INDIAN STATISTICAL INSTITUTE

SQC & OR Unit, Hyderabad

MS in Quality Management Science : 2014-16

III SEMESTER : Final EXAMINATION

Subject : Nonlinear Programming

October 26, 2015 **Max. Marks 50** **Duration: 3 Hours**

There are two sections - A and B. Answer as much as you can. The maximum you can score is 25 from each section.

Use a separate answer book for each section.

Notation: ln stands for natural logarithm.

Section A

1. (a) Define Hessian matrix of the function $f : \mathbb{R}^n \to \mathbb{R}$. (3)

   (b) Define a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ and show that its Hessian matrix is positive semidefinite. (2)

   (c) Let $f(x, y, u, v) = 100X^2 + 200y^2 + 300u^2 + 400v^2 - 6xy - 8xu + 10xv - 12yu - 10yv - 6uv$. Find the Hessian matrix of f. Is f convex? (justify) (3+2)

   (d) Consider a function $f : (0,\ \infty) \to \mathbb{R}$ be a convex function. Let $g(x) = f(x)$ if $x < 5$, and $g(x) = f(x) + 1$ for $x \geq 5$. Prove or disprove: $g(x)$ is a convex function. (5)

   (e) Let X be a discrete random variable taking values $1,\ 2,\ \ldots$ with finite expectation. Show that $ln(E(X)) \geq E(ln(X))$, where E stands for expectation. (5)

2. Let $f(x) = \frac{p^t x + \alpha}{q^t x + \beta},\ x \in S$, where p and q are non-zero vectors in $\mathbb{R}^n$ and α and β are real numbers. Assume that S is convex and that $q^t x + \beta > 0$ for all $x \in S$.

   (a) Show that f is quasiconvex. (5)

(b) Solve the following problem: (10)

$$\begin{aligned} \text{Maximize} \quad & \tfrac{-2x+y+2}{x+3y+4} \\ \text{subject to} \quad & \\ & -x+4 \leq 4 \\ & y \leq 6 \\ & 2x+y \leq 14 \\ & x, y \geq 0 \end{aligned}$$

Section B

3. Let $f : \mathbb{R}^n \to \mathbb{R}$. When do you say that f is an affine function? If f and $-f$ are convex, then, show that there exist $c \in \mathbb{R}^n$ and $k \in \mathbb{R}$ such that $f(x) = c^t x + k$ for all x. (1+4)

4. Consider the problem:

$$\begin{aligned} \text{Maximize} \quad & x_1 - x_2 + x_3 - x_4 + ln(x^t Jx) \\ \text{subject to} \quad & \\ & Ax \leq b, \\ & x \geq 0, \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4)^t$, $b = (4, 0, 2, -2)^t$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ -1 & 1 & -1 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

(a) Are there any a_{ij}s for which the problem has a feasible solution? (2)

(b) Assuming that the problem has a feasible solution, show that one of the extreme points is an optimal solution to the problem. (7)

(c) Can simplex method solve the problem ?(Justify). (3)

5. There is famous algorithm known as EM (Expectation Maximization) algorithm which is useful in finding maximum likelihood estimators. One of the critical steps

in this algorithm is finding a probability density that maximizes *Kullback-Leibler divergence* (also known as *relative entropy*). If p and q are two possible densities of a random variable X (assume X is a discrete random variable), then the relative entropy between p and q is defined by

$$H(p||q) = -E_p \left(ln \left(\frac{q(X)}{p(X)} \right) \right) = -\sum p(x_i) ln \left(\frac{q(x_i)}{p(x_i)} \right).$$

If $p(x)$ is the true density of X and it is approximated by the density $q(x)$, then $H(p||q)$ can be treated, in some sense, as a measure of gap in estimating p using q.

(a) Show that Kullback-Leibler is non-negative for all densities p and q. (7)

(b) Show that the gap, $H(p||q)$, is zero if, and only if, $q = p$. That is, the solution to the following optimization problem is $q = p$.

Minimize $H(p||q)$ over all densities q for a given p. (3)