A Very Short Evaluation of an Integral

In [1], the author used partial fractions to evaluate the integral of an odd power of $\sec\theta$. Here, we give a one-sentence evaluation of this integral:

$$\begin{aligned}
\int \sec^{2n+1}\theta\, d\theta &= \int \frac{\sec^{2n+1}\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta \\
&= \int \left(\frac{t+1/t}{2}\right)^{2n}\frac{dt}{t} \qquad\qquad (t := \sec\theta+\tan\theta) \\
&= \frac{1}{2^{2n}}\sum_{r=0}^{2n}\binom{2n}{r}\int t^{2r-1-2n}\,dt \\
&= \frac{1}{2^{2n}}\binom{2n}{n}\log t + \frac{1}{2^{2n}}\sum_{\substack{r=0\\ r\neq n}}^{2n}\binom{2n}{r}\frac{t^{2r-2n}}{2r-2n}+C \\
&= \frac{1}{2^{2n}}\binom{2n}{n}\log[\sec\theta+\tan\theta] \\
&\quad + \sum_{r=0}^{n-1}\binom{2n}{r}\frac{(\sec\theta+\tan\theta)^{2n-2r}-(\sec\theta-\tan\theta)^{2n-2r}}{2^{2n+1}(n-r)}+C.
\end{aligned}$$

In the second step, we have used the fact that $1/t = \sec\theta - \tan\theta$, and in the last step, for each r between 0 and $n-1$ we have combined the terms of the sum for r and $2n-r$. One can, of course, rewrite the expression in terms of $1 \pm \sin\theta$ easily because

$$(\sec\theta+\tan\theta)^2 = \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = \frac{1+\sin\theta}{1-\sin\theta}.$$

References

[1] D. J. Velleman, Partial fractions, binomial coefficients, and the integral of an odd power of $\sec\theta$, this Monthly **109** (2002) 746–749.

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