a_N = 1$, and c smooth on each $[a_i, a_{i+1}]$. For $a \in (0,1)$, let $w \in T_{c(a)}(M)$, and $\delta > 0$ such that $[a-\delta, a+\delta] \subset (0,1)$ be given. Then there exists a piecewise smooth variation $\phi : (-\epsilon, \epsilon) \times [0,1] \to M$ of c (see Definition 6.10.20) which further satisfies:

(i): $\partial_{s|s=a} := \frac{\partial \phi}{\partial s}(0,a) = w$

(ii): $\phi(s,t) \equiv c(t)$ for all $s \in (-\epsilon, \epsilon)$ and all $t \notin [a-\delta, a+\delta]$.

That is, ϕ perturbs c only in some (given) small neighbourhood of a, and in the direction w at the point a.

Proof: Clearly, if we achieve the assertion for $\delta_1 < \delta$, then we have achieved it for δ. Hence without loss of generality we may assume that δ is small enough to ensure that $c([a-\delta, a+\delta]) \subset U$, where U is a coordinate chart. Also, we don't disturb c outside $[a-\delta, a+\delta]$. This reduces the problem to the situation of a piecewise smooth curve in euclidean space. Let $\lambda : [a-\delta, a+\delta] \to \mathbb{R}$ be a smooth bump function with $0 \leq \lambda \leq 1$, $\lambda \equiv 1$ on $[a-\delta/2, a+\delta/2]$ and $\operatorname{supp} \lambda$ a compact subset of $(a-\delta, a+\delta)$. Consider the variation:

$$\begin{array}{rcl} \phi : (-\epsilon, \epsilon) \times [a-\delta, a+\delta] & \to & M \\ (s,t) & \mapsto & c(t) + s\lambda(t)w \end{array}$$

Clearly, for $t \notin [a-\delta, a+\delta]$, $\lambda(t) \equiv 0$, so $\phi(s,t) = c(t)$. Also for $t \in [a-\delta/2, a+\delta/2]$, we have $\lambda(t) \equiv 1$, so that $\phi(s,t) = c(t) + sw$, whence $\partial_{s|t=a} = \frac{\partial \phi}{\partial s}(0,a) = w$. This ϕ is the required variation, as is readily checked. $\square$

Corollary 6.10.23 (Extrema of the energy functional)**.** Let c be a piecewise smooth curve in a Riemannian manifold. Then c is an extremum of the energy functional iff c is a smooth geodesic.

Proof: Clearly, if c is a smooth geodesic, both the terms in the first variation formula vanish, and the assertion follows we have $\frac{dE(c_s)}{ds}(0) = 0$ for all piecewise smooth variations of c.

For the converse, we first claim that for all $t \in (a_i, a_{i+1})$, and all $0 \leq i \leq N-1$, we must have $\frac{Dc'(t)}{dt}(t) = 0$. If not, say for some $a_i < a < a_{i+1}$, we have $\frac{Dc'(t)}{dt}(a) = w \neq 0$. This implies that $\left\langle \partial_s(a), \frac{Dc'(t)}{dt}(a) \right\rangle > 0$. Now we choose $\delta > 0$ such that (a) $(a-\delta, a+\delta) \subset (a_i, a_{i+1})$ and (b) $\left\langle \partial_s(t), \frac{Dc'(t)}{dt}(t) \right\rangle > 0$ for $t \in (a-\delta, a+\delta)$. Let us choose a localised variation ϕ which satisfies the hypotheses of Lemma 6.10.22. We find from the first variation formula (since ∂_s vanishes in a neighbourhood of all the points a_i) that

$$\frac{1}{2}\frac{dE(c_s)}{ds}(0) = \int_{a-\delta}^{a+\delta} \left\langle \partial_s(t), \frac{Dc'(t)}{dt}(t) \right\rangle dt_{s=0} > 0$$

Hence it follows that $\frac{Dc'(t)}{dt}(t) = 0$ for all $t \neq a_i$, $0 \leq i \leq N-1$.

Now we claim that c must be smooth. For fix some a_i, and set $w := \lim_{t \to a_{i-}} c'(t) - \lim_{t \to a_{i+}} c'(t) \neq 0$, we can again construct (by the above Lemma 6.10.22) a localised variation concentrated in $(a_i - \delta, a_i + \delta)$, and $\partial_{s|s=a_i} = w$. If δ is small enough, then the first term of the first variation formula will contain a single positive term. Hence $w = 0$. $\square$

Remark 6.10.24. One could similarly want to characterise the curves for which the *length functional*:

$$L(c) = \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt$$

is extremal. We find on solving this variational problem that the extremal curves for this functional are just *reparametrisations* of geodesics, which is logical since the reparametrisation of a curve does not change its length. On the other hand, for a geodesic $c(t)$, we have:

$$\frac{d}{dt} \langle c'(t), c'(t) \rangle = 2 \left\langle \frac{Dc'(t)}{dt}, c'(t) \right\rangle = 0$$

by the geodesic equation. *Thus a geodesic has constant speed at all times* (where speed at t is the magnitude of the velocity vector $c'(t)$, i.e. $\langle c'(t), c'(t) \rangle^{\frac{1}{2}}$. It is customary to scale the parameter by a constant so that this constant speed is $= 1$, in which case it is parametrised by the *arc length* parameter:

$$s = \int_0^s dt = \int_0^s \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt$$

which is the length of the segment $c([0,s])$.

Now we will show that geodesics are locally minimising for the distance d defined in Definition 5.2.3. We will also finally prove the fact (mentioned after Definition 5.2.3) that the topology of M coincides with the metric space topology with respect to the distance d. We need some definitions and lemmas.

Definition 6.10.25. Let f be a smooth function on a Riemannian manifold M, with Riemannian metric $\langle\, ,\, \rangle$. We define a vector field $\operatorname{grad} f$ called the *gradient of* f by the formula:

$$\langle \operatorname{grad} f(x), Z\rangle = Z(f) = df(Z) \quad \text{for all } Z \in T_xM \text{ and all } x \in M$$

As an exercise, the reader can verify that in a coordinate chart U with local coordinates x_i, and local metric coefficients g_{ij}, with inverse matrix g^{kl}, we have the formula:

$$\operatorname{grad} f = \sum_{k,l} g^{kl}\left(\frac{\partial f}{\partial x_l}\right)\frac{\partial}{\partial x_k}$$

which shows that $\operatorname{grad} f$ is a smooth vector field.

Lemma 6.10.26. Let f be a smooth function on an open subset $U \subset M$, and $x \in M$ be a point.

(i): For each smooth curve $c : [\epsilon, r] \to U$ we have:

$$f(c(t)) - f(c(\epsilon)) = \int_\epsilon^t \langle \operatorname{grad} f, c'(s)\rangle\, ds$$

(ii): If $c : (0, r] \to U$ is a trajectory of $\operatorname{grad} f$, then

$$f(c(t)) = f(c(\epsilon)) + E_\epsilon^t(c)$$

where $E_\epsilon^t(c)$ denotes the energy of c between ϵ and t ($= \int_\epsilon^t \|c'(s)\|^2\, ds$).

(iii): If $\|\operatorname{grad} f\| \equiv 1$ at all points of U, then for any trajectory $c : (0, r] \to U$ of $\operatorname{grad} f$, we have:

$$f(c(t)) - f(c(\epsilon)) = t - \epsilon$$

(iv): Let f be as in (iii) above. Let $\sigma : [0, 1] \to M$ be a PS path, and $\sigma((0, 1]) \subset U$ and $\sigma(0) = x \in \overline{U}$. Assume that $f : U \to \mathbb{R}$ is continuous on $\overline{U}$. Then we have

$$L_0^t(\sigma) := \int_0^1 \|\sigma'(s)\|\, ds \geq f(\sigma(t)) - f(x)$$

In particular, if $c : (0, r] \to U$ is a trajectory of $\operatorname{grad} f$ with $\lim_{t\to 0} c(t) = x$, then for all PS paths σ satisfying $\sigma(0) = x = c(0), \sigma(r) = c(r)$, the arc lengths of σ and c satisfy the inequality:

$$L_0^r(\sigma) \geq r = L_0^r(c)$$

(Note that $L_0^r(c) = r$ in the last line above since $\|\operatorname{grad} f\| = \|c'\| \equiv 1$)

Proof: For a smooth curve c, the definition of $\operatorname{grad} f$ yields the relation $\langle \operatorname{grad} f, c'(s)\rangle = c'(s)(f) = \frac{d(f\circ c)}{ds}$, so that by the fundamental theorem of Calculus, we have:

$$\begin{aligned} f(c(t)) - f(c(\epsilon)) &= = \int_\epsilon^t \frac{d(f\circ c)}{ds} ds \\ &= \int_\epsilon^t \langle \operatorname{grad} f, c'(s)\rangle\, ds \end{aligned}$$

This implies (i).

(ii) is clear from (i), because for a trajectory c of $\operatorname{grad} f$, we have $c'(s) = \operatorname{grad} f(c(s))$.

For (iii), note that if $\|\operatorname{grad} f\| \equiv 1$ at all points of U, then for any trajectory $c : (0, r] \to U$ of $\operatorname{grad} f$, we have $E_\epsilon^t(c) = t - \epsilon$, and we are done by (ii).

For (iv), we take the limits as $\epsilon \to 0$ in all of the above. Then, if σ is any other PS path starting at x, the equality in (i) (using σ in place of c and noting $c'(s) = \text{grad } f$) will imply that

$$f(\sigma(r)) - f(x) = \int_0^r \langle c'(s), \sigma'(s)\rangle \, ds \leq \int_0^r \|\sigma'(s)\| \, ds = L_0^r(\sigma)$$

by the Cauchy-Schwarz inequality. Thus if the end points of σ, viz. $\sigma(r)$ and $\sigma(0) = x$ are the same as $c(r)$ and $c(0) = x$ respectively, where c is a trajectory of $\text{grad} f$, we get (using (iii) above) that $f(\sigma(r)) - f(x) = f(c(r)) - f(x) = r = L_0^r(c)$, and (iv) follows. □

Proposition 6.10.27. Let M be a Riemannian manifold, and let $x \in M$. Assume that the exponential map $\phi := \text{Exp}_x : V \to U$ is a diffeomorphism, where V is an open neighbourhood of $0 \in T_xM$, and U is an open neighbourhood x in M. Let $\overline{B(0,r)} \subset V$ be an closed ball or radius r. Then:

- (i): for each $v \in T_x(M)$ with $\|v\| = 1$, and each $\lambda \leq r$, the path $\text{Exp}_x(sv)$ (called *the radial geodesic in the direction* v) for $0 \leq s \leq \lambda$, is the path of minimal arc length joining x to $\text{Exp}_x(\lambda v)$.
- (ii): The distance function d introduced in Definition 5.2.3 is indeed a metric, and satisfies $d(x,y) = 0$ iff $x = y$. (We had skipped proving this fact at that time).
- (iii): For $\lambda < r$, the image $\text{Exp}_x(B(0,\lambda))$ is precisely the open ball $B_d(x,\lambda) := \{y \in M : d(x,y) < \lambda\}$, where d is the distance of Definition 5.2.3. In particular, a fundamental system of neighbourhoods of $x \in M$ is given by $\{B_d(0,\lambda)\}_{0<\lambda<r}$, and the metric topology on M defined by d coincides with the topology of M (as claimed in Definition 5.2.3).

Proof: We have already remarked that the derivative $D\text{Exp}_x(x)$ is the identity map, and so by the inverse function theorem, $\phi := \text{Exp}_x$ is a local diffeomorphism on some neighbourhood V of 0 in T_xM, and for some $r > 0$, we have $\overline{B(0,r)} \subset V$.

Consider the smooth function $f : U \setminus \{x\} \to \mathbb{R}$ defined by $f(y) = \left\|\phi^{-1}(y)\right\|$, where $\|v\|^2$, as usual, denotes $\langle v, v\rangle_x$. Note that f is not smooth, but merely continuous on U because of the corresponding fact about the norm map in any neighbourhood of 0. For v with $\|v\| = 1$, we will denote the geodesic $\phi(tv) = \text{Exp}_x(tv)$ by $c_v(t)$. We have the following:

Claim: The curve c_v is a trajectory of grad f. That is,

$$\text{grad} f(c_v(\lambda)) = \text{grad} f(\phi(\lambda v)) = c_v'(\lambda) \tag{27}$$

for all v with $\|v\| = 1$, and all $\lambda \leq r$.

Proof of Claim: For a vector $Z \in T_{\lambda v}(T_x(M)) = T_x(M)$, a path in $T_x(M)$ with initial velocity Z is the linear path $(\lambda v + sZ)$. We consider the 1-parameter variation of the geodesic $\phi(t\lambda v)$ given by $\psi(s,t) = \phi(t\lambda v + sZ))$. We have the natural vector fields along the geodesic $c_v(t\lambda) = \phi(t\lambda v)$ defined by:

$$\begin{aligned}
\partial_s(t) &:= \psi_*(0,t)\left(\frac{\partial}{\partial s}\right) = \left(\frac{d}{ds}\right)_{|s=0} (\phi(t\lambda v + sZ)) \\
\partial_t(t) &:= \psi_*(0,t)\left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}(\phi(t\lambda v)) = \lambda c_v'(t\lambda)
\end{aligned}$$

Now we have:

$$\begin{aligned}
\langle \text{grad} f, \phi_*(Z)\rangle_{c_v(\lambda)} &:= \phi_*(Z)(f) = D\phi(\lambda v)(Z)(f) = Z(f \circ \phi), \text{where } Z \in T_{\lambda v}(M)(T_x(M)) = T_x(M) \\
&= \left(\frac{d}{ds}\right)_{s=0} (\|(\lambda v + sZ)\| = \left(\frac{d}{ds}\right)_{s=0} (\langle \lambda v + sZ, \lambda v + sZ\rangle_x)^{\frac{1}{2}} \\
&= \frac{1}{2}(\|\lambda v\|)^{-1} (2 \langle \lambda v, Z\rangle_x) \\
&= \langle v, Z\rangle_{c_v(0)}
\end{aligned}$$

since $\|v\| = 1$. Note that the equality of the second line above follows from the fact that $\lambda v + sZ$ is a curve in $T_x(M)$ starting at λv and with initial velocity Z.

We now assert that the last quantity above is the same as $\langle c_v'(\lambda), \phi_*(Z)\rangle_{\phi(\lambda v)} = \langle c_v'(\lambda), \phi_*(Z)\rangle_{c_v(\lambda)}$

To see this, note that $\partial_s(1) = \phi_*(Z) = D\phi(\lambda v)(Z)$, and $\partial_s(0) = D\phi(x)(0) = Z$. Also $\partial_t(0) = \lambda v$, and $\partial_t(1) = \lambda c_v'(\lambda)$. Thus

$$\begin{aligned}
\langle c_v'(\lambda), \phi_*(Z)\rangle_{c_v(\lambda)} - \langle v, Z\rangle_{c_v(0)} &= \frac{1}{\lambda}\left(\langle \partial_t(1), \partial_s(1)\rangle_1 - \langle \partial_t(0), \partial_s(0)\rangle_0\right) \\
&= \frac{1}{\lambda}\int_0^1 \frac{d}{dt}\left(\langle \partial_t(t), \partial_s(t)\rangle_t\right) dt \\
&= \frac{1}{\lambda}\left(\int_0^1 \left(\langle \nabla_{\partial_t}\partial_t, \partial_s\rangle_t + \langle \partial_t, \nabla_{\partial_t}\partial_s\rangle_t\right) dt\right) \\
&= 0 + \frac{1}{\lambda}\int_0^1 \langle \partial_t, \nabla_{\partial_s}\partial_t\rangle_t \, dt \\
&= \frac{1}{2\lambda}\int_0^1 \left(\frac{d}{ds}\right)_{s=0} \langle \partial_t, \partial_t\rangle_t \, dt \\
&= 0
\end{aligned}$$

where we used the fact that ∂_t is the velocity field of a geodesic, so has zero acceleration, and that ∂_s and ∂_t commute (in the fourth line), and that it has constant norm (in the last line). To sum up, we have $\langle \operatorname{grad} f, \phi_*(Z)\rangle_{c_v(\lambda)} = \langle c_v'(\lambda), \phi_*(Z)\rangle_{c_v(\lambda)}$ for all $Z \in T_x(M)$, which proves that $(\operatorname{grad} f)(c(\lambda)) = c_v'(\lambda)$, and hence the Claim.

Getting back to the proof of the proposition, we now know that the trajectories of the vector field $\operatorname{grad} f$ on $U \setminus \{x\}$ are the geodesics $c_v(t)$ for $\|v\| = 1$ and $0 < t \leq r$ by the Claim (27). Thus by (ii) and (iii) of the Lemma 6.10.26, we have:

$$f(c_v(\lambda)) - f(c_v(\epsilon)) = E_\epsilon^\lambda(c_v) = \lambda - \epsilon$$

In particular, $E_0^\lambda(c_v) = \lambda = L_0^\lambda(c_v)$, because f is continuous on $\overline{U \setminus \{x\}}$, and we may take limits as $\epsilon \to 0$. This says that the energy of c_v over any time interval $[0, \lambda]$ is the same as its arc-length for $\lambda < r$. For any PS curve $\sigma : [0, r] \to U$ with $\sigma(0) = x$ and $\sigma(\lambda) = c_v(\lambda)$, (iv) of Lemma 6.10.26 will imply that $L_0^\lambda(\sigma) \geq r = L_0^\lambda(c_v)$. This implies (i) of the proposition.

Next we see the fact (ii) that we had postponed proving when introducing d, viz. that $d(x, y) = 0$ implies $x = y$. This is now quite clear, for let $x \neq y$. Since $\cap_{r>0} B(0, r) = \{0\}$, it follows that there is an $r > 0$ such that $\operatorname{Exp}_x(B(0, r))$ contains $x = \operatorname{Exp}_x(0)$ but not y. Let σ be any PS curve with $\sigma(0) = x$ and $\sigma(1) = y$. Then, for some $a \in (0, 1)$, we have $\sigma(a) = b \in \operatorname{Exp}_x(S_r)$, where $S_r = \{w \in T_x(M) : \|w\| = r\}$. Then, by (i), since radial geodesics are length minimising, we have $L_0^1(\sigma) \geq L_0^a(\sigma) \geq L_0^a(\operatorname{Exp}_x(w)) = \|w\| = r$ for some $w \in S_r$. Thus $d(x, y) \geq r > 0$. This implies (ii).

Finally we see (iii) as follows. For $\lambda < r$ we have $B(0, \lambda) \subset V$. From (i) above, since the radial geodesic $\operatorname{Exp}_x(tw)$ with initial velocity $w \in B(0, \lambda)$ is a curve of length $\|w\|$ joining x to $\operatorname{Exp}_x(w)$ for all $\|w\| < \lambda$, the distance $d(x, \operatorname{Exp}_x(w)) \leq \|w\| < \lambda$. Thus we have the inclusion $\operatorname{Exp}_x(B(0, \lambda)) \subset \{y \in M : d(y, x) < \lambda\}$. On the other hand, if y is in the right hand set, say $d(x, y) < \lambda$, we have a PS path $\sigma : [0, 1] \to M$ such that $\sigma(0) = x, \sigma(1) = y$, and with $L_0^1(\sigma) = \mu < \lambda$. If $y \notin \operatorname{Exp}_x(B(0, \lambda))$, then since $\sigma(0) = x$, the path σ would have to intersect $\operatorname{Exp}_x(S_\lambda)$, the image of the sphere of radius λ at some point $\sigma(a) = b$ for $a < 1$. But then $d(b, x) = \lambda$, so by (i) above, the arc length $L_0^a(\sigma) \geq \lambda > \mu = L_0^1(\sigma)$, a contradiction. Thus $y \in \operatorname{Exp}_x(B(0, \lambda))$.

The final assertion in (iii) now follows because (a) $\{B(0, \lambda)\}_{\lambda < r}$ is a fundamental system of neighbourhoods of 0 in $T_x(M)$ (b) Exp_x is a diffeomorphism of $B(0, r) \subset T_x(M)$ to its image in M, and (c) the fundamental system of neighbourhoods $\{B(0, \lambda)\}_{\lambda < r}$ of 0 map diffeomorphically (hence homeomorphically) to the family $\{B_d(x, \lambda)\}_{\lambda < r}$ where $B_d(x, \lambda) := \{y : d(y, x) < \lambda\}$. Thus the last collection is a fundamental set of neighbourhoods of x in M. Thus the topology on M is the metric topology from d. This proves our assertion (ii), and the proof of the proposition is complete. $\square$

We also have the following related:

Lemma 6.10.28. If U is as above, then for a smooth path $\sigma : [a,b] \to U \setminus \{x\}$, we have the inequality

$$d(\sigma(b), x) - d(\sigma(a), x) \leq L_a^b(\sigma)$$

with equality holding iff σ is a reparametrisation of a radial geodesic joining $\sigma(a)$ to $\sigma(b)$.

Proof: With the function f constructed in the last proposition, we have for any $y = \mathrm{Exp}_x(rv)$ and $\|v\| = 1$ that $d(y,x) = r = f(y)$. Thus the inequality follows from (iii), (iv) of 6.10.26.

To see when equality holds, write $\sigma(t) = \mathrm{Exp}_x(r(t)v(t))$, where $\|v(t)\| = 1$ for all t. Call the function $\mathrm{Exp}_x(rv(t))$ as $G(r,t)$. Then $\sigma(t) = G(r(t),t)$, and:

$$\dot{\sigma}(t) = \frac{\partial G}{\partial r} r'(t) + \frac{\partial G}{\partial t}$$

so that, because $\frac{\partial G}{\partial t}$ is orthogonal to $\frac{\partial G}{\partial r}$, and $\left\| \frac{\partial G}{\partial r} \right\| = 1$, we have:

$$\|\dot{\sigma}(t)\|^2 = |r'(t)|^2 + \left\| \frac{\partial G}{\partial t} \right\|^2 \geq |r'(t)|^2$$

Thus

$$\|\dot{\sigma}(t)\| \geq |r'(t)| \geq r'(t)$$

and integrating, we have the inequality above, after noting that $f(\mathrm{Exp}_x(r(t)v(t))) = r(t)$. If equality holds in this, then we must have $\frac{\partial G}{\partial t} \equiv 0$ for all t, that is $v(t)$ is a constant vector v, and also $r'(t)$ is positive, i.e. $r(t)$ is monotone. This yields $\sigma(t) = \mathrm{Exp}_x(r(t)v)$, i.e. it is a reparametrised geodesic. $\square$

Exercise 6.10.29. Let $x, y \in M$, and let σ be a PS path joining x and y which is of minimal length among all PS paths joining x and y. (That is, the arc length $L(\sigma) = d(x,y)$. Then σ is a geodesic. Such geodesics are called *minimal geodesics*.

Remark 6.10.30. Suppose x, y are two points on a Riemannian manifold M. Suppose that x and y are joined by a *unique geodesic* c. (For example if x, y lie in a geodesically convex neighbourhood). Let ϕ be an isometry which leaves x and y fixed. Since isometries preserve the metric, they preserve the Levi-Civita connection, and hence preserve geodesics. So $\phi(c)$ is another geodesic which joins x to y, and by the uniqueness of c, must be c. Thus c is left pointwise fixed by ϕ. This often enables us to determine geodesics on a Riemannian manifold without messing our hands with the non-linear system of geodesic ODE's, as we shall see below.

Example 6.10.31 (Geodesics on the Sphere)**.** Let $S^n \subset \mathbb{R}^{n+1}$ be given the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$. Then the orthogonal reflection R_H about any hyperplane H through the origin (i.e. n-dimensional vector subspace) on $\mathbb{R}^{n+1}$ is an isometry of S^n (because it is an isometry of $\mathbb{R}^{n+1}$ mapping S^n to itself!) So if x, y are two sufficiently near points (in fact, lying in the same open hemisphere), then there will be a unique geodesic c joining these two points, and consequently by remark 6.10.30 above, will be left pointwise fixed by the reflection R_H where H is the hyperplane H containing x and y. Thus the geodesic c will be contained in $S^n \cap H = S_H^{n-1}$. Now one can proceed with the hyperplane in S_H^{n-1} inductively, until one gets c to lie on circle, which is the successive intersection of S^n with a sequnce of such hyperplanes, i.e. the intersection of S^n with a 2-dimensional subspace of $\mathbb{R}^{n+1}$. Thus (the image of) c is the arc of a great circle through x and y. In fact, since any segment of a geodesic is a geodesic, the geodesics on S^n are precisely segments of great circles on S^n, parametrised by arc length. Note that for two points x, $y \in S^n$ which are not antipodal, i.e. $y \neq -x$, there are exactly two geodesics joining x to y, one of minimal length and the other of maximal length. For two antipodal points, there is a whole family of geodesics parametrised by S^{n-1} joining them. A geodesically convex neighbourhood of $x \in S^n$ is the open hemisphere centered at x (i.e. determined by the hyperplane $x^\perp$).

Example 6.10.32 (Geodesics in H^2)**.** We note that $x \mapsto -x$ and $y \mapsto y$ is an isometry of H^2 (see example 5.8.6 for notation). This leaves the y-axis (call it L) pointwise fixed, and so by the remark 6.10.30, all small geodesic segments between any two points $p, q \in L$ will consist of the segment of L between p and q. We can actually determine the parametrisation by taking $p = \sqrt{-1} = (0,1)$ as a reference point. The length of the segment $c(t) = (0, 1 + t(y-1))$ between $p = (0,1)$ and $q = (0,y)$ is precisely:

$$\begin{aligned} L(c) = s &= \int_0^1 \langle (y-1)\partial_y, (y-1)\partial_y \rangle^{\frac{1}{2}}\, dt \\ &= \int_0^1 \frac{y-1}{1+t(y-1)} dt \\ &= \log y \end{aligned}$$

Thus, reparametrising by arc length s, we have $c(s) = (0, e^s)$. This is therefore a geodesic.

To determine other geodesics, we hit this curve by various isometries from $G = SL(2,\mathbb{R})$. Under the isometry:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the point $iy = (0,y)$ goes to the point:

$$w = \frac{1}{c}\left(\frac{-d}{d^2 + c^2y^2} + \frac{icy}{d^2+c^2y^2} + a \right)$$

by an easy calculation using $ad - bc = 1$. Thus $w - \frac{a}{c} = \frac{1}{c}e^{i\theta}$ where $\theta = \tan^{-1}\frac{cy}{d}$. This shows that the image of L is a semicircular arc centred about $(\frac{a}{c}, 0)$ of radius $\frac{1}{c}$. Clearly by choosing a and c suitably, we can capture any semicircular arc centred about any desired point on the x-axis and of any desired radius, so all of these (suitably parametrised) are geodesics. Finally, since translation by a real number b is an isometry, (in fact corresponding to the element:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

of $SL(2,\mathbb{R})$), we get all vertical lines as geodesics by translating the geodesic L around. These may be treated as semicircular arcs meeting the x-axis orthogonally (that is, having their centre on the x-axis), and of infinite radius. Thus all semicircular arcs of finite or infinite radius, meeting the x-axis orthogonally (suitably parametrised) are geodesics. It turns out that these are all the geodesics.

Example 6.10.33 (Geodesic coordinates in the Disc Model D^2)**.** We recall the example 5.8.8 above. Our metric on D^2 is (after dropping the factor of 4 for convenience):

$$g(z) = \frac{dz d\bar{z}}{(1-|z|^2)^2}$$

Since the imaginary axis (suitably parametrised) constituted a geodesic in H^2, its image under the Cayley transform map ϕ, i.e. the x-axis will constitute a geodesic in D^2. Since rotations in $SU(1,1)$, i.e. the matrices:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

are isometries of D^2 which are rotations, they carry the ray

$$\{(t,0) : t \in \mathbb{R}_+\}$$

to any given ray through the origin in D^2. Thus all rays starting at the origin in D^2, suitably parametrised, are geodesics. To see what the parametrisation is, we compute the length of the radial segment $(t,0)$. Its velocity is $(1,0) = \partial_x$, and thus its speed at t is $(1-t^2)^{-1}$. Thus its length is:

$$s(t) = \int_0^t \frac{1}{1-t^2} dt = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$$

Thus $t = \tanh s$. Thus $\mathrm{Exp}(sv) = (\tanh s)v$ for any unit vector $v \in T_0(D^2)$. This gives us a "geodesic polar coordinate" chart on D^2. Since the metric at a point $te^{i\theta}$ is:

$$\frac{dx^2 + dy^2}{(1-t^2)^2} = \frac{dt^2 + t^2 d\theta^2}{(1-t^2)^2}$$

and $ds = (1-t^2)^{-1}dt$, and $\frac{t}{(1-t^2)} = \frac{1}{2}\sinh s \cosh s = \frac{1}{2}\sinh(2s)$, our metric reads

$$ds^2 + \frac{1}{4}(\sinh 2s)^2 d\theta^2$$

in these geodesic polar coordinates.

Example 6.10.34 (Geodesics in H^3)**.** By using the isometries which reflect through the 2-dimensional hyperplanes H that pass through the w-axis (see example 5.8.10), and using the remark 6.10.30, it follows that all geodesics in H^3 will lie on such hyperplanes. The induced Riemannian metric on any such hyperplane will be the 2-dimensional metric of the hyperbolic plane H^2. Thus it will be a geodesic of H^2. Thus by the previous example, the geodesics of H^3 are precisely semicircular arcs orthogonal to (centred at points of) the $w = 0$ hyperplane.

6.11. **Completeness and the Hopf-Rinow Theorem.** In this subsection, we shall prove that M is a complete metric space with respect to d if all geodesics are infinitely extendable (i.e. M is "geodesically complete").

Definition 6.11.1. We say that a Riemannian manifold M is *geodesically complete* if for each $x \in M$, the geodesic $\mathrm{Exp}_x(tX)$ is defined for all $t \in \mathbb{R}$, and all $X \in T_x(M)$.

Theorem 6.11.2 (Hopf-Rinow)**.** If M is connected and geodesically complete, then:

(i): Each pair of points in M can be joined by a minimal length geodesic.

(ii): The closure of every bounded set (with respect to the metric d) is compact. In particular, M is a complete metric space with respect to d.

Proof: Let $x, y \in M$, which is assumed to be geodesically complete. We first prove (i).

Around x, there is a closed ball $\overline{B(x,\delta)}$ such that the map $Exp_x : \overline{B(0,\delta)} \to \overline{B(x,\delta)}$ is a diffeomorphism, by the proposition 6.10.27 above. If $y \in \overline{B(x,\delta)}$, we are done by 6.10.27. If not, we proceed as follows.

Let $d(x,y) = r$ with $r > \delta$. We denote, as before, the sphere:

$$S_\delta = \mathrm{Exp}_x(\{v \in T_xM : \|v\| = \delta\})$$

Because S_δ is compact, and $y \notin \overline{B(x,\delta)}$, there is a point $z \in S_\delta$ of minimum distance from y. Write $z = \mathrm{Exp}_x(\delta v)$ where $\|v\| = 1$. Since M is geodesically complete, $\mathrm{Exp}_x(rv)$ is defined, and is a point in M. We claim that $\mathrm{Exp}_x(rv) = y$. Since the arc length of the curve $c_v(t) = \mathrm{Exp}_x(tv)$ from 0 to r ($= L_0^r(c_v)$) is $r = d(x,y)$, it will follow that this is the required minimal geodesic.

We first make the:

Claim:

$$d(c_v(t), y) = r - t \quad \text{for all } \ 0 \le t \le r$$

Then setting $t = r$ would prove that $y = c_v(r)$. First we note that this claim is true for $t = \delta$. For, every PS curve from x to y must meet S_δ, so:

$$\begin{aligned} r = d(x,y) &= \min\{L_0^1(c) : c \text{ a PS curve with } c(0) = x,\ c(1) = y\} \\ &= \min\{L_0^a(c) + L_a^1(c) : c(a) \in S_\delta\} \\ &\geq \delta + d(z,y) \end{aligned}$$

where $z = c_v(\delta) = \mathrm{Exp}_x(\delta v)$ is the point we chose above. This implies that $d(c_v(\delta), y) \leq r - \delta$. On the other hand, $r = d(x,y) \leq d(x, c_v(\delta)) + d(c_v(\delta), y) = \delta + d(c_v(\delta), y)$ by the triangle inequality. This shows that our claim is valid for $t = \delta$, and indeed for all $t \leq \delta$, by varying to smaller δ.

Let s be the supremum of all the $t \in [0, r]$ for which the claim above is valid. By continuity, it will follow that the claim is also valid for $t = s$. If we show that $s = r$, then we are done. Suppose that $s < r$. Then $c_v(s) \neq y$. Again choose a small enough ball $B(c_v(s), \epsilon)$ around $c_v(s)$ so that y lies outside it. Let $S_\epsilon(c_v(s))$ denote the image of $\{w : \|w\| = \epsilon\}$ under $\mathrm{Exp}_{c_v(s)}$. Let $p \in S_\epsilon(c_v(s))$ be the point nearest to y in that sphere. We claim that $p = \mathrm{Exp}_x((s+\epsilon)v)$. For we have by writing $p = \mathrm{Exp}_{c_v(s)}(\epsilon w)$ etc. and the identical argument to the one in the last para above that:

$$d(c_v(s), y) \geq d(p, y) + \epsilon$$

The triangle inequality implies $d(c_v(s), y) \leq d(c_v(s), p) + d(p, y) = \epsilon + d(p, y)$. Thus $d(p, y) = d(c_v(s), y) - \epsilon$.

From this, and the equality $d(c_v(s), y) = r - s$, it follows that $d(p, y) = r - s - \epsilon$. Thus $d(x, p) \geq d(x, y) - d(y, p) = r - (r - s - \epsilon) = s + \epsilon$.

Now choosing the PS curve ρ defined by $c_v(t)$ for $t \leq s$, and the curve $\mathrm{Exp}_{c_v(s)}(tw)$ for $t \leq \epsilon$, we have a curve of arc length $s + \epsilon$ joining x to p. Thus two facts result:

(i): $d(x, p) = s + \epsilon$.

(ii): ρ is a length-minimising curve from x to p.

From the exercise 6.10.29, it follows that ρ is a smooth geodesic. Since it coincides with $\mathrm{Exp}_x(tv)$ for $0 \leq t \leq s$, it coincides all the way upto $s + \epsilon$, that is $p = \mathrm{Exp}_x((s+\epsilon)v)$. That is $p = c_v(s+\epsilon)$. We had already seen above that $d(p, y) = r - s - \epsilon$, so this means $d(c_v(s+\epsilon), y) = r - s - \epsilon$, contradicting the maximality of s. Hence $s = r$, and (i) is proved.

To see (ii), it is enough to show that the closed ball $\overline{B(x,r)}$ is compact (for some fixed $x \in M$) for all $r > 0$. From (i), it follows that the ball $\overline{B(0,r)}$ is precisely the image $\mathrm{Exp}_x(\overline{B(0,r)})$ for all $r > 0$. But the last set is compact from the continuity of the map Exp_x, and by Heine-Borel for $T_x(M) \simeq \mathbb{R}^n$. Since every Cauchy sequence is a bounded sequence, and since all compact subsets of a metric space are sequentially compact, it follows that every Cauchy sequence is convergent, and M is complete as a metric space. This proves (ii), and the proposition. $\square$

Proposition 6.11.3. Every compact connected Riemannian manifold M is geodesically complete.

Proof: For each point $x \in M$, we use the remark 6.10.17 to get a neighbourhood $U(x)$ of x which is geodesically convex. Since M is compact, there is a Lebesgue number δ for this covering $\{U(x)\}_{x \in M}$. That is, every closed ball of radius δ in the manifold is contained in some $U(x)$. Now if $y \in M$ is any point, it follows that every point $z \in \overline{B(y,\delta)}$ can be joined to y by a geodesic of minimal length. This means that $\mathrm{Exp}_y : \overline{B(0,\delta)} \to \overline{B(y,\delta)}$ is a diffeomorphism for every $y \in M$.

Let $x \in M$, and $v \in T_x(M)$ with $\|v\| = 1$. We will show that $\mathrm{Exp}_x(tv) \in M$ for all $0 \leq t \leq m\delta$, for every positive integer m. This will clearly imply geodesic completeness.

Let $x_1 := \mathrm{Exp}_x(\delta v)$. Since $d(x_1, x) = \delta$, we have $x \in \overline{B(x_1, \delta)}$. Thus $x = \mathrm{Exp}_{x_1}(-\delta v_1)$ for some $v_1 \in T_{x_1}(M)$ with $\|v_1\| = 1$. We claim that $\mathrm{Exp}_x(tv) = \mathrm{Exp}_{x_1}((t-\delta)v_1)$ for all $0 \leq t \leq 2\delta$. The segments $\mathrm{Exp}_x(tv)$ for $0 \leq t \leq \delta$ and $\mathrm{Exp}_{x_1}((t-\delta)v_1)$ for $0 \leq t \leq \delta$ are both minimal length geodesics joining x and x_1, and hence coincide for $t \in [0, \delta]$. Thus, by uniqueness, it follows that $\mathrm{Exp}_x(tv) = Exp_{x_1}((t-\delta)v_1)$ for all $0 \leq t \leq 2\delta$.

Now repeat the argument by replacing x above with x_1, and x_1 by $\mathrm{Exp}_{x_1}(\delta v_1) = \mathrm{Exp}_x(2\delta v)$, and we are through by induction. $\square$

Corollary 6.11.4. If M is a compact connected Riemannian manifold, then for each point $x \in M$, the map $\mathrm{Exp}_x : T_x(M) \to M$ is surjective.

Proof: By Proposition 6.11.3 above, M is geodesically complete. By Theorem 6.11.2, every point y can be joined to x by a minimal geodesic, which implies that $y = \mathrm{Exp}_x(rv)$ for some unit vector $v \in T_x(M)$ and $r = d(x, y)$. $\square$

Corollary 6.11.5. Let G be a Lie group with a bi-invariant metric (for example if G is compact). Then the exponential map $\exp : \mathfrak{g} \to G$ (defining 1-parameter subgroups) is surjective.

Proof: For G, the maps Exp and exp coincide, as pointed out in the Proposition 6.10.18, the geodesics $\mathrm{Exp}_e(tv)$ through $1 \in G$ coincide with the one parameter subgroups $\exp(tv)$ for $v \in \mathfrak{g}$. Since 1-parameter subroups are defined for all t, it follows that G is geodesically complete. We are again through by the Hopf-Rinow Theorem 6.11.2. $\square$

7. Curvature

7.1. **Curvature form on a principal bundle.** Let $\pi : P \to M$ be a principal G-bundle, with a connection η, and associated $\mathfrak{g}$-valued connection 1-form ω. We also have the trivialisations $\Phi_i : \pi^{-1}(U_i) \to U_i \times G$, and the naturally arising smooth sections $\sigma_i : U_i \to \pi^{-1}(U_i)$ which are defined by $\Phi_i(\sigma_i(x)) = (x, 1)$ for $x \in U_i$. They are related by the transition formula $g_{ij}(x)\sigma_j(x) = \sigma_i(x)$ for $x \in U_i \cap U_j$. (see the subsections 6.3 and 6.4). The local connection 1-forms $\omega_i \in \Omega^1(U_i) \otimes \mathfrak{g}$ were defined as $\sigma_i^*\omega$.

Notation 7.1.1. Let α, β be smooth $\mathfrak{g}$-valued 1-forms on an open set $U \subset P$ for some smooth manifold P. Then we may write $\alpha = \sum_i \alpha_i X_i$ and $\beta = \sum_j \beta_j X_j$, where $\{X_i\}$ is a basis for $\mathfrak{g}$, and α_i, β_j are 1-forms on U. We define a $\mathfrak{g}$-valued 2- form on U by:

$$[\alpha, \beta] = \sum_{i,j} (\alpha_i \wedge \beta_j)[X_i, X_j]$$

Note that by this definition $[\alpha, \alpha]$ is not zero, but equal to $\sum_{i<j} \alpha_i \wedge \alpha_j [X_i, X_j]$. In fact, the reader can easily check that:

$$[\alpha, \beta] = [\beta, \alpha] \tag{28}$$

We also note that $[\alpha, \beta]$ is nothing but the image of $\alpha \otimes \beta$ under the composite map:

$$(\Omega^1(U) \otimes \mathfrak{g}) \otimes (\Omega^1(U) \otimes \mathfrak{g}) \simeq \Omega^1(U) \otimes \Omega^1(U) \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\wedge \otimes [\,,\,]} \Omega^2(U) \otimes \mathfrak{g}$$

Hence it does not depend on the basis $\{X_i\}$ chosen for $\mathfrak{g}$.

For example, if $G = GL(n, \mathbb{R})$ (or for that matter any group of matrices), and we use the basis of elementary matrices $\{E_{ij}\}_{1 \le i,j \le n}$ for $\mathfrak{g} = \mathfrak{gl}(n)$, then writing $\alpha = \sum_{i,j} \alpha_{ij} E_{ij}$, $\beta = \sum_{k,l} \beta_{kl} E_{kl}$, and noting that $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$, we see (using the repeated index Einstein summation convention) that

$$\begin{aligned}
[\alpha, \beta] &= (\alpha_{ij} \wedge \beta_{kl}(\delta_{jk} E_{il} - \delta_{li} E_{kj})) \\
&= (\alpha_{ij} \wedge \beta_{jl})\, E_{il} - (-\beta_{ki} \wedge \alpha_{ij})\, E_{kj} \\
&= \gamma_{il} E_{il}
\end{aligned}$$

where

$$\gamma_{il} := \sum_j (\alpha_{ij} \wedge \beta_{jl} + \beta_{ij} \wedge \alpha_{jl})$$

If $\beta = \alpha$, γ_{il} is the il-th "matrix entry" of the matrix product $2\alpha \wedge \alpha$, which is matrix multiplication but with usual product of scalar entries replaced by the wedge products. Some authors denote our $[\alpha, \beta]$ as $\alpha \wedge \beta$.

More generally, one defines $[\alpha, \beta]$ as a $\mathfrak{g}$-valued $(p+q)$-form, for α, β $\mathfrak{g}$-valued p and q-forms respectively on U, as follows:

$$\begin{aligned}
[\alpha, \beta] &= \sum_{i,j} (\alpha_i \wedge \beta_j)[X_i, X_j] \\
&= (-1)^{pq+1} \sum_{j,i} (\beta_j \wedge \alpha_i)[X_j, X_i] \\
&= (-1)^{pq+1} [\beta, \alpha]
\end{aligned} \tag{29}$$

Also, if $\alpha = \sum_i \alpha_i X_i \in \Omega^p(U) \otimes \mathfrak{g}$ is a $\mathfrak{g}$ valued p-form, then $d\alpha$ will be the $\mathfrak{g}$-valued $(p+1)$-form $\sum_i d\alpha_i X_i$. That is, it is the image of α under the map

$$d \otimes \mathrm{Id} : \Omega^p(U) \otimes \mathfrak{g} \to \Omega^{p+1}(U) \otimes \mathfrak{g}$$

applied to α.

Exercise 7.1.2. Show that for a $\mathfrak{g}$-valued 1-form ω,

$$[\omega, [\omega, \omega]] = 0$$

(Hint: use the Jacobi identity of $\mathfrak{g}$.)

Lemma 7.1.3. With the above definitions of d and $[\,,\,]$, we have:

$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^p[\alpha, d\beta]$$

for $\alpha \in \Omega^p(U)\otimes\mathfrak{g},\ \beta \in \Omega^q(U)\otimes\mathfrak{g}$.

Proof: By the $\mathbb{R}$ bilinearity of both sides, we may take $\alpha = \omega X$ and $\beta = \tau Y$ where ω, τ are $\mathbb{R}$-valued 1-forms and $X, Y \in \mathfrak{g}$. Then $[\alpha,\beta] = \omega\wedge\tau[X,Y]$, so that $d[\alpha,\beta] = d\omega\wedge\tau[X,Y] + (-1)^p\omega\wedge d\tau[X,Y] = [d\alpha,\beta]+(-1)^p[\alpha,d\beta]$, and we are done. $\square$

Lemma 7.1.4. If ω is a 1-form on an open subset U of a smooth manifold M, and X, Y smooth vector fields on U, then

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Proof: We first leave it as an exercise for the reader to verify that the right hand side satisfies linearity with respect to multiplication of functions, viz:

$$fX(\omega(gY)) - gY(\omega(fX)) - \omega([fX,gY]) = fg\left(X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\right)$$

for smooth functions f, g. Now we need verify the formula of the lemma on any any coordinate chart U, and by the observation above, take $X = \partial_i = \frac{\partial}{\partial x_i}$ and $Y = \partial_j = \frac{\partial}{\partial x_j}$. Then $[X,Y]\equiv 0$. Say $i<j$, without loss of generality, because both sides are skew-symmetric in X and Y.

Write $\omega = \sum_j \omega_j dx_j$, so that $X(\omega(Y)) = \partial_i\omega_j$ and $Y(\omega(X)) = \partial_j\omega_i$, so that the right side of our formula is $\partial_i\omega_j - \partial_j\omega_i$. On the other hand, by the definition of exterior derivative:

$$d\omega = \sum_j d\omega_j\wedge dx_j = \sum_{i<j}(\partial_i\omega_j - \partial_j\omega_i)dx_i\wedge dx_j$$

so that

$$d\omega(X,Y) = d\omega(\partial_i,\partial_j) = \partial_i\omega_j - \partial_j\omega_i$$

thus proving the lemma. The same formula is checked to be true for $\mathfrak{g}$-valued 1-forms. $\square$

Corollary 7.1.5. Let ω^R be the right-invariant Maurer- Cartan form on a Lie group G. That is, $\Omega^R(\hat{X}) = X$ for $X\in\mathfrak{g}$ and $\hat{X}$ the right- invariant vector field on G generated by X. (see 6.3.1). Then

$$d\omega^R = \frac{1}{2}[\omega^R,\omega^R]$$

Proof: Let $\{X_i\}$ be a basis for $\mathfrak{g}$, so that $\omega^R = \sum_j \hat{e}_j\otimes X_j$ for $\hat{e}_j$ the right-invariant $\mathbb{R}$ valued 1-forms satisfying $\hat{e}_i(\hat{X}_j) = \delta_{ij}$. Let $i<j$. Then, by the lemma 7.1.4 above:

$$\begin{aligned} d\omega^R(\hat{X}_i,\hat{X}_j) &= \hat{X}_i(X_j) - \hat{X}_j(X_i) - \omega^R([\hat{X}_i,\hat{X}_j]) \\ &= 0 - 0 - \omega^R([\hat{X}_i,\hat{X}_j]) \end{aligned}$$

Let $\tau : G\to G$ be the inversion map, i.e. $\tau(g) = g^{-1}$. Then $\tau\circ L_g(h) = h^{-1}g^{-1} = R_{g^{-1}}\circ\tau(h)$. Thus we have $\tau_*\circ L_{g*} = R_{g^{-1}*}\circ\tau_*$. Applying both sides to $X\in\mathfrak{g}$, and noting that $\tau_*(e) = -\mathrm{Id}$, we get $\tau_*\widetilde{X}(g) = -\hat{X}(g^{-1}) = -\hat{X}(\tau(g))$. That is, $\tau_*(\widetilde{X}) = -\hat{X}\circ\tau$. From this it follows that $[\hat{X}_i,\hat{X}_j] = [-\tau_*(\widetilde{X}_i), -\tau_*(\widetilde{X}_j)] = \tau_*[\widetilde{X}_i,\widetilde{X}_j] = \tau_*[\widetilde{X_i,X_j}] = -[\widehat{X_i,X_j}]$.

Thus we have:

$$d\omega^R(\hat{X}_i,\hat{X}_j) = \omega^R(\widehat{[X_i,X_j]}) = [X_i,X_j]$$

On the other hand

$$[\omega^R,\omega^R] = \sum_{k,l}(\hat{e}_k\wedge\hat{e}_l)[X_k,X_l]$$

from which it follows that $\frac{1}{2}[\omega^R, \omega^R](\hat{X}_i, \hat{X}_j) = [X_i, X_j]$. Since the vector fields $\hat{X}_i$ constitute a $C^\infty(G)$ basis for all the vector fields on G, we are done.

We note that an analogous argument with $\widetilde{X}_i$ will imply a similar formula for ω^L with a changed sign , viz. $d\omega^L + \frac{1}{2}[\omega^L, \omega^L] = 0$. □

Lemma 7.1.6. Let $\alpha \in \Omega^0(G) \otimes \mathfrak{g}$, where G is a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathrm{Ad}\, g\, \alpha$ denote the smooth $\mathfrak{g}$-valued function whose value at $g \in G$ is $\mathrm{Ad}\, g(\alpha(g))$. Then:

$$d(\mathrm{Ad}\, g\, \alpha) = \mathrm{Ad}\, g(d\alpha) + [\omega^R, \mathrm{Ad}\, g(\alpha)]$$

where ω^R is the right-invariant Maurer-Cartan form of G.

Proof: By the $\mathbb{R}$ linearity of both sides, we may assume that $\alpha = fX$, where $f \in \Omega^0(G)$ is asmooth function, and $X \in \mathfrak{g}$. Then $\mathrm{Ad}\, g(\alpha) = f \mathrm{Ad}\, g(X)$. Let $Z \in \mathfrak{g}$, and $\hat{Z}$ the corresponding right-invariant vector field on Z. Then $(\exp(tZ).g)$ will be a smooth curve passing through g, and with velocity $R_{g*}(Z) = \hat{Z}(g)$ at $t = 0$. Then

$$\begin{aligned}
d(\mathrm{Ad}\, g\alpha)(\hat{Z}(g)) &= \frac{d}{dt}_{|t=0} (f(\exp(tZ)g)(\mathrm{Ad}\exp(tZ)g)X) \\
&= \hat{Z}(g)(f)(\mathrm{Ad}\, g)X + f(g)\frac{d}{dt}_{|t=0} \mathrm{Ad}\exp(tZ)(\mathrm{Ad}\, g)X \\
&= df(\hat{Z}(g))(\mathrm{Ad}\, g)X + [Z, f(g)(\mathrm{Ad}\, g)X] \\
&= \left(\mathrm{Ad}\, g(d\alpha) + [\omega^R, \mathrm{Ad}\, g\, \alpha]\right)(\hat{Z}(g))
\end{aligned}$$

which proves our formula. □

Corollary 7.1.7. Let ω be a $\mathfrak{g}$-valued 1-form on a manifold P, and let $g : P \to G$ be a smooth function. If X_i are as above, and $\omega = \sum_i \omega_i X_i$, where $\omega_i \in \Omega^1(P)$, we denote by $\mathrm{Ad}\, g\, \omega$ the $\mathfrak{g}$-valued 1- form whose value at $x \in P$ is $\sum_i \omega_i(x) \mathrm{Ad}\, g(x)\, X_i$. Then:

$$d(\mathrm{Ad}\, g\, \omega) = \mathrm{Ad}\, g\, d\omega + [g^*\omega^R, \mathrm{Ad}\, g\, \omega]$$

Proof: Again, as before, we may assume $\omega = \omega_i X_i$, where ω_i is a $\mathbb{R}$-valued 1-form on P. Then, if we let α denote the $\mathfrak{g}$-valued constant function X_i on G, we have $g^*(\mathrm{Ad}\, g\, \alpha) = Ad(g(x))X_i$, so that $\mathrm{Ad}\, g\, \omega = \omega_i g^*(\mathrm{Ad}\, g\, \alpha)$. Then $d\alpha = 0$, and $d(\mathrm{Ad}\, g\, \alpha) = [\omega^R, \mathrm{Ad}\, g\, \alpha]$. Thus:

$$\begin{aligned}
d(\mathrm{Ad}\, g\, \omega) &= d\omega_i g^*(\mathrm{Ad}\, g\, \alpha) - \omega_i \wedge d(g^*(\mathrm{Ad}\, g\, \alpha) \\
&= \mathrm{Ad}\, g\, d\omega - \omega_i \wedge g^*(d(\mathrm{Ad}\, g\, \alpha)) \\
&= \mathrm{Ad}\, g\, d\omega - \omega_i \wedge g^*[\omega^R, \mathrm{Ad}\, g\, \alpha] \\
&= \mathrm{Ad}\, g\, d\omega - \omega_i \wedge \sum_j g^*\hat{e}_j [X_j, \mathrm{Ad}\, gX_i] \\
&= \mathrm{Ad}\, g\, d\omega + \sum_j (\omega_i \wedge g^*\hat{e}_j)[\mathrm{Ad}\, gX_i, X_j] \\
&= \mathrm{Ad}\, g\, d\omega + [\mathrm{Ad}\, g\, \omega, g^*\omega^R]
\end{aligned}$$

This proves the lemma. □

Let η be a connection on the principal G- bundle with connection form ω. Let $\phi : P \to P$ a gauge transformation, and let ω^ϕ denote the connection form of the gauge-transformed connection η^ϕ. We recall from 6.4.12 and 6.4.15 that:

$$\omega^\phi = (\mathrm{Ad}\, g)\omega + g^*\omega^R$$

Proposition 7.1.8 (Curvature form)**.** In the setting above, we have the identity:

$$-d\omega^\phi + \frac{1}{2}[\omega^\phi, \omega^\phi] = \operatorname{Ad} g \left(-d\omega + \frac{1}{2}[\omega, \omega] \right) \tag{30}$$

The $\mathfrak{g}$-valued 2-form $\Omega := -d\omega + \frac{1}{2}[\omega, \omega]$ is called the *curvature* of the connection η.

Proof: We have the formula above:

$$\omega^\phi = (\operatorname{Ad} g)\omega + g^*\omega^R$$

For brevity denote $\omega_1 = (\operatorname{Ad} g)\omega$. Then by the lemmas 7.1.7 and 7.1.5 we have:

$$\begin{aligned} d\omega^\phi &= d\omega_1 + d(g^*\omega^R) \\ &= \operatorname{Ad} g \, d\omega + [g^*\omega^R, \omega^1] + g^* d\omega^R \\ &= \operatorname{Ad} g \, d\omega + [g^*\omega^R, \omega_1] + \frac{1}{2}[g^*\omega^R, g^*\omega^R] \end{aligned}$$

On the other hand we have:

$$\begin{aligned} \frac{1}{2}[\omega^\phi, \omega^\phi] &= \frac{1}{2}[\omega^1, \omega^1] + \frac{1}{2}[\omega^1, g^*\omega^R] + \frac{1}{2}[g^*\omega^R, \omega^1] + \frac{1}{2}[g^*\omega^R, g^*\omega^R] \\ &= \frac{1}{2}[\operatorname{Ad} g \, \omega, \operatorname{Ad} g \, \omega] + [g^*\omega^R, \omega_1] + \frac{1}{2}[g^*\omega^R, g^*\omega^R] \\ &= \frac{1}{2}\operatorname{Ad} g[\omega, \omega] + [g^*\omega^R, \omega_1] + \frac{1}{2}[g^*\omega^R, g^*\omega^R] \end{aligned}$$

Thus we get:

$$-d\omega^\phi + \frac{1}{2}[\omega^\phi, \omega^\phi] = \operatorname{Ad} g \left(-d\omega + \frac{1}{2}[\omega, \omega] \right)$$

This proves the proposition. □

Corollary 7.1.9. Let $\pi : P \to M$ be as above, and let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be the local sections defined above on the trivialising coordinate opens U_i. Denote the pullbacks $\sigma_i^*(\Omega) \in \Omega^2(U_i) \otimes \mathfrak{g}$ as Ω_i. Then:

$$\Omega_i = \operatorname{Ad} g_{ij} \Omega_j$$

That is, Ω_i patch up to give a global smooth section of $\wedge^2(T^*M) \otimes P \times_{\operatorname{Ad}} \mathfrak{g}$.

Proof: As in the proof of proposition 6.4.18, view the transition functions g_{ij} as defining a gauge transformation of the principal bundle $P_{|U_i \cap U_j}$, and use the proposition 7.1.8 above. □

Remark 7.1.10 (Tensoriality of Ω)**.** We recall here that if $P \to M$ is a principal G-bundle, with transition functions $g_{ij} : U_i \cap U_j \to G$ coming from the local trivilisations, then for any associated vector bundle, the transition functions would be $\rho(g_{ij}$. (See the example 6.2.2). In particular the vector bundle $E = P \times_{\operatorname{Ad}} \mathfrak{g}$ would have transition functions $\operatorname{Ad}(g_{ij}) \in GL(\mathfrak{g})$. To give a global section of this bundle, we would have to give a collection of smooth functions $s_i : U_i \to \mathfrak{g}$ which satisfy $s_i = \operatorname{Ad}(g_{ij})s_j$ on $U_i \cap U_j$.

If we want to give a global section s of $(T^*(M)) \otimes E$, then for every smooth vector field X of M, we would have $s(X)$ as a smooth section of E (by pointwise evaluation), so that we would need a collection of maps $s_i(X) : U_i \to g$ which obeyed (i) $s_i(X)$ is linear in X, and its value at $x \in U_i$ depends only on $X(x)$, and (ii) $s_i(X) = \operatorname{Ad}(g_{ij})s_j(X)$. Similarly for sections of $\bigwedge^p(T^*M) \otimes E$.

The transformation formula for ω_i in 6.4.18 precludes them from patching up to give a global section of $T^*M \otimes E$, or for that matter, any bundle on M. The term $g_{ij}^*\omega^R$ show that *derivatives* of transition functions occur, and the values of ω_i, ω_j at $x \in U_i \cap U_j$ are *not related* by an automorphism depending only on the point x.

By contrast, the forms Ω_i defined above, by virtue of 7.1.9, patch up to give a global smooth section of $\bigwedge^2(T^*M) \otimes E$.

Proposition 7.1.11 (Bianchi Identity)**.** The curvature form Ω obeys the formula:

$$d\Omega - [\omega, \Omega] = 0$$

Proof: Since $\Omega = -d\omega + \frac{1}{2}[\omega, \omega]$, we get on taking d, and using the lemma 7.1.4 above:

$$\begin{aligned} d\Omega &= \frac{1}{2}[d\omega, \omega] - \frac{1}{2}[\omega, d\omega] \\ &= [\omega, \Omega - \frac{1}{2}[\omega, \omega]] - \frac{1}{2}[\Omega - \frac{1}{2}[\omega, \omega], \omega] \\ &= [\omega, \Omega] \end{aligned}$$

where we have used the exercise 7.1.2 to get $[\omega, [\omega, \omega]] = 0$, and the skew commutativity $[\Omega, \omega] = (-1)^{2.1+1}[\omega, \Omega] = -[\omega, \Omega]$ from 7.1.1. □

7.2. **Curvature and covariant differentiation.** Let $p : E = P \times_\rho V \to M$ be an associated smooth vector bundle to the principal bundle $\pi : P \to M$ with a connection η. In 6.6.2, 6.6.5 we introduced the covariant differentiation operator:

$$\nabla : \Gamma(U, E) \to \Omega^1(U) \otimes \Gamma(U, E)$$

coming from the connection. If U is a coordinate chart, and $\sigma : U \to \pi^{-1}(U)$ is a smooth local section, we had the natural associated sections $w_i(z) = [\sigma(z), e_i]$ of $E_{|U} := p^{-1}(U)$ (see the discussion immediately preceding proposition 6.6.3).

If we denote the frame $(w_1, .., w_k)$ of $E_{|U}$ by w, then the proposition 6.6.3 related the local connection form $\sigma^*\omega$ by the formula:

$$\nabla w_i = -\sum_j (\dot{\rho}(\sigma^*\omega))_{ji} w_j$$

Notation 7.2.1. Since the $\mathfrak{g}$-valued curvature form on P is defined as:

$$\Omega = -d\omega + \frac{1}{2}[\omega, \omega]$$

we will have $\sigma^*\Omega = -d\sigma^*\omega + \frac{1}{2}[\sigma^*\omega, \sigma^*\omega]$. Thus, applying $\dot{\rho}$, we have:

$$\dot{\rho}(\sigma^*\Omega) = d(-\dot{\rho}\sigma^*\omega) + \frac{1}{2}[-\dot{\rho}\sigma^*\omega, -\dot{\rho}\sigma^*\omega]$$

For the sake of convenience, call this $\mathfrak{gl}(k, \mathbb{R})$ valued 1-form $-\dot{\rho}\sigma^*\omega$ on U as ω, and the $\mathfrak{gl}(k, \mathbb{R})$ 2-form $\dot{\rho}\sigma^*\Omega$ as Ω. We also noted at the end of notation 29, that for a matrix valued 1-form ω, $[\omega, \omega] = 2\omega \wedge \omega$, where the right side is the matrix wedge product.

Then we have the two neater looking equations:

$$\nabla_X w_i = \sum_j \omega(X)_{ji} w_j \tag{31}$$

$$\Omega = d\omega + \omega \wedge \omega \tag{32}$$

The second equation above means that $\Omega(X, Y) = d\omega(X, Y) + \omega(X)\omega(Y) - \omega(Y)\omega(X)$, where the products on the right hand side denote matrix products.

Proposition 7.2.2. With the notations of 7.2.1 above, we have:

$$\sum_j \Omega(X, Y)_{ji} w_j = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) w_i$$

for smooth vector fields X, Y on U.

Proof: By the formula (32) above, and using the repeated summation convention,

$$\nabla_Y w_i = (\omega(Y))_{ji} w_j$$

so that, using Leibniz formula for covariant differentiation,

$$\begin{aligned}
\nabla_X \nabla_Y w_i &= X(\omega(Y)_{ji}) w_j + \omega(Y)_{ji} \nabla_X w_j \\
&= X(\omega(Y))_{ji} w_j + \omega(Y)_{ji} \omega(X)_{lj} w_l \\
&= [(X(\omega(Y)) + \omega(X)\omega(Y)]_{ji}\, w_j
\end{aligned}$$

Thus we will have:

$$\begin{aligned}
(\nabla_X \nabla_Y w - \nabla_Y \nabla_X) w_i &= [(X(\omega(Y)) - Y(\omega(X)) + [(\omega(X))(\omega(Y)) - \omega(Y)\omega(X)]]_{ji}\, w_j \\
&= [(d\omega(X,Y) + \omega([X,Y]) + [\omega \wedge \omega](X,Y)]_{ji}\, w_j \\
&= \Omega(X,Y))_{ji} w_j + \nabla_{[X,Y]} w_i
\end{aligned}$$

which proves the proposition. □

We recall the remark made at the end of definition 6.6.1 that parallel transport P_t^c depends on the curve c. We will presently see how curvature measures this dependency.

Proposition 7.2.3. Let $p : E \to M$ be a smooth vector bundle of rank k, with its parallel transport P coming from a connection (on the underlying principal bundle). Let U be a coordinate patch, and let $X = \partial_{x_1}$ and $Y = \partial_{X_2}$ be two coordinate vector fields, with $(x_1, x_2, .., x_n))$ being some coordinates on U. At any point $x = (a_1, .., a_n)$, we denote the trajectory $(a_1 + t, a_2, .., a_n)$ of X by $a(t)$, and the corresponding trajectory $(a_1, a_2 + s, .., a_n)$ of Y, by $b(s)$. Then for any smooth section σ of E on U:

$$\Omega(X,Y)\sigma = \lim_{s,t \to 0} \frac{P^a_{-t} P^b_{-s}\sigma - P^b_{-s} P^a_{-t}\sigma}{st}$$

Proof: The statement means that we are joining the point $x = (a_1, .., a_n)$ to $x + te_1 + se_2 = (a_1 + t, a_2 + s, .., a_n)$ by two paths: one being the path a from x to $x + te_1$ followed by b from $x + te_1$ to $x + te_1 + se_2$, and the other path is the one with the roles of a and b reversed.

We recall the definition of covariant differentiation, which implies that:

$$\begin{aligned}
P^b_{-s}\sigma(x + te_1 + se_2) &= \sigma(x + te_1) + s\nabla_Y \sigma(x + te_1) + o_2(x + te_1; s)\sigma \\
&= \sigma(x + te_1) + s\nabla_Y \sigma(x + te_1) + o_2(x; s)\sigma + o(st)
\end{aligned}$$

where the symbol $o(\epsilon)$ as usual, denotes a quantity which goes to zero faster than ϵ (i.e. $\lim_{\epsilon \to 0} \epsilon^{-1} o(\epsilon) \to 0$).

Now one iterates this and writes a similar "Taylor formula" for $P^a_{-t} P^b_{-s}\sigma$. Then take the difference with $P^b_{-s} P^a_{-t}$, and take limits on dividing by st to obtain $(\nabla_X \nabla_Y - \nabla_Y \nabla_X)\sigma = \Omega(X,Y)\sigma$, since $[X,Y] = 0$. We leave the details as an exercise. □

Definition 7.2.4. A connection η for which $\Omega \equiv 0$ is called a *flat connection.*

7.3. Some curvature computations.

Example 7.3.1 (Curvature of Cartan connection on a Lie group)**.** We recall from the example 6.4.25 that the connection form for the canonical Cartan connection is just $\omega = \pi_2^* \omega^R$, where $\pi_2 : P = G \times G \to G$ is the second projection, and ω^R is the right-invariant Maurer-Cartan 1-form of G. Thus $\Omega = -d\omega + \frac{1}{2}[\omega, \omega] = -\pi_2^* \left(d\omega^R - \frac{1}{2}[\omega^R, \omega^R] \right) = 0$ by the lemma 7.1.5 above. Hence the canonical Cartan connection has no curvature, i.e. is a *flat connection.*

The above showed that the 2-form Ω on P is $\equiv 0$. Alternatively, one could use the formula $\nabla_{\widetilde{X}} \widetilde{Y} = [\widetilde{Y}, \widetilde{X}]$, and the proposition 7.2.2 above, together with the Jacobi identity to show that the curvature form $\sigma^* \Omega$ is $\equiv 0$ on $M = G$.

Definition 7.3.2 (Riemannian curvature)**.** Let η be Levi-Civita connection on a Riemannian manifold. The curvature form Ω on M, taking values in $\mathfrak{gl}(n, \mathbb{R})$ satisfies:

$$\sum_j \Omega(X, Y)_{ji} w_j = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) w_i$$

as note in the proposition 7.2.2, where w_i is the natural frame in a coordinate chart coming from σ, which in this case is the coordinate frame ∂_i. Thus, if $Z = \sum_i Z_i \partial_i$ is a smooth vector field, we get the equation:

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z = \sum_j (\sum_i \Omega_{ji}(X, Y) Z_i) w_j$$

We denote, by standard convention, the vector field on the right hand side as $R(X, Y)Z$. That is $R(X, Y)\partial_i = \sum_j \Omega_{ji}(X, Y) w_j$. By the remark 7.1.10, it follows that R defines a global smooth section of

$$\bigwedge^2 (T^* M) \otimes F(TM)) \otimes_{\mathrm{Ad}} \mathfrak{gl}(n, \mathbb{R}) = \wedge^2 (T^* M) \otimes \hom(TM, TM)$$

(Convince yourself of the bundle identity $F(TM) \otimes_{\mathrm{Ad}} \mathfrak{gl}(n, \mathbb{R}) = \hom(TM, TM)$) R is called the *Riemann curvature tensor.*

In other words, the Riemann curvature tensor is a smooth section $R(\ ,\ )$ of $\bigwedge^2 T^* M \otimes \hom(TM, TM)$, such that for tangent vectors $X, Y \in T_x(M)$ and $x \in U_i$, the matrix of the linear transformation $R(X, Y) : T_x M \to T_x M$ with respect to the frame w_j of $TM_{|U_i}$ is $\Omega_i(X, Y)$.

Remark 7.3.3. Many authors (e.g. Milnor, Hicks) directly define $R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$, for vector fields X, Y, Z. Then, to show that it is actually a tensor, one needs to verify that $R(fX, gY)(hZ) = fgh(R(X, Y)Z)$ for smooth functions X, Y, Z. This is is a routine verification using the Leibniz formula for ∇, definition of commutator, which is left as the exercise below. We don't need this verification by 7.1.10 and the Proposition 7.2.2 above.

Exercise 7.3.4. Let f, g, h be smooth functions, and X, Y, Z vector fields on a smooth Riemannian manifold M. Show that:

$$R(fX, gY)hZ = fghR(X, Y)Z$$

One can directly see the significance of the curvature tensor using the above formula $R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$ as the definition of the curvature tensor. Firstly, if we first define parallel translation along a smooth curve with respect to a connection on a principal bundle $P \to M$ (as we have done in the foregoing sections), then the covariant derivative along this curve for sections of smooth associated bundles $E \to M$ is defined as the time derivative of this parallel translation (see Definition 6.6.2). However, if one defines covariant derivative directly as an $\mathbb{R}$-linear operator on sections obeying Leibniz's formula (as we have done in Definition

6.8.2), and then parallel transport (as we did in Proposition 6.10.7), we can again recover covariant derivative as a derivative of parallel transport. This is the content of:

Lemma 7.3.5. Let ∇ denote the Levi-Civita connection on a Riemannian manifold, and let $c : [0,1] \to M$ be a smooth curve in M. Let the covariant derivative $\frac{D}{dt}$ along c be defined as in Lemma 6.10.3. Let parallel transport along c be defined as in Proposition 6.10.7. Then, for a vector field X along c, we have:

$$\frac{DX}{dt}(0) = \lim_{t\to 0} \frac{P_{-t}X(t) - X(0)}{t}$$

Note that both vectors $P_{-t}X(t)$ and $X(0)$ reside in $T_{c(0)}(M)$, so the definition makes sense. (Compare Definition 6.6.2).

Proof: We use the repere mobile $\{e_i(t) = P_t e_i\}$ corresponding to an orthonormal frame $\{e_i = e_i(0)\}$ of $T_{c(0)}(M)$, as defined in Corollary 6.10.8. This means that $\frac{De_i(t)}{dt} \equiv 0$ for all i. This, of course means that $e_i = e_i(0) = P_{-t}e_i(t)$. Hence if $X(t) = \sum_i X_i(t)e_i(t) \in T_{c(t)}(M)$, we have

$$P_{-t}X(t) = \sum_i X_i(t)P_{-t}e_i(t) = \sum_i X_i(t)e_i(0)$$

Thus :

$$\lim_{t\to 0} \frac{P_{-t}X(t) - X(0)}{t} = \sum_i \lim_{t\to 0}\left(\frac{X_i(t) - X_i(0)}{t}\right) e_i(0) = \sum_i X_i'(0)e_i(0)$$

Further, since $e_i(t)$ are parallel along c, we have by Leibnitz's rule:

$$\frac{DX}{dt} = \sum_i X_i'(t)e_i(t)$$

from which it follows that $\frac{DX}{dt}(0) = \lim_{t\to 0} \frac{P_{-t}X(t)-X(0)}{t}$ and hence the lemma. □

Now we come to the motivation for defining Riemannian curvature. It measures non-commutativity of parallel transport.

Proposition 7.3.6. Let $\phi(s,t)$ be a parametrised surface in a smooth Riemannian manifold M, with $\phi(0,0) = x \in M$. Let Z be a smooth vector field in a neighbourhood U of $x \in M$, and let us denote $Z(s,t) := Z(\phi(s,t))$, $\partial_s := \phi_*\left(\frac{\partial}{\partial s}\right)$, $\partial_t := \phi_*\left(\frac{\partial}{\partial t}\right)$ for notational convenience. If we let P_{-s} and P_{-t} denote parallel transport along the curves $\phi(-,t)$ and $\phi(s,-)$, then we have:

$$R(\partial_s, \partial_t)Z(0,0) = \lim_{s,t\to 0}\left(\frac{P_{-s}P_{-t}Z(s,t) - P_{-t}P_{-s}Z(s,t)}{st}\right)$$

Note both $P_{-s}P_{-t}Z(s,t)$ and $P_{-t}P_{-s}Z(s,t)$ lie in $T_{\phi(0,0)}(M) = T_x(M)$, so the term on the right side makes sense. (Compare Proposition 7.2.3.)

Proof: By a parametrised surface we mean a smooth embedding $\phi : (-\epsilon,\epsilon) \times (-\delta,\delta) \to M$. The last Lemma 7.3.5 implies that:

$$\begin{aligned}
\nabla_{\partial_s}\nabla_{\partial_t}Z(0,0) &= \lim_{s\to 0}\left(\frac{P_{-s}(\nabla_{\partial_t}Z(s,0)) - (\nabla_{\partial_t}Z(0,0))}{s}\right) \\
&= \lim_{s\to 0}\frac{P_{-s}\left[\lim_{t\to 0}\frac{P_{-t}Z(s,t)-Z(s,0)}{t}\right] - \left[\lim_{t\to 0}\frac{P_{-t}Z(0,t)-Z(0,0)}{t}\right]}{s} \\
&= \lim_{s,t\to 0}\frac{P_{-s}P_{-t}Z(s,t) - P_{-s}Z(s,0) - P_{-t}Z(0,t) + Z(0,0)}{st} \qquad (33)
\end{aligned}$$

Similarly, by interchanging the roles of s and t we find:

$$\nabla_{\partial_s}\nabla_{\partial_t}Z(0,0) = \lim_{s,t\to 0}\frac{P_{-t}P_{-s}Z(s,t) - P_{-t}Z(0,t) - P_{-s}Z(s,0) + Z(0,0)}{st}$$

so that on subtracting the above equation from the equation (33), we have the result asserted. □

Proposition 7.3.7 (Properties of Riemannian curvature)**.** Let X, Y, Z, W be tangent vectors in $T_x M$, where M is a Riemannian manifold with Riemannian metric $\langle\, , \,\rangle$. Then:

(i): $R(X,Y)Z = -R(Y,X)Z$
(ii): $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
(iii): $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$
(iv): $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

Proof: (i) is clear from the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

(ii) is a routine verification using the fact that $\nabla_X Y - \nabla_Y X = [X,Y]$ and the Jacobi identity.

For (iii), we have, for vector fields X, Y, Z, W:

$$\begin{aligned}
\langle \nabla_X\nabla_Y Z, W\rangle &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle \\
&= X(Y\langle Z, W\rangle - \langle Z, \nabla_Y W\rangle) - \langle \nabla_Y Z, \nabla_X W\rangle \\
&= XY\langle Z, W\rangle - \langle \nabla_X Z, \nabla_Y W\rangle - \langle Z, \nabla_X\nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle
\end{aligned}$$

Interchanging X and Y and subtracting, we have:

$$\langle \nabla_X\nabla_Y Z, W\rangle - \langle \nabla_Y\nabla_X Z, W\rangle = [X,Y]\langle Z, W\rangle - \langle \nabla_X\nabla_Y W, Z\rangle + \langle \nabla_Y\nabla_X W, Z\rangle$$

Now using that $[X,Y]\langle Z,W\rangle = \left\langle \nabla_{[X,Y]}Z, W\right\rangle + \left\langle Z, \nabla_{[X,Y]}W\right\rangle$, we have (iii).

(iv) is also straightforward using (i), (ii) and (iii), and left as an exercise. □

Example 7.3.8 (Riemannian Curvature of Lie Groups)**.** Suppose G is a Lie-group with a bi-invariant metric $\langle\, , \,\rangle$. From Example 6.9.3 we have the formula:

$$\nabla_{\widetilde{X}}\widetilde{Y} = \frac{1}{2}[\widetilde{X}, \widetilde{Y}]$$

Thus

$$\begin{aligned}
R(\widetilde{X}, \widetilde{Y})\widetilde{Z} &= \frac{1}{4}\left([\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]] - [\widetilde{Y}, [\widetilde{X}, \widetilde{Z}]]\right) - \frac{1}{2}[[\widetilde{X}, \widetilde{Y}], \widetilde{Z}] \\
&= -\frac{1}{4}[\widetilde{Z}, [\widetilde{X}, \widetilde{Y}] + \frac{1}{2}[\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]] \\
&= \frac{1}{4}[\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]] \qquad (34)
\end{aligned}$$

Thus the Riemannian curvature of such a Lie group is a left-invariant tensor, as it should be, since the metric is left invariant. Thus the equation (34) may as well be written as $R(X,Y)Z = \frac{1}{4}[Z, [X,Y]]$, where X, Y and $Z \in \mathfrak{g} = T_e(G)$. It also shows how curvature measures the degree of non-abelian-ness of G. That is, if all triple commutators in G vanish (as happens in e.g. abelian groups, or more generally 2-step nilpotent groups), then the Levi-Civita connection for a bi-invariant metric on G is a *flat connection*, i.e. has no curvature. Another useful formula is:

$$\langle R(X,Y)Z, W\rangle = \frac{1}{4}\langle [Z, [X,Y]], W\rangle = -\frac{1}{4}\langle [X,Y], [Z,W]\rangle \qquad (35)$$

where the last equality follows from the ad-invariance of $\langle -, -\rangle$ on $\mathfrak{g}$.

Exercise 7.3.9. Using the Example 6.9.3, compute the Riemann curvature tensor of a Lie-Group G with respect to a *left-invariant* Riemannian metric on G.

There is an explicit formula relating the curvature tensor R with the Christoffel symbols of (17).

Proposition 7.3.10 (Curvature in local coordinates)**.** Let U be a coordinate patch, and ∂_i for $i = 1, .., n$ be the resulting coordinate vector fields on U. Note that they all commute on U. Writing (with repeated index summation convention):

$$R(\partial_i, \partial_j)\partial_k = R^h_{ijk}\partial_h$$

Then:

$$R^h_{ijk} = \partial_i\Gamma^h_{jk} - \partial_j\Gamma^h_{ik} + \Gamma^h_{il}\Gamma^l_{jk} - \Gamma^h_{jl}\Gamma^l_{ik}$$

where Γ^i_{jk} are the Christoffel symbols of the connection.

Proof: We have the relation (17), by which:

$$\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l$$

so that

$$\begin{aligned}\nabla_{\partial_i}\nabla_{\partial_j} &= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\nabla_{\partial_i}\partial_l \\ &= (\partial_i\Gamma^h_{jk} + \Gamma^l_{jk}\Gamma^h_{il})\partial_h\end{aligned}$$

Interchanging the roles of i and j, subtracting, and noting that $[\partial_i, \partial_j] = 0$, we have the result. □

Definition 7.3.11 (Sectional curvature)**.** Let $X, Y \in T_x(X)$ be two tangent vectors. We define the *sectional curvature in the plane of* X and Y as the quantity:

$$K(X,Y)(x) = \frac{-\langle R(X,Y)X, Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2} = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$

(the denominator represents the square of the area of the parallelogram spanned by X and Y in $T_x(X)$). Note that:

$$\langle R(\alpha X + \beta Y, \gamma X + \delta Y)\alpha X + \beta Y, \gamma X + \delta Y\rangle = (\alpha\delta - \beta\gamma)^2\langle R(X,Y)X, Y\rangle$$

by the skew-symmetry properties (i) and (iii) of proposition 7.3.7. The denominator $\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$ also changes by the same factor $(\alpha\delta - \beta\gamma)^2$ on changing X to $\alpha X + \beta Y$, and Y to $\gamma X + \delta Y$. Hence the quantity $K(X,Y)$ depends only on the plane spanned by X and Y.

Remark 7.3.12. Note that in M is 1-dimensional, its Riemann tensor is zero because $R(X,X)X = -R(X,X)X$ by (i) of Proposition 7.3.7, so it vanishes identically. For a surface, note that (again by (i) of Proposition 7.3.7), $R(X,X) = R(Y,Y) = 0$. So only $R(X,Y)$ can be non-zero for a surface. Again, by (iii) of 7.3.7, $\langle R(X,Y)X, X\rangle = 0$, and similarly $\langle R(X,Y)Y, Y\rangle = 0$. Thus the only non-zero quantity is possibly $\langle R(X,Y)X, Y\rangle$. That is, for a surface, the sectional curvature completely captures the Riemann curvature tensor, and is a scalar.

Example 7.3.13. If G is a Lie group with a bi-invariant metric, then the sectional curvature (with respect to the corresponding Levi-Civita connection is given by:

$$K(X,Y) = \frac{1}{4}\langle [X,Y], [X,Y]\rangle$$

for $X, Y \in \mathfrak{g}$ orthonormal vectors, from (35) above. As a corollary, the Lie algebra $\mathfrak{g}$ is abelian iff all the sectional curvatures of G are zero. In particular, if the group is connected, then since G is generated by a neighbourhood of e, and since $\mathrm{Ad}\exp(tX) = \exp(t\,\mathrm{ad}\,X)$, it will follow that:

Corollary 7.3.14. A connected Lie group with bi-invariant metric is abelian iff all its sectional curvatures are zero.

This puts a strong geometric condition on the (purely algebraic) group structure of a Lie group. As an application, we look at, say S^3, which is the group of quaternions of unit length and isomorphic as a Lie group to $SU(2)$, and a non-abelian Lie group. It is interesting to note that:

Corollary 7.3.15. There does not exist any abelian Lie group structure on S^3 compatible with its usual manifold structure.

Proof: If there were such an abelian group structure on S^3, we could easily put a bi-invariant metric on it. Indeed, by its abelian-ness, a left-invariant metric would become automatically bi-invariant. The corollary above would imply that all sectional curvatures for this bi-invariant Riemannian metric would be $\equiv 0$. Also, the compactness of S^3 would imply that it is geodesically complete (see Proposition 6.11.3). There is a fundamental theorem of Cartan-Hadamard that a simply connected complete Riemannian manifold of dimension n with all sectional curvatures ≤ 0 is diffeomorphic to $\mathbb{R}^n$. Thus, since S^3 is simply-connected, it would follow that S^3 diffeomorphic to $\mathbb{R}^3$, a contradiction. $\square$

Example 7.3.16 (Sectional curvatures of S^n). We first recall the parametrisation:

$$\begin{array}{rcl} h:(0,\pi)\times(0,2\pi) & \to & S^2 \\ (\theta,\phi) & \mapsto & (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \end{array}$$

of S^2, and note that the Riemannian metric on this coordinate patch $U := h((0,\pi)\times(0,2\pi))$ is given by $d\theta^2 + \sin^2\theta d\phi^2$, from (i) of 5.1.8. Thus $e_1 = \partial_\theta$ and $e_2 = (\sin\theta)^{-1}\partial_\phi$ constitute an orthonormal frame of TS^2 on U. This means a section $\sigma : U \to \pi^{-1}(U)$ of the (oriented) orthonormal frame bundle of TS^2, which we know is $\pi : SO(3) \to S^2$ by the Exercise 6.1.7. Since $\mathfrak{g}$ is the space of all 2×2 skew-symmetric matrices, we see that local connection form ω on U will be a smooth map:

$$\omega = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$$

where ω_{12} is a scalar valued 1-form on U. Also, by (32), we have the equations:

$$\begin{array}{rcl} \nabla e_1 & = & -\omega_{12}e_2 \\ \nabla e_2 & = & \omega_{12}e_1 \end{array}$$

In fact, if we wish to arrive at the above directly from the definiton of the Levi-Civita connection, we note that from the definition of the connection 1-forms (see Remark 6.8.4) we have:

$$\nabla e_1 = \omega_{11}e_1 + \omega_{21}e_2; \quad \nabla e_2 = \omega_{12}e_1 + \omega_{22}e_2$$

and the relation $X\langle e_i, e_j\rangle \equiv 0$ (since e_1, e_2 are orthonormal) and compatibility with the metric implies that

$$0 = \langle \nabla e_1, e_1\rangle = \langle \nabla e_2, e_2\rangle = \langle \nabla e_1, e_2\rangle + \langle \nabla e_2, e_1\rangle$$

which translates into $\omega_{11} = \omega_{22} = \omega_{12} + \omega_{21} = 0$.

Further, we have the torsion free condition:

$$\nabla_{e_1}e_2 - \nabla_{e_2}e_1 = \omega_{12}(e_1)e_1 + \omega_{12}(e_2)e_2 = [e_1, e_2]$$

Now $[e_1, e_2] = [\partial_\theta, (\sin\theta)^{-1}\partial_\phi] = -\cot\theta e_2$. Thus $\omega_{12}(e_1) = 0$ and $\omega_{12}(e_2) = -\cot\theta$ which implies $\omega_{12} = -\cot\theta e_2^* = -(\cot\theta\sin\theta)d\phi = -\cos\theta d\phi$. Thus, by (32)

$$\begin{array}{rcl} \Omega & = & d\omega + \omega\wedge\omega \\ & = & d\begin{pmatrix} 0 & -\cos\theta\, d\phi \\ \cos\theta\, d\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\cos\theta\, d\phi \\ \cos\theta\, d\phi & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\cos\theta\, d\phi \\ \cos\theta\, d\phi & 0 \end{pmatrix} \\ & = & \begin{pmatrix} 0 & \sin\theta\, d\theta\wedge d\phi \\ -\sin\theta\, d\theta\wedge d\phi & 0 \end{pmatrix} + 0 \\ & = & \begin{pmatrix} 0 & e_1^*\wedge e_2^* \\ -e_1^*\wedge e_2^* & 0 \end{pmatrix} \end{array}$$

Thus, since e_i is an orthonormal basis,

$$\begin{aligned} K(e_1, e_2) &= \langle R(e_1, e_2)e_2, e_1\rangle \\ &= \langle \Omega(e_1, e_2)e_2, e_1\rangle \\ &= (e_1^* \wedge e_2^*)\,\langle e_1, e_1\rangle \\ &= 1 \end{aligned}$$

Alternatively, directly from the formulas $\nabla e_1 = -\omega_{12} e_2$, and $\nabla e_2 = \omega_{12} e_1$ we have:

$$\nabla_{e_1} e_1 = 0;\ \nabla_{e_2} e_1 = \cot\theta e_2;\ \nabla_{e_1} e_2 = 0;\ \ \nabla_{e_2} e_2 = -\cot\theta e_1$$

from which we get that:

$$\nabla_{e_1}\nabla_{e_2} e_2 = \nabla_{e_1}(-\cot\theta e_1) = \mathrm{cosec}^2\theta e_1;\ \ \nabla_{e_2}\nabla_{e_1} e_2 = \nabla_{e_2}(0) = 0$$

and finally

$$\nabla_{[e_1,e_2]} e_2 = -\cot\theta \nabla_{e_2} e_2 = \cot^2\theta e_1$$

Thus

$$\begin{aligned} R(e_1, e_2)e_2 &= \nabla_{e_1}\nabla_{e_2} e_2 - \nabla_{e_2}\nabla_{e_1} e_2 - \nabla_{[e_1,e_2]} e_2 \\ &= \mathrm{cosec}^2\theta e_1 - \cot^2\theta e_1 = e_1 \end{aligned}$$

which implies that $K(e_1, e_2) = \langle R(e_1, e_2)e_2, e_1\rangle = \langle e_1, e_1\rangle = 1$

(One could have also done the above calculation with Christoffel symbols and the local formula for R.) This shows that the sectional curvature of S^2 is the constant 1. (This follows from the symmetry of S^2, for U could be rotated to any other location. Or alternatively, since the (scalar) sectional curvature is continuous on S^2, and $\equiv 1$ on U, and U is dense in S^2, it is $\equiv 1$ on S^2.

For S^n, we claim that for two unit vectors X, Y in $T_x(S^n)$, the quantity $R(X, Y)Y$ lies entirely in the span of X, Y. To see this, we successively use the isometric reflections about all hyperplanes containing x, X and Y, (similar to the argument for geodesics on S^2), and since this vector $R(X, Y)Y$ is invariant under all such isometric reflections, it lies in the span of x, X and Y. Now, since it is a tangent vector to S^n, it must lie entirely in the span of X and Y. Then the sectional curvature $K(X, Y)$ is precisely the sectional curvature of the intersection of the plane containing x, X, Y with S^n, which is an S^2 of radius 1, with its usual metric. Thus it is $\equiv 1$ at all points of S^n.

Example 7.3.17 (Sectional curvature of H^2)**.** We recall the metric:

$$\frac{dx^2 + dy^2}{y^2}$$

so that $\langle \partial_i, \partial_j\rangle = y^{-2}\delta_{ij}$ for $1 \le i, j \le 2$, where $\partial_1 := \frac{\partial}{\partial x}$, and $\partial_2 := \frac{\partial}{\partial y}$. Instead of mimicking what we did for S^2 above (using an orthonormal frame and connection forms), for variety, we do the calculation with a coordinate frame and Christoffel symbols.

Recalling from Example 6.9.9 that the Christoffel symbols for H^2 are:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0$$
$$\Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^1_{21} = -\Gamma^2_{22} = y^{-1}$$

From these relations it follows:

$$\nabla_{\partial_2}\partial_1 = -y^{-1}\partial_1,\ \nabla_{\partial_1}\partial_1 = y^{-1}\partial_2$$

so that

$$\nabla_{\partial_1}\nabla_{\partial_2}\partial_1 - \nabla_{\partial_2}\nabla_{\partial_1}\partial_1 = y^{-2}\partial_2$$

Thus $\langle R(\partial_1, \partial_2)\partial_1, \partial_2\rangle = y^{-4}$. Since $\langle \partial_1, \partial_1\rangle\,\langle \partial_2, \partial_2\rangle = y^{-4}$ and $\langle \partial_1, \partial_2\rangle^2 = 0$, we get the sectional curvature of H^2 to be -1 at all points.

Exercise 7.3.18. In the hyperbolic plane H^2, let $c(t) = (t, a)$ for some $a > 0$, the horizontal line starting at $(0, a)$. Compute the parallel transport operator P_t along c. What is the repere mobile $\{e_1(t), e_2(t)\}$ with initial value $\{e_1 = a\partial_x,\ e_2 = a\partial_y\}$?

Exercise 7.3.19 (Transformation law for Christoffel Symbols)**.** Let (ϕ, U) with local coordinate functions x_i and (ψ, V) with local coordinate functions y_j be two overlapping charts on a smooth Riemannian manifold M. If Γ^i_{jk} and $\widetilde{\Gamma}^p_{qr}$ denote the Christoffel symbols of the Levi-Civita connection with respect to the charts (ϕ, U) and (ψ, V) respectively, then show that on $U \cap V$ we have the transformation formula:

$$\Gamma^i_{jk} = \frac{\partial x_i}{\partial y_l}\frac{\partial^2 y_l}{\partial x_j \partial x_k} + \frac{\partial x_i}{\partial y_p}\frac{\partial y_q}{\partial x_j}\frac{\partial y_r}{\partial x_k}\widetilde{\Gamma}^p_{qr}$$

where repeated indices are summed over.

Exercise 7.3.20 (Curvature as an integrability condition)**.** Let (ϕ, U) be a local chart on M with coordinate functions x_i, as in the previous exercise. However let $V = U$, and ψ be a new chart (whose existence we wish to investigate) where we want all the new Christoffel symbols $\widetilde{\Gamma}^p_{qr} = 0$ for all p, q, r.

(i): Using the last exercise, show that this is equivalent to solving the system of second-order PDEs

$$\frac{\partial^2 y_l}{\partial x_j \partial x_k} = \Gamma^r_{jk}\frac{\partial y_l}{\partial x_r} \qquad \text{for } 0 \le j, k, l \le n = \dim M$$

(ii): Take partial derivative of both sides of the above equation with respect to x_s, and use the above equation again to show that we must have

$$\frac{\partial^3 y_l}{\partial x_s \partial x_j \partial x_k} = \left(\frac{\partial \Gamma^p_{jk}}{\partial x_s} + \Gamma^r_{jk}\Gamma^p_{sr}\right)\frac{\partial y_l}{\partial x_p}$$

(Since r is a dummy index that is summed over, it is replaced by p to get the first term on the right).

(iii) Interchanging s and j leaves the left side of the last relation unchanged, so it must also leave the right side unchanged. Hence a necessary condition for the solution of the system of PDE's in (i) above is that:

$$\frac{\partial \Gamma^p_{jk}}{\partial x_s} - \frac{\partial \Gamma^p_{sk}}{\partial x_j} + \Gamma^r_{jk}\Gamma^p_{sr} - \Gamma^r_{sk}\Gamma^p_{jr} = 0$$

Using Proposition 7.3.10, this just the assertion that the components $R^p_{sjk} \equiv 0$ all over U. It turns out (using a theorem of Frobenius) that this condition is also sufficient. Thus, on a coordinate chart, the Riemann curvature tensor is exactly the obstruction to finding a new system of coordinates wherein all the Christoffel symbols vanish, and covariant differentiation becomes ordinary differentiation.

7.4. **Surfaces in $\mathbb{R}^3$.** Let $M \subset \mathbb{R}^3$ be a smooth submanifold of dimension 2. We further assume that the manifold is *orientable*. We give M the induced Riemannian metric from the euclidean metric on $\mathbb{R}^3$ (as we did for S^2). We would like to compute curvatures, covariant derivatives etc. for M, in terms of the parametric representation of M over trivialising charts $h(U)$, where $h = (h_1, h_2, h_3)$ is a diffeo (local parametrisation) of M for some open subset $U \subset \mathbb{R}^2$ (e.g. the spherical polar coordinates of S^2 above.)

Because the manifold M is orientable, we can find a system of charts $\{(h_i, U_i)\}$ such that U_i are opens in $\mathbb{R}^2$, and $\det(h_j^{-1} \circ h_i) > 0$ (on $h_i^{-1}(h_i(U_i) \cap h_j(U_j))$ for all i and j. Thus for each such parametrisation

$$h = (h_1, h_2, h_3) : U \to h(U) \subset M \subset \mathbb{R}^3$$

the vectors:

$$\begin{aligned}
\partial_u &:= \frac{\partial h_1}{\partial u}\partial_1 + \frac{\partial h_2}{\partial u}\partial_2 + \frac{\partial h_3}{\partial u}\partial_3 \\
\partial_v &:= \frac{\partial h_1}{\partial v}\partial_1 + \frac{\partial h_2}{\partial v}\partial_2 + \frac{\partial h_3}{\partial v}\partial_3
\end{aligned}$$

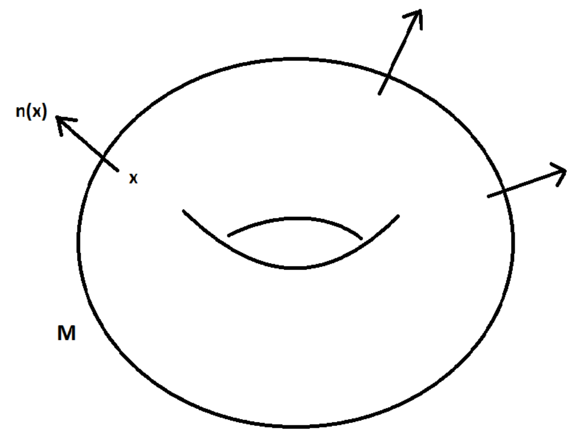

FIGURE 9. Unit normal field on M

where $\partial_1 = \frac{\partial}{\partial x}$ etc. are the standard coordinate fields on $\mathbb{R}^3$, constitute an oriented basis for $T_x(M)$ for all $x \in h(U)$. In particular, the cross-product vector

$$\hat{n}(u,v) = \frac{\partial_u \times \partial_v}{\|\partial_u \times \partial_v\|}$$

is a unit vector in $\mathbb{R}^3$, normal to M (i.e. $T_x(M)$) at all points $x = h(u,v) \in U$. In fact, this vector is globally well-defined independent of charts, as the reader can readily verify, from the orientability of M and the choice of the atlas $\{(h_i, U_i)\}$ as an oriented atlas. It is called the *unit normal field to M*.

Definition 7.4.1 (Gauss Map)**.** The smooth map

$$\hat{n} : M \to S^2$$

is called the *Gauss map*. This map captures all the curvature properties of M, as we shall see.

Definition 7.4.2 (1st and 2nd fundamental forms)**.** The induced Riemannian metric on M, in terms of the basis ∂_u, ∂_v introduced above, is

$$g(h(u,v)) = g_{11} du \otimes du + g_{12} du \otimes dv + g_{21} dv \otimes du + g_{22} dv \otimes dv$$

It is easily seen that

$$\begin{aligned} g_{11} &= \langle \partial_u, \partial_u \rangle = \|(\partial_u h_1, \partial_u h_2, \partial_u h_3)\|^2 \\ g_{22} &= \langle \partial_v, \partial_v \rangle = \|(\partial_v h_1, \partial_v h_2, \partial_v h_3)\|^2 \\ g_{12} = g_{21} &= \langle \partial_u, \partial_v \rangle = \langle (\partial_u h_1, \partial_u h_2, \partial_u h_3), (\partial_v h_1, \partial_v h_2, \partial_v h_3) \rangle \end{aligned}$$

where $\langle \, , \, \rangle$ always denotes the euclidean inner product in this subsection. This 2×2 symmetric matrix $[g_{ij}]$ (of the induced Riemannian metric) is called the *1st fundamental form*.

Let us consider the Gauss map $\hat{n} : M \to S^2$. We will drop the hat on n for brevity. Note that for $x \in M$, $Dn(x) : T_x(M) \to T_{n(x)} S^2$. But $T_{n(x)} S^2 = (\mathbb{R} n(x))^\perp = T_x(M)$. Thus $Dn(x) : T_x(M) \to T_x(M)$. It is called the *Weingarten map* at x, and denoted S_x. For $Y, Z \in T_x(M)$, the bilinear form:

$$S_x(Y,Z) := \langle S_x Y, Z \rangle$$

is called the *second fundamental form*. We shall see presently that it is symmetric, and encodes covariant differentiation on M.

Lemma 7.4.3. Let M be as above, with the normal vector field n as defined above. For a tangent vector X to $\mathbb{R}^3$, we denote by ∇_X the covariant differentiation on $\mathbb{R}^3$ with respect to the (Levi Civita connection) of the euclidean metric, and for $X \in T_x(M)$, D_X denotes the covariant derivative with respect to the (Levi-Civita connection) of the Riemannian metric g induced on M. Note that we can write $\langle\, ,\, \rangle$ without ambiguity. Let $S : TM \to TM$ denote the Weingarten map. Then:

(i): $S(X) = \nabla_X n$ for $X \in TM$.

(ii): $S(X,Y) = -\langle \nabla_X Y, n \rangle$

(iii): $S(X,Y) := \langle S(X), Y \rangle = \langle S(Y), X \rangle = S(Y,X)$ for all $X, Y \in TM$.

(iv): $\nabla_X Y = D_X Y - S(X,Y)n$ for $X, Y \in TM$.

Proof: We first note that (i) has to be interpreted, because n is only a vector field (section of the normal bundle $TM^\perp$) defined on M, and not on an open subset of $\mathbb{R}^3$. We write $n = \sum_{i=1}^3 n_i \partial_i$, where n_i are smooth functions on M, and ∂_i are the coordinate fields on $\mathbb{R}^3$. Then we mean:

$$\nabla_{\partial_u} n := \sum_{i=1}^{3} \partial_u(n_i)\partial_i + \sum_{i=1}^{3} n_i \nabla_{\partial_u} \partial_i$$

and similarly for $\nabla_{\partial_v} n$, and this makes sense since ∂_u is a vector field on M, acting on smooth functions n_i on M, and the field above is well defined at points of M (though we don't know yet whether it is tangent to M.) Now, $\nabla_{\partial_u} \partial_i$ is a linear combination of $\nabla_{\partial_i} \partial_j$. But $\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k = 0$ since all the Christoffel symbols for $\mathbb{R}^3$ are 0, by (17)

Hence $\nabla_{\partial_u} n = \sum_{i=1}^3 (\frac{\partial n_i}{\partial u})\partial_i = Dn(\partial_u) = S(\partial_u)$, and similarly for $\nabla_{\partial_v} n$. This proves (i).

To see (ii), note that $\langle Y, n \rangle \equiv 0$ for a vector field Y on M. Thus $0 = X\langle Y, n \rangle = \langle \nabla_X Y, n \rangle + \langle Y, \nabla_X n \rangle = \langle \nabla_X Y, n \rangle + \langle Y, S(X) \rangle$. That is, $S(X,Y) = \langle S(X), Y \rangle = -\langle \nabla_X Y, n \rangle$. The assertion (ii) follows.

For (iii), note that (ii) and the torsionlessness of the connection imply that

$$S(X,Y) - S(Y,X) = -\langle \nabla_X Y, n \rangle + \langle \nabla_Y X, n \rangle = -\langle [X,Y], n \rangle$$

But since X, Y are vector fields on M, $[X,Y]$ is a vector field on M, and thus $\langle [X,Y], n \rangle \equiv 0$. This proves (iii).

To see (iv), *define* a covariant derivative operator on TM by $D_X Y = \nabla_X Y + S(X,Y)n$. By (ii) above, $D_X Y$ is the component of $\nabla_X Y$ perpendicular to n, hence lies in $T_x(M)$. The Leibniz formula for D_X easily follows from the Leibniz formula for ∇. Further

$$D_X Y - D_Y X = \nabla_X Y + S(X,Y)n - \nabla_Y X - S(Y,X)n = [X,Y]$$

by the fact that ∇ is torsionless, and S is symmetric. Thus D is torsionless. To see compatibility with the metric, for vector fields X, Y, Z on M,

$$\begin{aligned} X\langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ &= \langle D_X Y - S(X,Y)n, Z \rangle + \langle Y, D_X Z - S(X,Z)n \rangle \\ &= \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \end{aligned}$$

since n is orthogonal to Y and Z. Thus D_X is *the* Levi-Civita connection on TM, by the fact that the Levi-Civita connection is unique (see Proposition 6.9.1). This proves the lemma. □

Corollary 7.4.4. Let $h = (h_1, h_2, h_3) : U \to M$ be the parametric representation (coordinate chart) above. Then the second fundamental form is given by:

$$\begin{aligned} S(\partial_u, \partial_u) &= -\left\langle \frac{\partial^2 h}{\partial u^2}, n \right\rangle \\ S(\partial_u, \partial_v) &= S(\partial_v, \partial_u) = -\left\langle \frac{\partial^2 h}{\partial u \partial v}, n \right\rangle \\ S(\partial_v, \partial_v) &= -\left\langle \frac{\partial^2 h}{\partial v^2}, n \right\rangle \end{aligned}$$

Proof: In (ii) of the above Proposition 7.4.3, we saw that $S(X, Y) = -\langle \nabla_X Y, n \rangle$. Since

$$X = \partial_u := h_* \left(\frac{\partial}{\partial u} \right) = \sum_i (\partial_u h_i) \partial_i$$

then by the first para in the proof of (i) above

$$\nabla_{\partial_u} \partial_u = \sum_{i=1}^{3} \partial_u (\partial_u h_i) \partial_i$$

which proves that $S(\partial_u, \partial_u) = -\langle \nabla_{\partial_u} \partial_u, n \rangle = -\left\langle \frac{\partial^2 h}{\partial u^2}, n \right\rangle$. The proofs of the other identities are identical. $\square$

Example 7.4.5. Consider the example of the sphere, with the parametrisation of (i) in 5.1.8. Thus:

$$\begin{aligned} h : (0, \pi) \times (0, 2\pi) &\to S^2 \\ (u, v) &\mapsto (\sin u \cos v, \sin u \sin v, \cos u) \end{aligned}$$

Clearly $n(x) = n(h(u, v)) = h(u, v)$.

Then

$$\begin{aligned} \partial_u &= \partial_u h = (\cos u \cos v, \cos u \sin v, -\sin u) \\ \partial_v &= \partial_v h = (-\sin u \sin v, \sin u \cos v, 0) \\ \partial_u^2 h &= (-\sin u \cos v, -\sin u \sin v, -\cos u) \\ \partial_v \partial_u h &= (-\cos u \sin v, \cos u \cos v, 0) \\ \partial_v^2 h &= (-\sin u \cos v, -\sin u \sin v, 0) \end{aligned}$$

So that the matrix of S with respect to $\{\partial_u, \partial_v\}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$$

That is, the second fundamental form is precisely the metric (=1st fundamental form) which it should be since the Weingarten map S is the identity map.

Definition 7.4.6 (Principal curvatures)**.** The 2 eigenvalues $\lambda_1(x), \lambda_2(x)$ of the symmetric linear (Weingarten) map $S_x : T_x(M) \to T_x(M)$ (= the eigenvalues of the matrix of the second fundamental form $[S_x(e_i, e_j)]$ where e_i are an *orthonormal frame* of $T_x(M)$) are real numbers at each x, in fact smooth functions of x, and are called the *principal curvatures of* M *at* x. $\frac{1}{2}(\lambda_1(x) + \lambda_2(x))$ is called the *mean curvature* at x. The product $\lambda_1(x)\lambda_2(x)$ is called the *scalar* or *Gauss* curvature at x.

For example, in the case of the sphere S^2 above, the principal curvatures were both =1.

Exercise 7.4.7. Show that the Gauss curvature is $\frac{\det[S(f_i, f_j)]}{\det[\langle f_i, f_j \rangle]}$ where $\{f_1, f_2\}$ is *any* frame of $T_x(M)$.

Exercise 7.4.8 (Curvatures of Torus in $\mathbb{R}^3$)**.** Consider the chart :

$$h(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u)$$

as described in Exercise 2.1.11. Prove that in this chart :

(i): The first fundamental form (metric) is given by

$$g = du^2 + (2+\cos u)^2 dv^2$$

(ii): The unit normal field is given by :

$$n(u,v) = -(\cos u \cos v, \cos u \sin v, \sin u)$$

(iii): Compute the matrix entries $S(e_i, e_j)$ of the second fundamental form with respect to the orthonormal basis $e_1 = \partial_u$ and $e_2 = (2+\cos u)^{-1}\partial_v$ of $T_p(M)$, where $p = h(u,v)$.

(iv): At the point $p = h(u,v)$, use (iii) above to compute the mean and scalar curvatures at p.

Proposition 7.4.9 (Theorema Egregium of Gauss)**.** Let M be an orientable surface in $\mathbb{R}^3$ as above. Then the Gauss curvature is equal to the sectional curvature of the induced metric on M. In particular, the Gauss curvature, which is defined in terms of the embeddings h, is an intrinsic isometric invariant of the Riemannian manifold M.

Proof: By the Remark 7.3.12, we just need to calculate $\langle R(X,Y)Y, X\rangle$ for a frame X, Y of $T_x(M)$. It is most convenient to take X and Y to be coordinate fields (e.g. ∂_u, ∂_v), so that $[X,Y] = 0$.

By (i) of Lemma 7.4.3 we have $S(X) = \nabla_X n$, and using (iii) of the same lemma, we compute:

$$\begin{aligned}
\nabla_X \nabla_Y Y &= \nabla_X \left[D_Y Y - S(Y,Y)n\right] \\
&= \nabla_X D_Y Y - S(Y,Y)\nabla_X n - X(S(Y,Y))n \\
&= D_X D_Y Y - S(D_Y Y, X)n - S(Y,Y)S(X) - X(S(Y,Y))n \\
&= D_X D_Y Y - S(Y,Y)S(X) - \left[X(S(Y,Y) + S(D_Y Y, X)\right] n
\end{aligned}$$

Thus, since $\langle n, X\rangle = 0$, we have:

$$\langle \nabla_X \nabla_Y Y, X\rangle = \langle D_X D_Y Y, X\rangle - S(Y,Y)S(X,X) \tag{36}$$

Similarly

$$\begin{aligned}
\nabla_Y \nabla_X Y &= \nabla_Y \left[D_X Y - S(X,Y)n\right] \\
&= D_Y D_X Y - S(D_X Y, Y)n - Y(S(X,Y))n - S(X,Y)\nabla_Y n \\
&= D_Y D_X Y - S(X,Y)S(Y) - \left[S(D_X Y, Y) + Y(S(X,Y))\right] n
\end{aligned}$$

Taking inner product with X, and noting $\langle n, X\rangle = 0$, we have

$$\langle \nabla_Y \nabla_X Y, X\rangle = \langle D_Y D_X Y, X\rangle - S(X,Y)^2 \tag{37}$$

Subtracting (37) from (36), we get:

$$\langle \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y, X\rangle = \langle D_X D_Y Y - D_Y D_X Y, X\rangle - S(Y,Y)S(X,X) + S(X,Y)^2$$

But since $[X,Y] = 0$, the left hand side is the sectional curvature of $\mathbb{R}^3$ in the plane spanned by X, Y, and hence 0. The right side is $\langle R(X,Y)Y, X\rangle - \Delta$, where Δ is the determinant of the matrix of S with respect to the basis $\{X, Y\}$. Dividing by $\langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2$, on both sides and using the Exercise 7.4.7, we are done. □

Proposition 7.4.10 (Gauss-Mainardi-Codazzi relations)**.** In the setting of the last proposition, we have the following relations:

(i): $D_X S(Y) - D_Y S(X) = S([X,Y])$ (relation between connection D and 2nd fundamental form).

(ii): $R(X,Y)Z = S(Y,Z)S(X) - S(X,Z)S(Y)$ (relation between curvature and 2nd fundamental form)

Proof: We have, from (i) of the Lemma 7.4.3 that $S(X) = \nabla_X n$. Thus:

$$\begin{aligned} D_X S(Y) &= D_X(\nabla_Y n) = \nabla_X(\nabla_Y n) + S(X, \nabla_Y n)n \\ &= \nabla_X \nabla_Y n + \langle S(X), \nabla_Y n\rangle\, n = \nabla_X \nabla_Y n + \langle \nabla_X n, \nabla_Y n\rangle\, n \end{aligned}$$

Interchanging X, Y and subtracting, and noting that on $\mathbb{R}^3$, flatness implies $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$, we have that

$$D_X S(Y) - D_Y S(X) = \nabla_{[X,Y]} n = S([X,Y])$$

which proves (i).

For (ii), let X, Y, Z be vector fields on M. We recall the proof of (36) in 7.4.9, and show in identical fashion that

$$\nabla_X \nabla_Y Z = D_X D_Y Z - S(Y,Z)S(X) - [X(S(Y,Z)) + S(D_Y Z, X)]\, n$$

Interchanging X, Y, and subtracting, we get by using (iv) of Lemma 7.4.3 and flatness of the connection $\mathbb{R}^3$:

$$\begin{aligned} D_{[X,Y]}Z - S([X,Y],Z)n &= \nabla_{[X,Y]}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \\ &= D_X D_Y Z - D_Y D_X Z - S(Y,Z)S(X) + S(X,Z)S(Y) - \alpha n \end{aligned}$$

where α is some scalar. Taking the tangential component of this equation kills all the terms containing n, and we get:

$$D_{[X,Y]}Z = D_X D_Y Z - D_Y D_X Z - S(Y,Z)S(X) + S(X,Z)S(Y)$$

from which we have (ii). □

7.5. **Gauss-Bonnet for surfaces.** Before we get on to the Gauss-Bonnet Theorem for surfaces, we need the following:

Lemma 7.5.1. Let $\{e_1, e_2\}$ be an orthonormal frame field in a coordinate chart U of a Riemannian manifold of dimension 2. Let the matrix of connection forms with respect to this frame be denoted by $[\omega_{ij}]$. As we saw in Example 7.3.16, this matrix is skew-symmetric. Then :

(i):

$$R(e_1, e_2)e_2 = [d\omega_{12}(e_1, e_2)]\, e_1$$

(ii): If we denote the volume form on U as $dA = e_1^* \wedge e_2^*$, and the scalar curvature function as k, then

$$k\, dA = \langle R(e_1, e_2)e_2, e_1\rangle\, e_1^* \wedge e_2^* = d\omega_{12}$$

Proof: We recall from the computation in Example 7.3.16 that :

$$\nabla e_1 = \omega_{21} e_2 = -\omega_{12} e_2; \quad \nabla e_2 = \omega_{12} e_1$$

because of the orthonormality of the frame $\{e_1, e_2\}$. Thus

$$\begin{aligned} R(e_1, e_2)e_2 &= \nabla_{e_1} \nabla_{e_2} e_2 - \nabla_{e_2} \nabla_{e_1} e_2 - \nabla_{[e_1,e_2]} e_2 \\ &= \nabla_{e_1}(\omega_{12}(e_2)e_1) - \nabla_{e_2}(\omega_{12}(e_1)e_1) - \omega_{12}([e_1,e_2])e_1 \\ &= [e_1(\omega_{12}(e_2)]\, e_1 + \omega_{12}(e_2)\nabla_{e_1} e_1 - [e_2(\omega_{12}(e_1)]\, e_1 - \omega_{12}(e_1)\nabla_{e_2} e_1 - \omega_{12}([e_1,e_2])e_1 \\ &= (e_1(\omega_{12}(e_2) - e_2(\omega_{12}(e_1) - \omega_{12}([e_1,e_2]))\, e_1 - (\omega_{12}(e_1)\omega_{21}(e_2) - \omega_{12}(e_2)\omega_{21}(e_1))\, e_2 \\ &= [d\omega_{12}(e_1, e_2)]e_1 \end{aligned}$$

where we have used the Exercise 5.6.10 for computing $d\omega_{12}$, and the fact that $\omega_{12} = -\omega_{21}$. This proves (i)

From this it follows that the sectional curvature $K(e_1, e_2) = \langle R(e_1, e_2)e_2, e_1\rangle = d\omega_{12}(e_1, e_2)$. By the Gauss Theorema Egregium 7.4.9, this is the scalar curvature k, so that $k\, dA = k e_1^* \wedge e_2^* = d\omega_{12}(e_1, e_2) e_1^* \wedge e_2^* = d\omega_{12}$ and (ii) follows. □

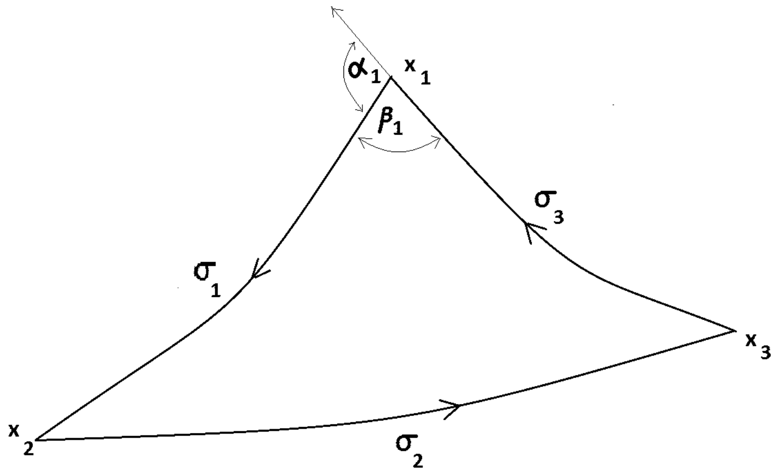

FIGURE 10. A Geodesic Triangle

Definition 7.5.2. A *geodesic triangle* in a Riemannian manifold is a triangle whose edges are geodesics. Similarly geodesic polygons can be defined. If $x_1 \in M$ is a vertex of a geodesic triangle $\triangle x_1x_2x_3$, at which the geodesic edges σ_1 and σ_3 meet, with $\sigma_3(1) = \sigma_1(0) = x_1$ (see the Figure 10), the *external angle* α_1 at x_1 is defined as:

$$\cos^{-1}\left(\frac{\langle \sigma_1'(0), \sigma_3'(1)\rangle}{\|\sigma_1'(0)\|\,\|\sigma_3'(1)\|}\right)$$

The *internal angle* at x_1 is defined as $\beta_1 = \pi - \alpha_1$.

Theorem 7.5.3 (Local Gauss-Bonnet)**.** Let M be an oriented Riemannian manifold of dimension 2, and let $\triangle x_1x_2x_3$ be a geodesic triangle in a coordinate chart U which is diffeomorphic to $\mathbb{R}^2$, with external angles $\alpha_i = \pi - \beta_i$ (where β_i are the internal angles) at x_i for $i = 1, 2, 3$. Let k be the scalar (=sectional) curvature function of M, and let dA denote the volume (=area) form on M. Then:

$$\int_\triangle k dA = 2\pi - \sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i - \pi$$

where the left side denotes the integral over the region bounded by σ_i.

Proof: We let $\sigma_1, \sigma_2, \sigma_3$ denote the geodesic segments x_1x_2, x_2x_3, x_3x_1 respectively. We let $c : S^1 \to M$ denote the PS curve which is σ_i on the segments $[0, s_1], [s_1, s_2], [s_2, 2\pi]$ of $[0, 2\pi]$ respectively. For simplicity we assume σ_i are parametrised by arc length, so that the arc length of c is 2π, and σ_i' is of unit length everywhere. We will denote the velocity vector (which is defined at all s except $0, s_1, s_2$) by T.

Let e_1, e_2 be an orthonormal frame field on U. Then, the connection form ω on U is a skew-symmetric 2×2 matrix with entry $\omega_{21} = -\omega_{12}$. Thus

$$\nabla_T e_1 = \omega_{21}(T)e_2 = -\omega_{12}(T)e_2$$

and hence $-\omega_{12}(T) = \langle \nabla_T e_1, e_2\rangle$. If we integrate the left side on the 1-chain $c = \sigma_1 + \sigma_2 + \sigma_3$, we note that U being diffeomorphic to $\mathbb{R}^2$, the simple closed curve c bounds a bounded region $\triangle$ (Jordan-Brouwer Separation

Theorem), so we get by Stokes formula, and (ii) of the previous Lemma 7.5.1 that

$$\begin{aligned} -\int_c \omega_{12} &= \int_{\triangle}(-d\omega_{12}) \\ &= -\int_{\triangle} kdA \\ &= \sum_i \int_{\sigma_i} \langle \nabla_T e_1, e_2 \rangle \end{aligned}$$

Now we need to understand the last expression. Since σ_i are geodesics, we have:

$$\begin{aligned} 0 = \nabla_T T &= \nabla_T \left[\langle T, e_1\rangle e_1 + \langle T, e_2\rangle e_2\right] \\ &= \sum_{i=1}^{2} (T(\langle T, e_i\rangle)e_i + \langle T, e_i\rangle \nabla_T e_i) \end{aligned}$$

so that, on taking inner-product with e_1, e_2, and noting that $\nabla_T e_2$ is a multiple of e_1, and $\nabla_T e_1$ is a multiple of e_2, we have the two equations:

$$\langle \nabla_T e_2, e_1\rangle \langle T, e_2\rangle + T(\langle T, e_1\rangle) = 0 \quad (38)$$

$$\langle \nabla_T e_1, e_2\rangle \langle T, e_1\rangle + T(\langle T, e_2\rangle) = 0 \quad (39)$$

Now since $\langle \nabla_T e_2, e_1\rangle = \omega_{12}(T) = -\langle \nabla_T e_1, e_2\rangle$, we multiply the first of the equations above with $\langle T, e_2\rangle$, the second with $\langle T, e_1\rangle$ and subtract the first from the second, and use $\langle T, e_1\rangle^2 + \langle T, e_2\rangle^2 = \|T\|^2 = 1$, to get:

$$\langle \nabla_T e_1, e_2\rangle = \langle T, e_2\rangle T(\langle T, e_1\rangle) - \langle T, e_1\rangle T(\langle T, e_2\rangle) = T_2\frac{dT_1}{ds} - T_1\frac{dT_2}{ds}$$

where $T_i := \langle T, e_i\rangle$. But the right hand side is precisely the pullback $T^*(ydx - xdy) = -T^*\alpha$, where $\alpha = xdy - ydx$ is the volume form of S^1, and $T : c \to S^1$ is the piecewise smooth map, taking $c(s)$ to $(T_1(s), T_2(s))$.

It is a result of Hopf (called the *Hopf Umlaufsatz*, see W. Klingenberg, *A Course in Differential Geometry*, p. 24) that the integral $\int_c T^*\alpha = 2\pi - \sum_{i=1}^{3} \alpha_i$. This proves the result. □

Proposition 7.5.4 (Global Gauss-Bonnet)**.** Let M be a compact oriented surface. Then:

$$\int_M kdA = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

Proof: It is a theorem (deep) that a surface as above can be triangulated via geodesic triangles. That is, there is a finite subset $V \subset M$ (called *vertices*), and triangles (homeomorphic to 2-simplices) called *faces* with vertices coming from V, such that M is the union of these triangles, and such that a pair of triangles can meet at either at a single vertex, or along a unique geodesic common edge, or not meet at all. Further, every geodesic edge is a common edge of exactly two faces. Finally, there exists a geodesic triangulation finer than any given open covering, in particular a covering by open charts.

Then, if we denote the number of vertices, edges, faces by v, e, f respectively, one definition of the *Euler characteristic* is $\chi(M) = v - e + f$. That it does not depend on the triangulation, and is a topological invariant follows from homology theory. If we count three edges to each face, then we would get $3f$, but in this counting, each edge would have been counted twice, since it is a common edge to two faces. Thus, $2e = 3f$, or $e = \frac{3f}{2}$.

Denote by F the set of faces. Then note that at each vertex, the sum of the internal angles β_i at x of all the faces having x as a vertex, is 2π. Thus by the previous Theorem 7.5.3, we have Thus:

$$\begin{aligned} \int_M kdA &= \sum_{\triangle x_1x_2x_3 \in F} (\beta_1 + \beta_2 + \beta_3 - \pi) \\ &= 2\pi v - \pi f = 2\pi(v - \frac{3f}{2} + f) \\ &= 2\pi(v - e + f) = \chi(M) \end{aligned}$$

thus proving the assertion. □

Remark 7.5.5. To compute $\chi(M)$, it is not necessary to have a geodesic triangulation, any triangulation will do. The result above says that there is a *global topological invariant* governing the total curvature (integral of the scalar curvature) of a surface, which is a *geometric* property. In particular, there is a topological obstruction to putting a metric of prescribed scalar curvature on a compact oriented manifold.

For example, since $\chi(S^2) = 2$, one cannot put a Riemannian metric on S^2 whose scalar curvature is everywhere ≤ 0. Similarly, on a torus T^2, whose Euler characteristic is zero, one cannot put any Riemannian metric whose scalar curvature is everywhere positive, or everywhere negative. It turns out that the result above generalises to oriented Riemannian manifolds of all even dimension. Odd dimensional oriented Riemannian manifolds have Euler characteristic zero.

Exercise 7.5.6. Let M be a compact surface in $\mathbb{R}^3$, with Riemannian metric induced from the Euclidean metric in $\mathbb{R}^3$. Prove that there exists a point $p \in M$ such that the second fundamental form $S_p(-,-)$ is positive definite. Consequently, both mean and scalar curvatures at p are strictly positive. As a corollary, no compact surface of everywhere non-positive curvature embeds isometrically in $\mathbb{R}^3$. (*Hint:* Assume without loss that $0 \notin M$, and by compactness of M, find a point $p \in M$ where the function $f(x) = \|x\|^2$ on M reaches a global maximum. Using a chart around p, and the usual necessary conditions for p to be a local maximum for f, show that $S_p(X,X) \geq \langle X, X \rangle$ for all $X \in T_p(M)$. You may appeal to the Corollary 7.4.4 to prove this).

8. Jacobi Fields

8.1. **Variations of geodesics.** Let M be a Riemannian manifold, and $x \in M$ be a point. Let $\gamma : [0,a] \to M$ be a geodesic (parametrised by arc-length), with $\gamma(0) = x$. A *variation of geodesics* is a smooth map:

$$\begin{array}{rcl} \phi : (-\delta, \delta) \times [0,a] & \to & M \\ (s,t) & \mapsto & \phi(s,t) \end{array}$$

such that $\phi(0,t) = \gamma(t)$ and for each s, $\gamma_s := \phi(s,)$ is a geodesic. For example, if $\gamma(t) = \mathrm{Exp}_x(tX)$ for some tangent vector $X \in T_x(M)$, and if $W \in T_x(M)$ is another tangent vector, we can consider the family γ_s of geodesics:

$$\gamma_s(t) = \phi(s,t) := \mathrm{Exp}_x(tX + sW)$$

parametrised by $s \in (-\delta, \delta)$, with $\gamma_0 = \gamma$, and $\gamma_s(0) = \mathrm{Exp}_x(sW)$.

Definition 8.1.1 (Jacobi Field of a variation)**.** By the *Jacobi field of the variation* ϕ, we mean the vector field along γ defined as:

$$J(t) := \frac{\partial \phi(s,t)}{\partial s}_{|s=0}$$

Proposition 8.1.2. The Jacobi field $J(t)$ satisfies the following 2nd- order linear differential equation:

$$\frac{D^2 J}{dt^2} + R(J,T)T = 0$$

where $T(t) := \dot{\gamma}(t)$. The Jacobi field $J(t)$ is uniquely determined by the values $J(0)$ and $J(a)$.

Proof: To get the differential equation, note that the vector fields $\partial_s := \phi_*(s,t)(\partial_s)$ and $\partial_t := \phi_*(s,t)(\partial_t)$, which are vector fields "along" $(-\delta,\delta) \times (0,a)$ (i.e. smooth sections of ϕ^*TM) commute (why?). Note also that the restrictions of ∂_s and ∂_t to $(0) \times (0,a)$ are precisely $J(t)$ and $T(t)$ respectively. In particular, J and T commute. Thus:

$$\begin{array}{rcl} R(\partial_s, \partial_t)\partial_t & = & \frac{D}{ds}\frac{D\partial_t}{dt} - \frac{D}{dt}\frac{D\partial_t}{ds} \\ & = & -\frac{D}{dt}\frac{D\partial_t}{ds} \end{array}$$

But $\frac{D\partial_t}{ds} - \frac{D\partial_s}{dt} = [\partial_s, \partial_t] = 0$, so that we have, on setting $s = 0$, the relation:

$$\frac{D^2 J}{dt^2} + R(J,T)T = 0$$

which is the required ODE. The uniqueness given the boundary conditions follows from the existence and uniqueness theorem of linear 2nd-order ODE's. □

Definition 8.1.3 (Jacobi Field)**.** Let $\gamma : [0,a] \to M$ be a geodesic, and let J be a smooth vector field along γ. Then we say J is a *Jacobi field* if J satisfies:

$$\frac{D^2 J}{dt^2} + R(J,T)T = 0$$

where $T = \dot{\gamma}(t)$. Clearly, J is uniquely determined by either (i) $J(0)$ and $J(a)$ or (ii) $J(0)$ and $\frac{DJ}{dt}(0)$.

More generally, we shall say J is a *broken Jacobi field* if there exists a subdivision $0 = t_0 < t_1 < t_2 < ... < t_k = a$ of $[0,a]$ such that $J_{|[t_i,t_{i+1}]}$ is a smooth Jacobi field for each i. We do not require J to be even continuous on $[0,a]$. If J is a broken Jacobi field along γ, it is uniquely determined by $J(0)$, $J(a)$, $J(t_i+)$ and $J(t_i-)$ for $i = 1,..,k-1$.

Remark 8.1.4 (Broken vs smooth Jacobi fields)**.** Let J be a broken Jacobi field along a geodesic γ, and in the notation of the definition above, assume that for each i, we have (i)$J(t_i+) = J(t_i-)$, and (ii)$\frac{DJ}{dt}(t_i+) = \frac{DJ}{dt}(t_i-)$. Then the broken Jacobi field J is smooth. This is clear, by the hypothesis above, the 0-th and 1-st order derivatives from the left and right agree at t_i, and by the ODE of 8.1.2, all derivatives of J of order ≥ 2 from the left and right also agree at the points t_i.

Lemma 8.1.5. Let $\gamma : [0,a] \to M$ be a geodesic, with $\gamma(0) = x$. Then every Jacobi field along γ is the Jacobi field of some geodesic variation.

Proof: Let J be a Jacobi field along γ. We first assume $\gamma(t) = \mathrm{Exp}_x(tX)$. This is possible if a is small enough, for example such that γ lies entirely in a neighbourhood U of x such that $\mathrm{Exp}_x : U_1 \to U$ is a diffeomorphism for some neighbourhood U_1 of 0 in T_xM, and $aX \in U_1$.

Let $\rho : (-\delta, \delta) \to M$ be a path such that $\rho(0) = x$, and $\dot{\rho}(0) = J(0)$. Let $X(s)$ be a smooth vector field along ρ such that:

$$\begin{aligned} X(0) &= X \\ \frac{DJ}{dt}(0) &= \frac{DX}{ds}(0) \end{aligned} \tag{40}$$

(prove that such a vector field exists). Consider the variation:

$$\phi(s,t) = \mathrm{Exp}_{\rho(s)}(tX(s))$$

Note that for a fixed s, $\phi(s,-)$ is a geodesic, so that ϕ is a geodesic variation. Clearly, by (40), $\phi(0,t) = \mathrm{Exp}_x(tX) = \gamma(t)$. Also,

$$\frac{\partial\phi(s,0)}{\partial s} = \frac{\partial \mathrm{Exp}_{\rho(s)}(0)}{\partial s} = \dot{\rho}(s) \tag{41}$$

Now, we also have

$$\frac{\partial\phi}{\partial t}(s,0) = X(s) = \partial_t(s,0)$$

so that $\frac{D\partial_t(s,0)}{ds} = \frac{DX(s)}{ds}$. But since $[\partial_s, \partial_t] = 0$, we have:

$$\frac{D\partial_s(s,0)}{dt} = \frac{D\partial_t(s,0)}{ds} = \frac{DX(s)}{ds}$$

Denote the Jacobi field of the variation ϕ by $J_1(t) = \partial_s(0,t)$. Since $\frac{D\partial_s(0,t)}{dt} = \frac{DJ_1}{dt}(t)$, the above calculation implies $\frac{DX(s)}{ds}(0) = \frac{DJ_1}{dt}(0)$. But by (40), $\frac{DX(s)}{ds}(0) = \frac{DJ}{dt}(0)$, so we have:

$$\frac{DJ_1}{dt}(0) = \frac{DJ}{dt}(0) \tag{42}$$

Both J and J_1 satisfy the 2nd order Jacobi ODE of 8.1.2. The equation (41) implies that $J_1(0) = \dot{\rho}(0) = J(0)$. Combined with (42), the uniqueness part of the ODE theorem implies that $J(t) \equiv J_1(t)$ for all $t \in [0,a]$.

In the general case, we break $[0,a]$ into a partition $0 = t_0 < t_1 < .. < t_k = a$ such that each segment $[t_i, t_{i+1}]$ is a small geodesic segment of the kind considered above. We construct geodesic variations over each segment $[t_i, t_{i+1}]$, and patch them up. The details are left to the reader. □

Lemma 8.1.6 (Second Variation Formula)**.** Let $\gamma : [0,a] \to M$ be a continuous curve, and let $0 = t_0 < t_1 < ... < t_k = a$ be a partition of $[0,a]$ such that $\gamma_{|[t_i,t_{i+1}]}$ is a smooth geodesic, for each $0 \leq i \leq k-1$. (That is, γ is a *broken geodesic*), with velocity field $T(t)$ on the intervals of smoothness $[t_i, t_{i+1}]$. Let:

$$\begin{aligned} \phi : (-\delta,\delta) \times (-\epsilon,\epsilon) \times [0,a] &\to M \\ (r,s,t) &\mapsto \phi(r,s,t) \end{aligned}$$

be a two parameter variation which is continuous, and smooth on $(-\delta,\delta)\times(-\epsilon,\epsilon)\times[t_i,t_{i+1}]$ for each $0\leq i\leq k-1$, and such that $\phi(0,0,t)=\gamma(t)$. Let $Y(t)$ and $Z(t)$ denote the (piecewise smooth) vector fields along γ defined by $\frac{\partial\phi}{\partial r}(0,0,t)$ and $\frac{\partial\phi}{\partial s}(0,0,t)$ respectively. Denote by:

$$E(r,s):=\sum_{i=0}^{k-1}\left(\int_{t_i}^{t_{i+1}}\langle\partial_t\phi(r,s,t),\partial_t\phi(r,s,t)\rangle\,dt\right)$$

the energy functional. Then,

$$\frac{1}{2}E_{**}(Y,Z):=\frac{1}{2}\frac{\partial^2 E}{\partial r\partial s}(0,0)=$$

Proof:

We note that the vector fields "along" $(-\delta,\delta)\times(-\epsilon,\epsilon)\times[t_i,t_{i+1}]$ defined by $\partial_r:=\frac{\partial\phi}{\partial r}$, $\partial_s:=\frac{\partial\phi}{\partial s}$, and $\partial_t:=\frac{\partial\phi}{\partial t}$ are smooth sections of ϕ^*TM, and commute, for each $0\leq i\leq k-1$. Differentiating once:

$$\frac{\partial E(r,s)}{\partial s}=2\sum_i\int_{t_i}^{t_{i+1}}\left\langle\frac{D\partial_t}{\partial s},\partial_t\right\rangle dt$$

So that

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 E(r,s)}{\partial r\partial s}&=\sum_i\left(\int_{t_i}^{t_{i+1}}\left\langle\frac{D\partial_t}{\partial s},\frac{D\partial_t}{\partial r}\right\rangle dt+\int_{t_i}^{t_{i+1}}\left\langle\frac{D}{\partial r}\frac{D\partial_t}{\partial s},\partial_t\right\rangle dt\right)\\&=\sum_i\left(\int_{t_i}^{t_{i+1}}\left\langle\frac{D\partial_s}{\partial t},\frac{D\partial_r}{\partial t}\right\rangle dt+\int_{t_i}^{t_{i+1}}\left\langle\frac{D}{\partial r}\frac{D\partial_s}{\partial t},\partial_t\right\rangle dt\right)\\&=\sum_i\left(\int_{t_i}^{t_{i+1}}\left(\frac{d}{dt}\left\langle\partial_r,\frac{D\partial_s}{\partial t}\right\rangle-\left\langle\partial_r,\frac{D^2\partial_s}{\partial t^2}\right\rangle\right)dt+\int_{t_i}^{t_{i+1}}\left\langle\frac{D}{\partial r}\frac{D\partial_s}{\partial t},\partial_t\right\rangle dt\right)\end{aligned}$$

Now denote the "jumps":

$$\Delta_i\left\langle Y,\frac{DZ}{\partial t}\right\rangle:=\left\langle Y(t_i+),\frac{DZ}{\partial t}(t_i+)\right\rangle-\left\langle Y(t_i-),\frac{DZ}{\partial t}(t_i-)\right\rangle$$

and use the relation

$$\frac{D}{\partial r}\frac{D\partial_s}{\partial t}=\frac{D}{\partial t}\frac{D\partial_s}{\partial r}+R(\partial_r,\partial_t)\partial_s$$

and observe that:

$$\left\langle\frac{D}{\partial t}\frac{D\partial_s}{\partial r},\partial_t\right\rangle=\frac{d}{dt}\left\langle\frac{D\partial_s}{\partial r},\partial_t\right\rangle-\left\langle\frac{D\partial_s}{\partial r},\frac{D\partial_t}{\partial t}\right\rangle$$

so that when $s=r=0$, $\partial_t=T$, and since $\nabla_T T=0$, we have:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 E}{\partial r\partial s}(0,0)&=\left\langle Y(a),\frac{DZ}{\partial t}(a)\right\rangle-\left\langle Y(0),\frac{DZ}{\partial t}(0)\right\rangle-\sum_{i=1}^{k-1}\Delta_i\left\langle Y,\frac{DZ}{\partial t}\right\rangle\\&-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left(\left\langle Y,\frac{D^2Z}{\partial t^2}\right\rangle-\langle T,R(Y,T)Z\rangle\right)\end{aligned}$$

Now, from the relation $\langle T,R(Y,T)Z\rangle=\langle T,R(Z,T)Y\rangle=-\langle Y,R(Z,T)T\rangle$ (identities (iii) and (iv) of the proposition 7.3.7), we finally have:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 E}{\partial r\partial s}(0,0)&=\left\langle Y(a),\frac{DZ}{\partial t}(a)\right\rangle-\left\langle Y(0),\frac{DZ}{\partial t}(0)\right\rangle-\sum_{i=1}^{k-1}\Delta_i\left\langle Y,\frac{DZ}{\partial t}\right\rangle\\&-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left(\left\langle Y,\frac{D^2Z}{\partial t^2}+R(Z,T)T\right\rangle\right)\end{aligned}$$

and the proposition follows. $\square$