Applications of de Branges-Rovnyak decomposition to Graph Theory \(^1\)

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Reminiscences of Bieberbach conjecture

\[ \mathbb{D} := \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \]

\[ S := \{ f \in \text{Hol}(\mathbb{D}) : f \text{ is injective and } f(z) = z + \sum_{n=2}^{\infty} c_n z^n \} \]

**Bieberbach conjecture**

- If \( f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S \), then \( |c_n| \leq n \)?

- In 1984, de Branges gave the solution with Hilbert space operator theory.

- However, since his original proof was very complicated, his operator theory method has been forgotten.
In this talk

- We deal with increasing sequences of graphs from the viewpoint of Hilbert space operator theory.

- As results, two different types of inequality are given.

- Our scheme gives a toy model of de Branges’ solution to the Bieberbach conjecture (in fact, this is my motivation).
Preliminaries from Graph theory

Graph
We deal with simple graphs (no loops, no multi-edges and no direction).

\[ V = \{1, 2, 3, 4\}, \quad E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}, \quad G = (V, E) \]

Laplace matrix

\[
L = \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 2 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}
\]  

(→ spectral graph theory).
Inequality 1

Setting

- $V$: a finite set of vertices (fixed),

- $G_j = (V, E_j)$: connected and simple graphs

  \[ G_1 \subset \cdots \subset G_n \quad \text{(i.e. } E_1 \subset \cdots \subset E_n \text{)} \]

- $\gamma(G)$: the number of connected components of $G$. 

We have an operator theory proof of this inequality.
Inequality 1

Setting

- $\mathcal{V}$: a finite set of vertices (fixed),
- $G_j = (\mathcal{V}, E_j)$: connected and simple graphs
  
  \[ \text{s.t. } G_1 \subset \cdots \subset G_n \quad (\text{i.e. } E_1 \subset \cdots \subset E_n). \]
- $\gamma(G)$: the number of connected components of $G$.

Inequality for $\gamma$

\[ \sum_{j=1}^{n-1} \gamma(G_{j+1} - G_j) \leq \gamma(G_n - G_1) + (n - 2)|\mathcal{V}| \]

\[ (G_{j+1} - G_j := (\mathcal{V}, E_{j+1} \setminus E_j)). \]

- We have an operator theory proof of this inequality.
Inequality 2

Setting

- \( V \): a finite set (fixed),
- \( G_j = (V, E_j) \) \((j = 0, 1)\): connected and simple graphs s.t. \( G_0 \subseteq G_1 \) (i.e. \( G_0 \) is a subgraph of \( G_1 \))
- \( L_j \): Laplace matrix of \( G_j \),
- \( K_j = (P + L_j)^{-1} \) (where \( P := \text{proj ker } L_j \) in \( \ell^2(V) \))
Inequality 2

Setting

- $V$: a finite set (fixed),

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- $L_j$: Laplace matrix of $G_j$,

- $K_j = (P + L_j)^{-1}$ (where $P := \text{proj ker } L_j \text{ in } \ell^2(V)$)

Trivial observation

- $G_0 \subset G_1 \Rightarrow L_0 \leq L_1 \iff P + L_0 \leq P + L_1 \iff K_0 \geq K_1.$

- Many graph theorists are interested in spectral property of $L$. We shall improve $K_0 \geq K_1$ in the next page.
Inequality 2

Theorem (S-Suda)

If \( G_0 \subset G_1 \) (simple, connected and having the same vertex set), then \( \forall c \in \ell^2(V) \)

\[
0 \leq \langle L_0(K_0 - K_1)\tilde{c}, (K_0 - K_1)\tilde{c} \rangle_{\ell^2(V)} \leq \langle (K_0 - K_1)c, c \rangle_{\ell^2(V)},
\]

where

\[
\tilde{c} := \frac{1}{|G|} \sum_{g \in G} c \circ g,
\]

the averaged vector of \( c \) with respect to

\( G = \text{Aut}(G_0) \cap \text{Aut}(G_1). \)
Remarks

- These two inequalities are derived from de Branges-Rovnyak theory.

- Our scheme gives general method for finding inequalities (but it is rather complicated).

- There is a proof of Inequality 1 with graph theory (Ozeki).

- We have a simple proof of Inequality 2 without de Branges-Rovnyak theory.
Our idea

Hilbert space $\mathcal{H}_G$

- For functions $u$ and $v$ on $V$ (in fact, $u$ and $v$ are vectors),

$$\langle u, v \rangle_{\mathcal{H}_G} := \langle (l_{\ell^2(V)} + L_G)u, v \rangle_{\ell^2(V)} \quad (L_G: \text{Laplacian of } G).$$

- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$
Our idea

Hilbert space $\mathcal{H}_G$

- For functions $u$ and $v$ on $V$ (in fact, $u$ and $v$ are vectors),

$$\langle u, v \rangle_{\mathcal{H}_G} := \langle (I_{\ell^2(V)} + L_G)u, v \rangle_{\ell^2(V)} \quad (L_G: \text{Laplacian of } G).$$

- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$

Translation from $G$ to $\mathcal{H}$

$$G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n$$

$$\downarrow$$

$$\mathcal{H}_{G_1} \hookleftarrow \mathcal{H}_{G_2} \hookleftarrow \cdots \hookleftarrow \mathcal{H}_{G_{n-1}} \hookleftarrow \mathcal{H}_{G_n}$$

Can this sequence be telescoped?

(Note: $\mathcal{H}_{G_{j+1}}$ is not a Hilbert subspace of $\mathcal{H}_{G_j}$)
Our idea

Hilbert space $\mathcal{H}_G$

- For functions $u$ and $v$ on $V$ (in fact, $u$ and $v$ are vectors),
  $$\langle u, v \rangle_{\mathcal{H}_G} := \langle (\ell^2(V) + L_G)u, v \rangle_{\ell^2(V)} \quad (L_G: \text{Laplacian of } G).$$
- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$

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Can this sequence be telescoped?
(Note: $\mathcal{H}_{G_{j+1}}$ is not a Hilbert subspace of $\mathcal{H}_{G_j}$)

Our answer
Use de Branges-Rovnyak theory.
A review of de Branges-Rovnyak theory

Pull-back construction

- $\mathcal{H}, \mathcal{K}$: Hilbert spaces
- $T: \mathcal{H} \to \mathcal{K}$ any bounded linear operator,
- $\langle Tx, Ty \rangle_T := \langle Px, Py \rangle_\mathcal{H}$ $(P := P_{(\ker T)\perp})$,
- $\mathcal{M}(T) := (T\mathcal{H}, \langle \cdot, \cdot \rangle_T)$ is a Hilbert space

$\therefore$ $T\mathcal{H} \cong \mathcal{H}/\ker T \cong (\ker T)\perp$. 
A review of de Branges-Rovnyak theory

Fundamental theorem

If $T : \mathcal{H} \to \mathcal{K}$ and $\|T\| \leq 1$, then

1. $\mathcal{K} = \mathcal{M}(T) + \mathcal{H}(T)$ (where $\mathcal{H}(T) := \mathcal{M}(\sqrt{I_{\mathcal{K}} - TT^*})$),

2. $\|z\|_{\mathcal{K}}^2 \leq \|x\|_{\mathcal{M}(T)}^2 + \|y\|_{\mathcal{H}(T)}^2$ if $z = x + y \in \mathcal{M}(T) + \mathcal{H}(T)$

3. $\forall z \in \mathcal{K} \quad \exists! x_z \in \mathcal{M}(T) \quad \exists! y_z \in \mathcal{H}(T)$

s.t. $z = x + y$ and $\|z\|_{\mathcal{K}}^2 = \|x_z\|_{\mathcal{M}(T)}^2 + \|y_z\|_{\mathcal{H}(T)}^2$

• See Ando’s lecture notes or Sarason’s red book for details.
How to use de Branges-Rovnyak theory in graph theory

Increasing family of graphs
\[ G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n \]

Embedding of Hilbert spaces
\[ \mathcal{H}_{G_1} \leftarrow T_{1,2} \mathcal{H}_{G_2} \leftarrow T_{2,3} \cdots \leftarrow T_{n-2,n-1} \mathcal{H}_{G_{n-1}} \leftarrow T_{n-1,n} \mathcal{H}_{G_n} \]

Telescoping
\[ T_1 := I_{\mathcal{H}_{G_1}}, T_{j+1} := T_j T_{j,j+1}, \]
\[ \mathcal{H}(T_n) = \sum_{j=1}^{n-1} \mathcal{M}(\sqrt{T_j T_j^* - T_{j+1} T_{j+1}^*}). \]

Trivial estimate
\[ \dim \mathcal{H}(T_n) \leq \sum_{j=1}^{n-1} \dim \mathcal{M}(\sqrt{T_j T_j^* - T_{j+1} T_{j+1}^*}) \]
\[ (\Rightarrow \text{Inequality 1}). \]
How to use de Branges-Rovnyak theory in graph theory

Time evolution of graphs
\[ G_0 \subset G_1 \rightarrow G_0 \subset G_r \subset G_t \subset G_1 \ (0 \leq r \leq t \leq 1). \]

Continuous chain of Hilbert spaces
\[ \mathcal{H}_{G_0} \leftrightarrow \mathcal{H}_{G_r} \leftrightarrow \mathcal{H}_{G_t} \leftrightarrow \mathcal{H}_{G_1} \ (0 \leq r \leq t \leq 1). \]

Quasi-orthogonal integrals

\[ \mathcal{H}(T_{rt}) = \int_r^t \mathcal{M}(T_{rs}\Delta(s)) \ ds \ (0 \leq r \leq t \leq 1). \]

\[ \left\| \int_r^t T_{rs}\Delta(s)f(s) \ ds \right\|_{\mathcal{H}(T_{rt})}^2 \leq \int_r^t \left\| \Delta(s)f(s) \right\|_{\mathcal{M}(\Delta(s))}^2 \ ds \]
\[ (\Rightarrow \text{Inequality 2}). \]
Summary 1

The following inequalities are derived from de Branges-Rovnyak theory (discrete and continuous cases):

- If $G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n$, then
  \[
  \sum_{j=1}^{n-1} \gamma(G_{j+1} - G_j) \leq \gamma(G_n - G_1) + (n - 2)|V|.
  \]

- If $G_0 \subset G_1$, then
  \[
  0 \leq \langle L_0(K_0 - K_1)\tilde{c}, (K_0 - K_1)\tilde{c} \rangle_{\ell^2(V)} \leq \langle (K_0 - K_1)c, c \rangle_{\ell^2(V)}
  \]
  \[
  (c \in \ell^2(V)).
  \]
Our scheme

\[
\begin{align*}
\text{increasing sequences of non-negative matrices} & \quad \downarrow \quad \text{input} \\
de \text{ Branges-Rovnyak theory (quasi-orthogonal integrals)} & \quad \downarrow \quad \text{output} \\
\text{inequalities} & 
\end{align*}
\]

This device is similar to that many identities are implied from formulas in Fourier analysis.