Uncertainty Principle

The uncertainty principle in Quantum Mechanics says that position and momentum cannot be simultaneously localized. There are many mathematical formulations of this principle. Here are a few:

1. If f is a unit vector in $L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ then $\int (x-a)^2 |f(x)|^2 \, dx \int (y-b)^2 \left|\hat{f}(y)\right|^2 dy \geq \frac{1}{4}$.

2. Let $f \in L^1$. Then f and $\hat{f}$ cannot both have compact support

3. If f is a non-zero element of $L^2(\mathbb{R})$ then $m\{x : f(x) \neq 0\}$ and $m\{x : \hat{f}(x) \neq 0\}$ cannot both be finite.

4. If f is a measurable function such that $|f(x)| \leq Ae^{-\alpha x^2}$ and $\left|\hat{f}(y)\right| \leq Be^{-\beta y^2}$ for all x where α, β are positive numbers with $\alpha\beta > 1$ then $f = 0$ a.e.

5. (Beurling) $f \in L^2$, $\int\int |f(x)| \left|\hat{f}(x)\right| e^{|xy|} dx dy < \infty$ implies $f = 0$ a.e..

Proof of 1: we prove below that if the left side of the inequality is finite then f is absolutely continuous, $f' \in L^2$ and $\hat{f'}(x) = -ix\hat{f}(x)$ a.e.. Assuming this we have

$$\int x^2 |f(x)|^2 \, dx \int y^2 \left|\hat{f}(y)\right|^2 dy = \int x^2 |f(x)|^2 \, dx \int \left|-yi\hat{f}(y)\right|^2 dy = \int x^2 |f(x)|^2 \, dx \int |f'(y)|^2 \, dy$$

(because $\|f'\|_2^2 = \left\|\hat{f'}\right\|_2^2$). Thus $\int x^2 |f(x)|^2 \, dx \int y^2 \left|\hat{f}(y)\right|^2 dy \geq (\int \left|xf'(x)\bar{f}(x)\right| dx)^2$.

Now note that

$$x(f(x)\bar{f}(x))' = xf'(x)\bar{f}(x) + xf(x)\bar{f}'(x) = 2x \operatorname{Re}\{f'(x)\bar{f}(x)\} \geq -2\left|xf'(x)\bar{f}(x)\right|.$$

Hence $\int_\alpha^\beta \left|xf'(x)\bar{f}(x)\right| dx \geq -\frac{1}{2}\int_\alpha^\beta x(f(x)\bar{f}(x))' dx = -\frac{1}{2}xf(x)\bar{f}(x)|_{-\alpha}^{\beta} + \frac{1}{2}\int_\alpha^\beta f(x)\bar{f}(x) dx$.

Since the integrals on both sides converge in L^2 as $\alpha \to -\infty$ and $\beta \to \infty$ it follows that $xf(x)\bar{f}(x)|_{-\alpha}^{\beta}$ also converges to a finite limit as $\alpha \to -\infty$ and $\beta \to \infty$. This limit has to be 0 because, otherwise, $|x| |f(x)|^2$ is bounded below which contradicts the fact that $f \in L^2$. Now $\int x^2 |f(x)|^2 \, dx \int y^2 \left|\hat{f}(y)\right|^2 dy \geq (\int \left|xf'(x)\bar{f}(x)\right| dx)^2 \geq \frac{1}{4}\int f(x)\bar{f}(x) dx = \frac{1}{4}$.

Lemma

If $f \in L^2$ and $\int y^2 \left|\hat{f}(y)\right|^2 dy < \infty$ then f is absolutely continuous, $f' \in L^2$ and $\hat{f'}(x) = ix\hat{f}(x)$ a.e..

Proof of the lemma: let $\phi(x) = \int\limits_0^x g(t)dt$ where $g \in L^2$ is such that $\hat{g}(y) = iy\hat{f}(y)$. Such a function g exists because $iy\hat{f}(y) \in L^2$. Assume that $\phi \in L^2$. This fact is established below. [See Dym and McKean's proof of Heisenberg's inequality] Now $\int_\alpha^\beta e^{-itx}\phi(x)dx = \int_\alpha^\beta e^{-itx} \int\limits_0^x g(t)dtdx = \frac{e^{-itx}}{-it}\int\limits_0^x g(t)dt|_\alpha^\beta - \int_\alpha^\beta \frac{e^{-itx}}{-it} g(x)dx$.
Noting that $\int_\alpha^\beta e^{-itx}\phi(x)dx$ and $\int_\alpha^\beta \frac{e^{-itx}}{-it} g(x)dx$ converge in L^2 norm as $\alpha \to -\infty$ and $\beta \to \infty$ it follows that $\frac{e^{-itx}}{-it}\int\limits_0^x g(t)dt|_\alpha^\beta$ also converges as $\alpha \to -\infty$ and $\beta \to \infty$. Hence $\frac{e^{-itx}}{-it}\int\limits_0^x g(t)dt|_\alpha^\beta$ also converges as $\alpha \to -\infty$ and $\beta \to \infty$ to an L^2 function (of t). If this limit is not zero on a set of positive measure then $e^{-it\beta}$ and $e^{-it\alpha}$ converge for all t in a set of positive measure which is false. This gives $\hat{\phi}(t) = \frac{\hat{g}(t)}{it} = \hat{f}(t)$ a.e.. So $\phi = f$ a.e. which implies that f is absolutely continuous and $f' = g \in L^2$. Also $\hat{f'}(t) = \hat{g}(t) = it\hat{f}(t)$. This proves the lemma.

Back to the proof of the theorem:

Now let $g(x) = f(x+a)e^{-ibx}$. Then $\|g\|_2 = 1$ and we have $\int x^2 |g(x)|^2 dx \int y^2 \left|\hat{g}(y)\right|^2 dy \geq \frac{1}{4}$. This gives $\int (x-a)^2 |f(x)|^2 dx \int (y-b)^2 \left|\hat{g}(y-b)\right|^2 dy \geq \frac{1}{4}$. But $\hat{g}(t) = \hat{f}(t+b)$ so $\int (y-b)^2 \left|\hat{g}(y-b)\right|^2 dy = \int (y-b)^2 \left|\hat{f}(y)\right|^2 dy$ and this completes the proof when f a C^∞ function with compact support.

Property 2 is easy: the Fourier inversion formula shows that f extends to an entire function and hence its zeros are isolated.

Property 3 can be proved using Poisson Summation Formula. (See my notes)

Property 4 follows from Property 5. We do not prove Property 5 here. A reference for this proof is: Lars Hormander, "A uniqueness Theorem Of Beurling for Fourier Transform Pairs", Ark. Math, 29, 237-240.

Proof of Heisenberg's inequality from Dym and McKean:

Let $f \in L^2$ and $\int x^2 |f(x)|^2 dx < \infty, \int y^2 \left|\hat{f}(y)\right|^2 dy < \infty$. Note that $\hat{f} \in L^1$ because $(\int_{\{|y|>1\}} \left|\hat{f}(y)\right| dy)^2 \leq \int_{\{|y|>1\}} y^2 \left|\hat{f}(y)\right| dy \int_{\{|y|>1\}} \frac{1}{y^2} dy < \infty$ and $\hat{f} I_{\{|y|\leq 1\}} \in L^2([-,1,1]) \subset L^1([-,1,1])$. Thus f is a continuous function. There exists a sequence $\{\alpha_n\} \to \infty$ such that $\alpha_n |f(\alpha_n)|^2 + \alpha_n |f(-\alpha_n)|^2 \to 0$. For, otherwise, $\liminf_{x\to\infty}\{x|f(x)|^2 + x|f(-x)|^2\} > 0$ and $\int\{|f(x)|^2+|f(-x)|^2\}dx = \infty$ which is a contradiction. We claim that there is a sequence $\{f_n\}$ in $\mathcal{S}$ such that $\int (1+y^2)\left|\hat{f_n}(y) - \hat{f}(y)\right|^2 dy \to 0$. For this just note that C^∞ functions with compact support are dense in $L^2((1+y^2)dy)$ and any C^∞ function with compact support is the Fourier transform of some function in $\mathcal{S}$. Let g be a function in L^2 such that $\hat{g}(y) = iy\hat{f}(y)$. Such a function exists because $iy\hat{f}(y) \in L^2$. Note that $\|f_n - f\|_2^2 + \|f_n' - g\|_2^2 = \|f_n - f\|_2^2 + \left\|(f_n')\hat{} - \hat{g}\right\|_2^2 = \left\|\hat{f_n}(y) - \hat{f}(y)\right\|_2^2 + \left\|iy\hat{f_n}(y) - iy\hat{f}(y)\right\|_2^2 = \int (1+y^2)\left|\hat{f_n}(y) - \hat{f}(y)\right|^2 dy \to 0$. Thus, $f_n \to f$ and $f_n' \to g$ in L^2.

Also, since $\hat{f} \in L^1$ and $\hat{f_n} \in L^1$ $|f_n(x) - f(x)|^2 = (\frac{1}{\sqrt{2\pi}}\left|\int e^{itx}(\hat{f_n}(y) - \hat{f}(y))\right| dt)^2 \leq \frac{1}{\sqrt{2\pi}}(\int (1+y^2)\left|\hat{f_n}(y) - \hat{f}(y)\right|^2)(\int \frac{1}{1+y^2}dy)$. Thus, $f_n \to f$ uniformly.

Remark: use these facts we prove that the function ϕ defined in the earlier proof is indeed an L^2 function: Clearly $\int_0^x f_n'(t)dt \to \int_0^x g(t)dt$ for each x. Hence $\phi(x) = \lim[f_n(x) - f_n(0)] = f(x) - f(0)$. If $f(0) = 0$ it follows that $\phi = f \in L^2$. For the general case let $f_1(x) = f(x) - f(0)e^{-x^2/2}$. Then $f_1(0) = 0, f_1 \in L^2$ and $y\hat{f_1}(y) = y\hat{f}(y) - f(0)e^{-t^2/2} \in L^2$. If the lemma above holds in the special case $f(0) = 0$ we can conclude that f_1 is is absolutely continuous, $f_1' \in L^2$ and $\hat{f_1'}(x) = ix\hat{f_1}(x)$ a.e.. This shows that f is is absolutely continuous, $f' \in L^2$ and $\hat{f'}(x) = ix\hat{f}(x)$ a.e..

Back to Dym and McKean's proof: $\int x^2|f(x)|^2 dx \int y^2\left|\hat{f}(y)\right|^2 dy = \int x^2|f(x)|^2 dx \int \left|\hat{g}(y)\right|^2 dy = \int x^2|f(x)|^2 dx \int |g(y)|^2 dy$

$\geq (\int \left|xf(x)\overline{g(x)}\right| dx)^2$. (1)

Now $\int x[g(x)\overline{f(x)} + f(x)\overline{g(x)}]dx = \lim_{n\to\infty}\int_{-\alpha_n}^{\alpha_n} x[g(x)\overline{f(x)} + f(x)\overline{g(x)}]dx$

$= \lim_{n\to\infty}\lim_{k\to\infty}\int_{-\alpha_n}^{\alpha_n} x[f_k'(x)\overline{f_k(x)} + f_k(x)\overline{f_k'(x)}]dx$ (because $f_k \to f$ and $f_k' \to g$

in L^2). Hence $\int x[g(x)\overline{f(x)} + f(x)\overline{g(x)}]dx = \lim_{n\to\infty}\lim_{k\to\infty}\int_{-\alpha_n}^{\alpha_n} x[|f_k|^2]'(x)dx =$ $\lim_{n\to\infty}\lim_{k\to\infty}\{x\,|f_k|^2\,(x)|_{-\alpha_n}^{\alpha_n} - \int_{-\alpha_n}^{\alpha_n} |f_k(x)|^2\,dx] = -1$ since $\|f\|_2 = 1$ and $\lim_{n\to\infty}\lim_{k\to\infty}\alpha_n\,|f_k|^2\,(\alpha_n) =$ $\lim_{n\to\infty}\alpha_n\,|f|^2\,(\alpha_n) = 0$ and $\lim_{n\to\infty}\lim_{k\to\infty}\alpha_n\,|f_k|^2\,(-\alpha_n) = \lim_{n\to\infty}\alpha_n\,|f|^2\,(-\alpha_n) = 0$ by our choice of $\{\alpha_n\}$. We now get $\frac{1}{2} = -\frac{1}{2}\int x[g(x)\overline{f(x)} + f(x)\overline{g(x)}]dx =$ $-\,\mathrm{Re}\int x[g(x)\overline{f(x)}dx \leq \int \left|xg(x)\overline{f(x)}\right| dx$ and $\frac{1}{4} \leq (\int \left|xg(x)\overline{f(x)}\right| dx)^2 \leq \int x^2\,|f(x)|^2\,dx \int y^2 \left|\hat{f}(y)\right|^2 dy$ by (1). Equality holds if and only if $g(x) = xf(x)$ a.e. implies $\int_0^x f_n'(t)dt \to$ $\int_0^x g(t)dt = \int_0^x tf(t)dt$ and so $f(x) - f(0) = \int_0^x tf(t)dt$ for all x which implies $f(x) = ce^{dx^2}$ for some real numbers c and d. Of course, $d < 0$ because $f \in L^2$. Conversely if f is of this type then equality holds in Heisenberg's inequality. The inequality $\int (x-a)^2\,|f(x)|^2\,dx \int (y-b)^2 \left|\hat{f}(y)\right|^2 dy \geq \frac{1}{4}$ follows be changing f to $f(x+a)e^{-ibx}$.

Alternative proof of the lemma above viz.:

If $f \in L^2$ and $\int y^2 \left|\hat{f}(y)\right|^2 dy < \infty$ then f is absolutely continuous, $f' \in L^2$ and $\hat{f'}(x) = ix\hat{f}(x)$ a.e..

We have $f_n(x) = f_n(0) + \int_0^x f_n'(t)dt$. This gives $f(x) = f(0) + \int_0^x g(t)dt$ since $f_n \to f$ pointwise and $f_n' \to g$ in L^2. This completes the proof!