Paper: 12

Elements of Maths 2 - End-Sem Question Paper

MS LIS First Year

June 13, 2016

Instructions: Answer as many questions as you can. The maximum you can score is 60 marks. Marks corresponding to each question is indicated in bold. Maximum time allotted is 3 hrs.

(1) (a) Calculate the number of binary relations between two finite sets X and Y in terms of their cardinalities.

(b) Provide an example of a binary relation on $\mathbb{Z}$ which is reflexive and transitive but not symmetric. Justify your answer.

[3+3]

(2) Let X, Y be two sets such that $X \subsetneq Y$ and let $f : X \to Y$ be a function.

(a) If X and Y are finite sets, show that f cannot be onto.

(b) Can f be onto if both X and Y are infinite sets? Justify.

[3+3]

(3) Calculate the following limits:

(a) $\lim_{x \to \infty} \sqrt{x^2 + x} - x$

(b) $\lim_{x \to 3} \frac{x^2 - 7x + 12}{x - 3}$

[2+2]

(4) Prove or disprove that there exists a function $f : (0,1) \to [0,1]$ which is onto and continuous.

[6]

(5) (a) Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at $x = 0$ and that satisfy $f(x) - f(x/2) = x/2 \ \forall \ x \in \mathbb{R}$.

(b) Provide an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is **discontinuous** at $x = 0$ and that satisfies $f(x) - f(x/2) = x/2 \ \forall \ x \in \mathbb{R}$. Justify your answer.

(Hint: Observe that x and $x/2$ are either both rational or both irrational. Piecewise definitions of the function on rational numbers and irrational numbers gives a solution to the second part of the question).

[6+3]

(6) Determine the number of real roots of the polynomial: $x^8 - x^7 + x^2 - x + 15$.

[6]

(7) Does there exist a continuous function $f : [0,2] \to \mathbb{R}$ which is differentiable on $(0,2)$ satisfying $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2\ \forall x \in (0,2)$? Justify.

[6]

(8) A field has the shape of a rectangle with two semicircles attached at opposite sides. Find the radius of the semicircles if the field is to have maximum area, the perimeter of the field equals 100, and the width of the field (twice the radius of the semicircles) is at most 18.

[6]

(9) If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, C_1, \cdots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has atleast one real root between 0 and 1.

[6]

(10) Suppose $f : [1,3] \to \mathbb{R}$ is a piecewise linear on both $[1,2]$ and $[2,3]$ with $f(1) = 2$, $f(2) = 5$ and $f(3) = 8$. Compute $\int_1^3 f(x)dx$.

[4]

(11) Show that $\sum_{k=2}^{100} \frac{1}{k^2} \leq \frac{99}{100}$.

[6]

(12) Find the value of x at which $f(x) = \int_x^{100} (\mid t \mid + 2)\, dt$ takes its maximum on the interval $[-5, 100]$.

[3]

(13) Evaluate $\int_0^3 \frac{2x}{(x^2+1)^2} dx$.

[2]

(14) Compute the average value of $f(x) = x^3$ over the interval $[0,8]$?

[2]