GROUPS ACTING ON TREES

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1 Introduction

This is an expanded version of the notes of talks given by us in the instructional workshop on geometric group theory held in the Indian Institute of Technology in Guwahati, India during December 2002. The aim here is to give a self-contained presentation of what generally goes by the name of Bass-Serre theory. This theory studies the structure of groups acting simplicially on simplicial trees. The aim is to recover information about the group from its action. A principal result of this theory is that there is a combinatorially defined notion of a 'graph of groups' associated to this action and of a fundamental group of such an object and that the given group can be identified naturally with this fundamental group. This is in analogy with the topological result where a group acting properly discontinuously on a simply-connected space can be recovered as the fundamental group of a suitable quotient space. This theory has now been profitably generalised by several people like Alperin, Bass, Culler, Morgan and Shalen to groups acting on nonarchimedean trees and to general ?-trees for ordered abelian groups. The presentation here is based on Serre's classical book titled Trees, Springer Verlag, (1990) but is meant to be more elementary; it is aimed at advanced undergraduate and beginning graduate students. In the last section, we discuss some results on more general ?-trees. In another article, the second author uses these to discuss some generalisations as well as some interesting applications due to Culler, Formanek, Narain Gupta, Morgan, Procesi, Shalen, Sidki, Tits and Voigtmann. For further reading, one may consult the book by I.Chiswell [C] and the articles [CS1],[CS2] by M.Culler P.Shalen. For the basic results on combinatorial group theory, two excellent references are [MKS] and [LS] but some of these are recalled in the first two sections in a form convenient for us. The first author wrote sections 1.2, 1.5 and 1.7 while the second author wrote sections 1.1,1.3,1.4,1.6,1.8,1.9. [vskip]¡LaTeX¿

¡/LaTeX¿3mm Chiswell points out that the terminology in the subject is an invitation to punsters. However, we shall refrain from using phrases such as X-rays and ?-rays while discussing rays in trees X and graphs ?. We shall also not mention the dictum that the theory of general 'trees' could be thought of as a 'branch' of representation theory. Nor will we point out as to why the relation with 'root' systems is not surprising. We shall simply 'leave' it to the imagination of the reader.

2 Free products

The notions of free groups, free products, and of free products with amalgamation come naturally from topology. For instance, the fundamental group of the union of two path-connected topological spaces joined at a single point is isomorphic to the free product of the individual fundamental groups.

The Seifert- van Kampen theorem asserts that if $X = V \cup W$ is a union of path-connected spaces with $V \cap W$ non-empty and path-connected, and if the homomorphisms $\pi_1(V \cap W) \to \pi_1(V)$ and $\pi_1(V \cap W) \to \pi_1(W)$ induced by inclusions, are injective, then $\pi_1(X)$ is isomorphic to the free product of $\pi_1(V)$ and $\pi_1(W)$ amalgamated along $\pi_1(V \cap W)$.

All these notions have found many group-theoretical applications. Recall that:

If $G_i, i \in I$ are groups, then a group G along with injective homomorphisms $\phi_i : G_i \to G$ is said to be their free product if, for every group H and homomorphisms $\theta_i : G_i \to H$, there is a unique homomorphism $\phi : G \to H$ so that $\phi \circ \phi_i = \theta_i$ for all $i \in I$.

In other words, G has the universal repelling property with respect to homomorphisms from G_i's to groups.

To construct G, one starts with a presentation $< X_i | R_i >$ of each G_i and takes $< X | R >$ as a presentation of G where X is the disjoint union of the X_i's and R is the union of the R_i's. The homomorphisms $\phi : G_i \to G$ are, therefore, simply inclusions. The uniqueness of such a free product G upto isomorphism follows from the uniqueness property of ϕ above.

One writes $G = *_{i \in I} G_i$. If I is a finite set, say, $I = \{1, 2, \cdots, n\}$, then it is customary to write $G = G_1 * G_2 * \cdots * G_n$.

For example, the *free group of rank* r is the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of r copies of $\mathbb{Z}$.

More generally, a free group $F(X)$ on a set X is the free product $*_{x \in X} < x >$.

The group $PSL(2, \mathbb{Z})$ is the free product of a cyclic group of order 2 and a cyclic group of order 3.

Recall that if A is a group, $G_i, i \in I$ is a family of groups and $\alpha_i : A \to G_i$ $(i \in I)$ are injective homomorphisms, then a group G is said to be the free product of G_i's amalgamated along A, if there are homomorphisms $\phi_i : G_i \to G$ satisfying $\phi_i \circ \alpha_i = \phi_j \circ \alpha_j$ for all $i, j \in I$ such that the following universal property holds: for every group H and homomorphisms $\theta_i : G_i \to H$ with $\theta_i \circ \alpha_i = \theta_j \circ \alpha_j$ for all $i, j \in I$, there is a unique homomorphism $\theta : G \to H$ with $\phi \circ \phi_i = \theta_i$.

One denotes G by $*_A G_i$ if there is no confusion as to what the maps α_i are.

Sometimes, the maps α_i are taken to be not necessarily injective and still

the above definition can be carried out. Note that, if α_i are trivial, then $*_A G_i = *G_i$, the free product.
The construction is as follows. If $G_i =< X_i | R_i >$, $i \in I$, then

$$G :=< \sqcup_{i \in I} X_i | \cup R_i \cup \cup_{i,j} R_{ij} >$$

where $R_{ij} = \{\alpha_i(a)\alpha_j(a)^{-1}; a \in A\} >$.
The uniqueness of G upto isomorphism is clear once again by the uniqueness of θ. An example is $SL(2, \mathbb{Z}) = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$.
The fundamental group of the Mobius strip is isomorphic to $\mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$.
A free product with amalgamation could be the trivial group even if the groups $\alpha_i(A)$ are not.
For example, let $\alpha_1 : \mathbb{Z} \to PSL(2,Q)$ be an injective homomorphism and let $\alpha_2 : \mathbb{Z} \to \mathbb{Z}/2$ be the natural homomorphism. Then, $G_1 *_{\mathbb{Z}} G_2 = \{1\}$.
Finally, we recall the notion of HNN extensions named after G.Higman, B.H.Neumann & H.Neumann. The construction is akin to adjoining elements to fields to get field extensions.
Let $G =< X | R >$ be a group and let A be a subgroup. For an injective homomorphism $\phi : A \to G$, the HNN extension of G with respect to ϕ is the group $G^* =< X \cup \{t\} | R \cup \{tat^{-1}\phi(a)^{-1}\} >$.
It is a fact that G^* is independent of the presentation of G chosen and that G embeds naturally into G^*. Also, given two elements a, b of equal order in a group G, this construction enables one to embed the group G into a bigger group in which a, b are conjugate. The HNN construction also finds a natural topological interpretation.
For, suppose V and W are open, path-connected subspaces of a path-connected space X and suppose that there is a homeomorphism between V and W inducing isomorphic embeddings of $\pi_1(V)$ and $\pi_1(W)$ in $\pi_1(X)$. One constructs

a space Y by attaching the handle $V \times [0,1]$ to X, identifying $V \times \{0\}$ with V and $V \times \{1\}$ with W. Then, the fundamental group $\pi_1(Y)$ of Y is the HNN extension of $\pi_1(V)$ relative to the isomorphism between its subgroups $\pi_1(V)$ and $\pi_1(W)$.
We now define these notions in a more general sense so as to be useful in the long run.

3 Amalgams

3.1 Preliminary definitions

The notion of an amalgamated product of two groups is a particular case of a more general notion of direct limit of groups which we now define.

Definition 1. *Let $\{G_i\}_{i \in I}$ be a collection of groups. Here I is some indexing set. For each $i, j \in I$, let F_{ij} be a set of homomorphisms of G_i to G_j, i.e., $F_{ij} \subset \mathrm{Hom}(G_i, G_j)$. The direct limit of this system $G = \lim G_i$ is a group G equipped with homomorphisms $\phi_i \in \mathrm{Hom}(G_i, G)$ such that given any group H and any homomorphisms $\psi_i \in \mathrm{Hom}(G_i, H)$ such that $\psi_j \circ f_{ij} = \psi_i$ for all $i, j \in I$ and for all $f_{ij} \in F_{ij}$, there is a unique homomorphism $\theta \in \mathrm{Hom}(G, H)$ such that $\theta \circ \phi_i = \psi_i$ for all $i \in I$.*

The above definition is depicted in the following diagram as the existence of a unique dotted arrow making the diagram commute.

$$
\begin{tikzcd}
G_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & G_j \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& G \arrow[d, dashed] & \\
& H &
\end{tikzcd}
\tag{2}
$$

The following proposition assures us of the existence and uniqueness of direct limits.

Proposition 3. *Given a collection of groups $\{G_i\}_{i\in I}$ and a collection F_{ij} of homomorphisms as above, the direct limit G exists and is unique up to unique isomorphism.*

Proof. The uniqueness part of the assertion is a standard argument using *universal definitions* as in the definition of the direct limit and is left as an exercise for the reader. We now show existence.

Let $\mathcal{S} = \cup_i G_i$ be the disjoint union of all the G_i's. Let $F(\mathcal{S})$ be the free group on $\mathcal{S}$. Let $\mathcal{N}$ be the normal subgroup of $F(\mathcal{S})$ generated by the relations

$$
\begin{aligned}
R_1 &= \{xyz^{-1} : x, y, z \in G_i \text{ for some } i \text{ such that } xy = z\} \\
R_2 &= \{xy^{-1} : x \in G_i, y \in G_j \text{ and } f_{ij}(x) = y \text{ for some } i, j \text{ and } f_{ij}\}
\end{aligned}
$$

Now take $G = F(\mathcal{S})/\mathcal{N}$. The homomorphism $\phi_i : G_i \to G$ is the canonical one; thanks to the relation R_1. The universal definition of free groups ensures that G is indeed a candidate for 'the' direct limit of the G_i's. $\square$

Remark 4. If the system we consider consists of three groups G_1,G_2 and A and homomorphisms $f_1 : A \to G_1$ and $f_2 : A \to G_2$ then the direct limit of this system is denoted

$$G_1 \underset{A}{*} G_2$$

This is called *the group obtained by amalgamating A in G_1 and G_2 via f_1 and f_2.*

Remark 5. Similarly we can consider a collection of groups $\{G_i\}_{i\in I}$ and another group A such that A is a subgroup of all the groups G_i. In this case the direct limit which is called *the group obtained by amalgamating A in the groups G_i's* and will be denoted as $\underset{A}{*}\, G_i$. In section 3.3 we will analyze such amalgamated products and prove a structure theorem on how elements of this group will look like.

Remark 6. The *free product* $G_1 * G_2$ of two groups G_1 and G_2 is simply the group obtained by amagamating the trivial group in G_1 and G_2.

3.2 Examples

Example 7. The free product of $\mathbb{Z}/2\mathbb{Z}$ and and $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to the infinite dihedral group D_∞.

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \simeq D_\infty := \{x, y : x^2 = 1, xy = y^{-1}x\}.$$

Example 8. With respect to the canonical maps from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ we get

$$\mathbb{Z}/2\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z}/3\mathbb{Z} = (0).$$

Example 9. Consider any injective map from $\mathbb{Z}$ to $\mathrm{PSL}_2(\mathbb{Q})$ and the canonical map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ then we have

$$\mathrm{PSL}_2(\mathbb{Q}) \underset{\mathbb{Z}}{*} \mathbb{Z}/2\mathbb{Z} = (0).$$

Example 10. This example realizes $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ as amalgams. It will be proved later in Section 6.4 after we study how amalgamated groups are characterized as groups acting on trees with certain special properties.

(i) $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \simeq \mathrm{PSL}_2(\mathbb{Z})$

(ii) $\mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z} \simeq \mathrm{SL}_2(\mathbb{Z})$

Example 11 (Nagao). Let K be a field and let $K[X]$ be the polynomial ring in one variable X with coefficients in K. Let $G = \mathrm{GL}_2$ and let B be the standard Borel subgroup consisting of upper triangular matrices in G. Then

$$G(K[X]) = G(K) \underset{B(K)}{*} B(K[X]).$$

Example 12 (Ihara). Let F be a non-Archimedean local field. Let $\mathfrak{P}$ be the maximal ideal of the ring of integers $\mathcal{O}$ of F. Let $G = \mathrm{SL}_2(F)$. Let $K = \mathrm{SL}_2(\mathcal{O})$ and let I be the subgroup of elements of K which are upper triangular modulo $\mathfrak{P}$. Then

$$G = K \underset{I}{*} K.$$

Section 8 is devoted entirely to this example.

Example 13 (Rational version of Ihara's example). For a prime number p let $\Gamma_0(p)$ be the subgroup of elements of $\mathrm{SL}_2(\mathbb{Z})$ which are upper triangular modulo p. Let $\mathbb{Z}[1/p]$ be the subring of $\mathbb{Q}$ containing all rational numbers whose denominators is some power of p. Then

$$\mathrm{SL}_2(\mathbb{Z}[1/p]) = \mathrm{SL}_2(\mathbb{Z}) \underset{\Gamma_0(p)}{*} \mathrm{SL}_2(\mathbb{Z}).$$

Example 14 (Margulis-Tits). The group $\mathrm{SL}_3(\mathbb{Z})$ is not an amalgam of the form $G_1 *_A G_2$ for any three groups G_1, G_2 and A such that $G_1 \neq A \neq G_2$.

This will be proved in section 8, but, in fact, it is actually true in a very general setting. Let F be a number field and let S be a finite set of primes of F. Let $\mathcal{O}(S)$ denote the ring of S-integers of F. If G is a simple Chevalley group of F-rank at least 2 then the group $G(\mathcal{O}(S))$ is not an amalgam. This is a deep theorem due to Margulis and Tits.

3.3 The main structure theorem

In this section we consider the following situation whose direct limits will be the object of study. (See Remark 5.) Let $\{G_i\}$ *be a collection of groups and let* A *be a subgroup of all the* G_i*'s. Let* $f_i : A \to G_i$ *be simply the inclusion map. The direct limit of this system is denoted* $G = \underset{A}{*} G_i$*. Recall that* G *comes equipped with homomorphisms* $\phi_i : G_i \to G$ *and* $\phi : A \to G$*. We will now describe what elements of* G *look like. Towards this end we need some notations.*

Let S_i *be a set of coset representatives for* A *in* G_i*. Assume that* $1 \in S_i$*. Hence*

$$G - A = \cup_{s \in S_i - \{1\}} As$$

We call a sequence $(i_1, \dots, i_n)$ *of indices an admissible sequence if* $i_k \neq i_{k+1}$ *for* $1 \leq k \leq n-1$*. Let* $\alpha = (i_1, \dots, i_n)$ *henceforth denote an admissible sequence. A reduced word of type* α *is a symbol*

$$m = (a; s_1, \dots, s_n)$$

where $a \in A$ *and* $s_k \in S_{i_k} - \{1\}$ *for* $1 \leq k \leq n$*.*

Theorem 15. *Let* $G = \underset{A}{*} G_i$ *and the rest of the notations be as above. Given any* $g \in G$ *there exists a unique reduced word* $m = (a; s_1, \dots, s_n)$ *such that*

$$g = \phi(a)\phi_1(s_1)\dots\phi_n(s_n).$$

Corollary 16. *With notations as above, all the homomrphisms* ϕ_i *and* ϕ *are injective. Hence suppresing the maps we write that given any* $g \in G$ *there exists a unique* $a \in A$ *and a unique sequence of elements* $s_k \in S_{i_k} - \{1\}$ *with* $i_k \neq i_{k+1}$ *such that*

$$g = as_1 \dots s_n.$$

Theorem 15 has another formulation which is also very useful. We need more notations to state this reformulation. Let $G'_i = G_i - A$*. For an admissible sequence* $\alpha = (i_1, \cdots, i_n)$ *as before let*

$$\widetilde{G_\alpha} = G'_{i_1} \times \cdots \times G'_{i_n}.$$

Note that A^{n-1} *acts on the set* $\widetilde{G_\alpha}$ *as*

$$(a_1, \dots, a_{n-1}) \cdot (x_1, \dots, x_n) = (x_1 a_1, a_1^{-1} x_2 a_2, \dots, a_{n-1}^{-1} x_n).$$

Let G_α *denote the quotient*

$$G_\alpha = \widetilde{G_\alpha}/A^{n-1} = G'_{i_1} \overset{A}{\times} \cdots \overset{A}{\times} G'_{i_n}.$$

The homomorphisms ϕ *and* ϕ_i*'s determine canonically a map* $\phi_\alpha : G_\alpha \to G$*. If* α *is the empty sequence then* $\widetilde{G_\alpha} = G_\alpha = A$ *and* $\phi_\alpha = \phi$*.*

Theorem 17. *With the notations as above we get that all maps* ϕ_α *induce a bijection from the disjoint union* $\cup_\alpha G_\alpha$ *onto* G*.*

Remark 18. Theorem 17 can be verbally formulated as that for every $g \in G$ one of the following is true:

(i) g is in A

(ii) g is in some G_i but not in A, i.e., $g \in G'_i$.

(iii) There is some uniquely determined sequence $\alpha = (i_1, \dots, i_n)$ and elements $g_k \in G'_{i_k}$ such that $g = g_1 \dots g_n$. The g_i's are not uniquely determined, and in fact for any $a_1, \dots, a_{n-1}$ we have $g = g_1 \dots g_n = (g_1 a_1)(a_1^{-1} g_2 a_2) \dots (a_{n-1} g_n)$.

However, in all cases, we may talk of the *length* of an element $g \in G$. For example, in case (1), g has length 0, in case (2) it has length 1 and in case (3) it has length n.

Proof. (Of Theorem 15.) Let X stand for the set of all reduced words $m = (a; s_1, \dots, s_n)$. Let $\varphi : X \to G$ denote the map

$$\varphi(m) = \varphi((a; s_1, \dots, s_n)) = \phi(a)\phi_{i_1}(s_1) \dots \phi_{i_n}(s_n).$$

We want to show that φ is a bijection.

To this end, we define an action of G on X, i.e., we need to give a homomorphism $G \to \mathrm{Aut}(X)$. By the definition of direct limits it suffices to define an action of each G_i on X such that they are all compatible which in this case boils down to saying that the induced action on A is independent of i.

Fix an $\in I$. Let

$$Y_i = \{(1; s_1, \dots, s_n) \in X : i_1 \neq i\}.$$

Consider the two maps

$$\begin{aligned} A \times Y_i &\longrightarrow X \\ (a, (1; s_1, \dots, s_n)) &\longmapsto (a; s_1, \dots, s_n) \\ A \times S_i - \{1\} \times Y_i &\longrightarrow X \\ (a, s, (1; s_1, \dots, s_n)) &\longmapsto (a; s, s_1, \dots, s_n) \end{aligned}$$

Clearly the images of these maps are disjoint and their union is all of X. Using these maps we get

$$\begin{aligned} X &= A \times Y_i \amalg A \times S_i - \{1\} \times Y_i \\ &= (A \amalg A \times S_i - \{1\}) \times Y_i \\ &= G_i \times Y_i \end{aligned}$$

We use this identification and define an action of G_i on X. In particular, it is easy to check that the induced action on A is given by :

$$a' \cdot (a; s_1, \ldots, s_n) = (a'a; s_1, \ldots, s_n)$$

for all $a' \in A$ and for reduced words $(a; s_1, \ldots, s_n)$. Hence we get an action of G on X.

We use this action and consider the map $\psi : G \to X$ given by

$$\psi(g) = g \cdot (1)$$

where (1) is the empty word. This map ψ is a candidate for the inverse of φ.

To begin with we show that $\psi \circ \varphi = 1_X$. This will prove that φ is injective and hence we will get uniqueness. Once we have injectivity, we can identify X with its image $\varphi(X)$ as a subset of G. Further, injectivity implies that each f_i is injective, and so $G_i \subset X$ for all i and hence $G \subset X$. This proves that φ is surjective. It suffices now to prove $\psi \circ \varphi = 1_X$ which can be seen as :

$$\begin{aligned} \psi \circ \varphi(a; s_1, \ldots, s_n) &= \psi(\phi(a)\phi_{i_1}(s_1) \ldots \phi_{i_n}(s_n)) \\ &= \phi(a)\phi_{i_1}(s_1) \ldots \phi_{i_n}(s_n)(1) \\ &= \phi(a)\phi_{i_1}(s_1) \ldots \phi_{i_{n-1}}(s_{n-1})(1; s_n) \\ &= \cdots = \phi(a)(1; s_1, \ldots, s_n) = (a; s_1, \ldots, s_n). \end{aligned}$$

□

Proof. (Of the equivalence of Theorem 15 and Theorem 17.)

Assume Theorem 15 and so any $g \in G$ is uniquely written in the form $g = \phi(a)\phi_{i_1}(s_1)\ldots\phi_{i_n}(s_n)$ for a reduced word $(a; s_1, \ldots, s_n)$ of type $\alpha = (i_1, \ldots, i_n)$. Then g lies in the set G_α and we think of g being represented by $g = g_1 g_2 \ldots g_n$ where $g_1 = as_1$ and $g_i = s_i$ for all $i \neq i_1$. This gives Theorem 17.

Now assume Theorem 17 and let $g \in G_\alpha$ which is written as $g = g_1 g_2 \ldots g_n$. If α is empty then $g = a \in A$ and g corresponds to the reduced word (a). If $\alpha = (i)$ then $g \in G_i'$ and can be uniquely written as $g = as$ with $a \in A$ and $s \in S - \{1\}$ and so g corresponds to the reduced word $(a; s)$. Now assume that $n \geq 2$.

Recall, by definition of G_α we may replace this expression by any expression of the form $g = (g_1 a_1)(a_1^{-1} g_2 a_2) \ldots (a_{n-1}^{-1} g_n)$ which implies that we can take $g_i = s_i \in S_i - \{1\}$ if $i = i_k$ and $k \geq 2$ and further $g_1 \in G_{i_1}'$ can be uniquely written as $g_1 = as_1$ with $s_1 \in S_1 - \{1\}$. Hence g corresponds to $(a; s_1, \ldots, s_n)$. $\square$

3.4 Some applications to abstract group theory

Proposition 19. *Let $G = \mathop{*}_A G_i$. Any element of G of finite order can be conjugated inside one of the G_i. In other words, if all the G_i's are torsion-free then so is G.*

Proof. Let $g \in G = \mathop{*}_A G_i$. Using Theorem 17 write $g = g_1 \ldots g_n$. Let $l(g) = n$ be the length of g. If $l(g) \leq 1$ then $g \in G_i$ for some i. If $l(g) \geq 2$ we say g is *cyclyically reduced* if $i_1 \neq i_n$.

We now show inductively that that any g is conjugate to either an element of some G_i or to a cyclically reduced element. Assume that $l(g) = n \geq 2$ and that we have shown this for all elements of length at most $n-1$. Suppose g is not cyclically reduced then $i_1 = i_n$ and so conjugating by g by g_1^{-1} we get $g = g_1 \dots g_n \sim g_2 \dots g_{n-1}(g_n g_1)$ and the length of $g_2 \dots g_{n-1}(g_n g_1)$ is at most $n-1$.

Now take any $g \in G$ which is of finite order. Since all the G_i are torsion free, we get that no conjugate of g is in any G_i. We may replace g by a conjugate and assume that it is cyclically reduced. We leave it to the reader to the check that in this case, for any $r \geq 1$ we have that the length of g^r is rn and so g cannot have been an element of finite order unless $n = 0$, i.e., $g = 1$. □

Proposition 20. *If G_1 and G_2 are two finite groups then their free product $G_1 * G_2$ contains a free subgroup of index $o(G_1)o(G_2)$. In particular, the free product of two finite groups admits a faithful finite-dimensional representation.*

Proof. Consider the direct product $G_1 \times G_2$ of G_1 and G_2. The inclusion maps from the G_i into $G_1 \times G_2$ gives a canonical homomorphism from the free product $G_1 * G_2$ to $G_1 \times G_2$. Clearly this map is surjective. Let K be the kernel of this homomorphism.

Let S be the set of commutators in $G_1 * G_2$ given by

$$S = \{xyx^{-1}y^{-1} : x \in G_1, y \in G_2\}.$$

Let N be the subgroup of $G_1 * G_2$ generated by S. Clearly N is contained in the kernel of the homomorphism. In fact, using the universal definitions of direct product and free product it is easy to see that $N = K$. It suffices now to prove that S is a free subset of $G_1 * G_2$.

To this end, it suffices to show that for any sequence $s_1, \dots, s_n \in S$ with $s_i = a_i b_i a_i^{-1} b_i^{-1}$ and any sequence $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ with the condition that if $\epsilon_k = -\epsilon_{k+1}$ then $s_k \neq s_{k+1}$, the element $g = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ is not the identity element. In fact we will show that

(i) $l(s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}) \geq n+3$.

(ii) If $\epsilon_n = 1$ (resp. $\epsilon_n = -1$) then g ends with $a_n^{-1}b_n^{-1}$ (resp. $a_n b_n$).

This can be seen using induction. Without loss of generality assume that $\epsilon_n = 1$. (The argument for the case $\epsilon_n = -1$ is similar.) If $n = 1$ then there is nothing to prove. Let $n \geq 2$.

If $\epsilon_{n-1} = 1$ then we may write g as

$$g = t_1 \dots t_p a_{n-1}^{-1} b_{n-1}^{-1} a_n b_n a_n^{-1} b_n^{-1}$$

with $p \geq n$ by induction hypothesis. Hence $l(g) = (p+2)+4 \geq n+6 > n+3$ and g ends with $a_n^{-1}b_n^{-1}$.

If $\epsilon_{n-1} = -1$ then we may write g as

$$g = t_1 \dots t_p b_{n-1}^{-1} a_{n-1}^{-1} a_n b_n a_n^{-1} b_n^{-1}$$

with $p \geq n$ by induction hypothesis. Now if $a_{n-1} \neq a_n$ then $l(g) = p+5 \geq n+5 > n+3$. If $a_{n-1} = a_n$ then $l(g) = p+3 \geq n+3$ and in either of these two cases g ends with $a_n^{-1}b_n^{-1}$. □

Proposition 21 (HNN-construction). *Let G be a group. Let A be a subgroup of G. Let $\theta : A \to G$ be any injective homomorphism of A into G. Then there exists a group $\mathcal{G}$ containing G and an element $s \in \mathcal{G}$ such that the inner automorphism of $\mathcal{G}$ detrmined by s when restricted to A gives θ, i.e.,*

$$\mathrm{Int}(s)|_A = \theta.$$

Actually there is a universal group and an element $(\mathcal{G}, s)$ with this property and this is called the HNN-extension of the data (G, A, θ).

Proof. There are two ways to construct the group $\mathcal{G}$.

1st Proof. Consider the following system of groups and group homomophisms. Let $G_n = G$ and $A_n = A$ for all integers n.

$$\begin{tikzcd}
\cdots G_{n-1} & & G_n & & G_{n+1} & & G_{n+2} \cdots \\
& A_n \arrow[ul, "1"'] \arrow[ur, "\theta"] & & A_{n+1} \arrow[ul, "1"'] \arrow[ur, "\theta"] & & A_{n+2} \arrow[ul, "1"'] \arrow[ur, "\theta"] &
\end{tikzcd} \tag{22}$$

Let $\widetilde{G}$ be the direct limit of this system. Let $u_n : G_n \to G_{n+1}$ be the canonical shift homomorphism. Let $u : \widetilde{G} \to \widetilde{G}$ be the induced homomorphism. Then it is easily seen that u extends the map θ.

Let $\mathcal{G} = \widetilde{G} \rtimes \langle u \rangle$ be the semi-direct product of $\widetilde{G}$ with the cyclic group $\langle u \rangle$ generated by u. Now take s as the element u in the semi-direct product.

2nd Proof. Let S be the infinite cyclic group on the symbol α. Let $\widetilde{G}$ be the free product $G * S$. Let N be the normal subgroup of $\widetilde{G}$ generated by all elements of the form

$$\{\alpha a \alpha^{-1} \theta(a)^{-1} : a \in A\}.$$

Let $\mathcal{G} = \widetilde{G}/N$ and let s be the image of α in $\mathcal{G}$. It is easy to see that $(\mathcal{G}, s)$ is the HNN-extension associated to the data (G, A, θ). $\square$

3.5 Exercises

Exercise 23. Show that

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \simeq D_\infty := \{x, y : x^2 = 1, xy = y^{-1}x\}.$$

Exercise 24. Let m, n be two relatively prime integers. Then with respect to the canonical homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ show that

$$\mathbb{Z}/n\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z}/m\mathbb{Z} = (0).$$

Exercise 25. Let G be a simple group which admits $\mathbb{Z}$ as a subgroup. Then with respect to the canonical homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ show that

$$G \underset{\mathbb{Z}}{*} \mathbb{Z}/n\mathbb{Z} = (0).$$

Exercise 26. Determine all finite order elements of $PSL(2,\mathbb{Z})$. Give an example of a subgroup of $PSL(2,\mathbb{Z})$ of index 6. Is it free?

Exercise 27. Let H be a subgroup of $G = G_1 \underset{A}{*} G_2$. Assume that $A \cdot H = G$. Let $B = A \cap H$ and let $H_i = G_i \cap H$ for $i = 1, 2$. Show that H is generated by H_1 and H_2 and can be identified with $H_1 \underset{B}{*} H_2$. Use this to deduce the rational version of Ihara's Example 13 from Example 12.

Exercise 28. Show that every group G can be embedded in a group K which has the property that all the elements of the same order are conjugate.

Exercise 29. Let $f_1 : A \to G_1$ and $f_2 : A \to G_2$ be two homomorphisms and let $G = G_1 \underset{A}{*} G_2$ be the corresponding amalgam. Define subgroups A^n, G_1^n and G_2^n of A,G_1 and G_2 recursively by the following conditions:

(i) $A^1 = \{1\}$, $G_1^1 = \{1\}$ and $G_2^1 = \{1\}$.

(ii) A^n is the subgroup generated by $f_1^{-1}(G_1^{n-1})$ and $f_2^{-1}(G_2^{n-1})$.

(iii) G_i^n is the subgroup of G_i generated by $f_i(A^n)$.

Let A^∞, G_i^∞ be the unions of the A^n and G_i^n respectively. Show that f_i induces an injection from A/A^∞ into G_i/G_i^∞. Further G may be identified as the amalgam

$$G = G_1/G_1^\infty \underset{A/A^\infty}{*} G_2/G_2^\infty.$$

Exercise 30. Let g, h be elements of $\underset{A}{*} G_i$ of lengths n, m respectively and of types $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$ respectively. Show that $l(gh) \leq n + m$ and that equality holds if and only if $i_n \neq j_m$, in which case gh is of type $(i_1, \ldots, i_n, j_1, \ldots, j_m)$.

Exercise 31. Show that the two constructions of the HNN-extension $(\mathcal{G}, s)$ for the data (G, A, θ) given in the proof of Proposition 21 are equivalent to each other.

4 Trees

Starting with this section, we shall see how groups can be studied geometrically by means of graphs associated to them. Let us start with the definition of what we mean by graphs.

Definition 1. *A graph consists of:*

(i) a non-empty set X (called vertices),

(ii) a set Y (called oriented edges),

(iii) a map $Y \to X \times X$ which sends any $e \in Y$ to the pair $(o(e), t(e))$ of its origin vertex $o(e)$ and the terminal vertex $t(e)$ and

(iv) a map from Y to itself which sends each edge e to its inverse edge $\bar{e}$ which is different from e and has its origin and terminus switched and so that $\bar{\bar{e}} = e$.

A graph is usually conveniently represented by a diagram. Each vertex is marked by a point or a bullet and each edge is represented by an arrow from its origin vertex to its terminal vertex.

It is usual to draw only one of e and $\bar{e}$ and one understands that each line joining two vertices of a graph corresponds to a pair of vertices.
A graph is finite if both X, Y are finite and is said to be locally finite if only finitely many edges start or end at each vertex.
A path in a graph is a concatenation $e_1 \cdots e_n$ of edges where e_i starts at the vertex where e_{i-1} ends for $i = 2, \cdots, n$. One says that the path is from $o(e_1)$ to $t(e_n)$.
A circuit is a path as above where e_n ends at the origin vertex of e_1. A circuit of length 1 is called a loop.
A graph is said to be connected if each pair of vertices is contained in some path.
From the definition of a graph, it is clear that the set Y can be written as a disjoint union $Y_+ \sqcup \bar{Y}_+$. A choice of Y_+ is called an orientation of the graph. Note that once Y_+ has been chosen, the set $Y = Y_+ \sqcup \bar{Y}_+$.
For example, if G is any group and S is any subset, one has an oriented graph known as the Cayley graph defined as follows. The set X of vertices is the set of elements of G. The set $Y_+ = G \times S$ with $o(g, s) = g$ and $t(g, s) = gs$. For instance, if $G = \mathbb{Z}/3\mathbb{Z}$ and $S = \{1\}$, then the graph is a triangle.
A morphism from a graph (X, Y) to a graph (X', Y') is a mapping $\alpha : X \to X'$ which takes edges to edges and the origin and terminus of $\alpha(e)$ are, respectively, the images of the origin and the terminus of e. One defines two graphs (X, Y) and (X', Y') to be isomorphic if there are graph-morphisms $\alpha : (X, Y) \to (X', Y')$ and $\beta : (X', Y') \to (X, Y)$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps.

If Z is a CW complex of dimension 1, then one can naturally associate a graph to it by taking 0-cells to be the vertices and the 1-cells to be the set of pairs of edges. Clearly, this graph has no loops and no circuits of length

2. *The complex Z is known as a geometric realization of the corresponding graph. The endpoints of an edge e determine the set $\{e, \bar{e}\}$. Therefore, for any edge e, one also refers to the pair $\{e, \bar{e}\}$ of edges as a geometric edge.*

Lemma 2. *A graph is isomorphic to one which arises from a CW complex of dimension 1 if, and only if, it has no circuit of length 1 or 2.*

Proof. If a graph is isomorphic to another which has no circuits of length ≤ 2, then the original graph itself evidently has the same property. For such a graph, one can define the corresponding geometric realization as the set X of vertices of the graph and the 0-cells and the 1-cells as the set of subsets $\{P, Q\}$ where P, Q are either adjacent vertices or $P = Q$. The converse is obvious. □

Our interest is particularly in the Cayley graph $\Gamma(G, S)$ associated to a group G and a subset S. We have :

Proposition 3. *(a) The subgroup $< S >$ gives the connected component of $\Gamma(G, S)$ at the vertex corresponding to the identity element and the left cosets of $< S >$ are in bijection with the various connected components.*
In particular, $\Gamma(G, S)$ is connected if, and only if, $G =< S >$.

(b) $\Gamma(G, S)$ is isomorphic to the graph associated to a CW complex of dimension 1 if, and only if, $S \cap S^{-1} = \emptyset$.

(c) $\Gamma(G, S)$ contains a loop if, and only if, $1 \in S$.

Proof. To prove (a), observe that edges of $\Gamma(G, S)$ join an element g of G to an element of the form gs or gs^{-1} for some $s \in S$. Thus, the connected component of any vertex g consists of all elements of $g < S >$.

For (b) and (c), note that there are elements $s, t \in S$ with $st = 1$ if, and only if, there is a circuit of length 1 (if $s = 1$) or 2 (if $s \neq 1$). By the lemma, the assertion follows. $\square$

Now, we shall study a special class of graphs called trees which will be crucial in our geometric study of groups. The notion of distance between two vertices as the length of the shortest path will make sense for trees and one can study trees as metric spaces. Let us start with the definition of a tree.

Definition 4. *A connected graph without any circuits is called a tree.*

Note that the Cayley graph of $G = \mathbb{Z}$ and $S = \{1\}$ is simply an infinite path and in particular a tree. Evidently, any tree has a geometric realization which is a CW complex of dimension 1.

Exercise. *Prove that a graph Γ is finite (respectively, locally finite) if and only if its realization Real Γ is compact (respectively, locally compact).*

Here is a rather interesting exercise :

Exercise. *Show that an infinite, locally finite, connected graph contains an infinite, injective path.*

A path $e_1 \cdots e_n$ in a tree is called a geodesic if $e_{i+1} \neq \bar{e}_i$ for any i. We have the following very important property of a tree.

Proposition 5. *For any two vertices P, Q in a tree Γ, there is a unique geodesic $e_1 \cdots e_n$ from P to Q. Moreover, all the vertices $o(e_i)$ are distinct.*

Proof. Obviously, since a tree is connected and since there are no circuits, there does exist a geodesic joining any two vertices. If $e_1 \cdots e_n$ is any geodesic and if $o(e_i) = o(e_j)$ for some $i < j$, then the path $e_i \cdots e_{j-1}$ would be a circuit, a contradiction to the fact that we have a tree. Finally, if $e_1 \cdots e_n$ and $f_1 \cdots f_m$ are two geodesics from P to Q, then the path $e_1 \cdots e_n \bar{f}_m \cdots \bar{f}_1$

would be a nontrivial circuit at P unless $e_n = f_m$. By induction, it would follow that $m = n$ and that $e_i = f_i$ for all i. $\square$

Definition 6. *The distance $l(P,Q)$ between two vertices P, Q of a tree is the length n of the geodesic $e_1 \cdots e_n$ from P to Q. We shall use the notation $\overline{PQ}$ to denote this geodesic.*
The set of vertices of a tree forms a metric space under the above distance function. An easy exercise is :
Exercise. *Let Γ be a tree and let P, Q, R be vertices. If $P' \in \overline{QR}$, prove that*

$$\overline{PP'} \leq Sup(\overline{PQ}, \overline{PR}).$$

One can define the diameter of a tree Γ to be the supremum of $l(P,Q)$ as P, Q vary. If the diameter of a tree is finite, the tree is said to be bounded. Clearly, any finite tree is bounded.

Given a tree (X, Y), the ball of radius n centred at a vertex P is the set $X_n(P)$ of vertices Q such that $l(P,Q) = n$. Note that $X_0(P) = \{P\}$. Also, given P, each point $P_n \in X_n(P)$ has a unique predecessor $P_{n-1} \in X_{n-1}(P)$. Therefore, there are maps $f_{n,P} : X_n(P) \to X_{n-1}(P)$ and the subsets $X_n(P)$ form an inverse system and their union over all n is the set of all vertices of the tree. All the pairs $e, \bar{e}$ of edges of the tree can be recovered from this inverse system as the pairs $\{Q, f_{n,P}(Q)\}$ for $n \geq 0$.
Let X be the vertices of a tree Γ and let X' be a subset of X. Then, every subtree which contains X' also contains all the geodesics which have their extremities in X'. Therefore, the set of all vertices and all edges contained in the geodesics of Γ which have their extremities in X' form a subtree called the subtree generated by X'. In particular, every tree is an increasing union of its finite subtrees. An easy exercise is:

Exercise. *Let* $\Gamma = (X, Y)$ *be a tree and let* Γ' *be the subtree generated by a subset* X' *of* X*, then diameter of* $\Gamma' =$ *diameter of* X'*.*

If P, Q *are vertices of a tree* Γ*, then the subtree* $\Gamma(P, Q)$ *generated by the set* $\{P, Q\}$ *has a geometric realization which is homeomorphic to the closed interval* $[0, n]$ *where* $l(P, Q) = n$*. Since such an interval is contractible, and since the realization of* Γ *is the union of realizations of subtrees of the form* $\Gamma(P, Q)$ *for vertices* P, Q*, it follows that the realization of any tree is a contractible space.*

It is very convenient to "build" any tree from subtrees ultimately starting with a single vertex. We try to understand this now.
For any graph $\Gamma = (X, Y)$*, and any vertex* $P \in X$*, we define the subgraph* $\Gamma - P$ *to be the graph obtained by dropping* P *from the vertex set* X*, and dropping all edges in* Y *which either start or end at* P*. Let us denote by* Y_P*, the subset of* Y *containing edges which end at* P*. Thus,* $\Gamma - P = (X \setminus \{P\}, Y \setminus (Y_P \cup \bar{Y}_P)$*. One calls a vertex* P *of a graph* Γ *to be a terminal vertex if there is at most one edge ending at* P*; it is said to be isolated if no edge ends at* P*. The special nature of such vertices is brought out by :*

Proposition 7. *Let* P *be a vertex of a graph* Γ *at which a unique edge ends. Then,*

- *(a)* Γ *is connected if, and only if,* $\Gamma - P$ *is connected.*
- *(b) Every circuit of* Γ *is contained in* $\Gamma - P$*.*
- *(c)* Γ *is a tree if, and only if,* $\Gamma - P$ *is a tree.*

Proof. As a unique edge e ends at P, the edges of $\Gamma - P$ form the set $Y \setminus \{e, \bar{e}\}$ and so, (a) is clear.

To prove (b), we observe that the vertices which are part of a circuit have atleast 2 edges ending in them and are, therefore, different from P and so, the whole circuit is contained in $\Gamma - P$.

Finally, (c) follows directly from (a) and (b). $\square$

Corollary 8. *Any maximal tree Λ in a connected graph Γ contains all the vertices of Γ.*

Proof. Suppose not. Then, we can find a vertex P of Γ which is not in Λ and an edge e joining P to a vertex Q of Λ. But, the graph Δ obtained by including the vertex P along with those of Λ and the edges $e, \bar{e}$ along with the edges of Λ is such that $\Delta - P = \Lambda$. So, Δ is also a tree by (c) above. This contradicts the maximality of Λ and proves that the assumption cannot hold. $\square$

The following is the key fact which shows how to obtain any finite tree from a single vertex. Note that a tree of diameter 0 is just a point and a tree of diameter 1 is just two vertices joined to each other by the two edges e and $\bar{e}$.

Proposition 9. *Let $\Gamma = (X, Y)$ be a tree of finite diameter n. Then,*

(a) if $n \geq 2$, then dropping all terminal vertices from X gives rise to a subtree of diameter $n - 2$.

(b) there exist terminal vertices.

Proof. Clearly, (a) implies (b) as from what have already observed to be the structures of trees of diameters 0 and 1.

To prove (a), let us notice that if P, Q are non-terminal vertices, then the geodesic joining them does not contain any terminal vertices. Therefore, it can be exntended both ways to produce a geodesic of length $l(P,Q)+2$. Therefore, $l(P,Q) \leq n-2$. This means that the subtree Γ' obtained by dropping the terminal vertices has diameter at most $n-2$. However, one can remove from a geodesic of length n in Γ, the first and the last edges to obtain a geodesic of length $n-2$ in Γ'. This proves (b). $\square$

Corollary 10. *The tree Γ' obtained from a tree Γ by removing every terminal vertex and the two edges containing it, is preserved by any automorphism of Γ. In particular, if Γ has finite diameter, then every automorphism of Γ fixes a vertex or a geometric edge $\{e, \bar{e}\}$ according as whether the diameter is even or odd.*

Proof. Since every automorphism of Γ must carry a terminal vertex to a terminal vertex, it has to carry the edge ending at the first terminal vertex to either the edge ending in the second or to its inverse. Thus, Γ' is preserved. Finally, by (a) above, this means that every automorphism of Γ of diameter n preserves a subtree of diameter $n-2[n/2]$. This completes the proof. $\square$

Here is a nice exercise which also proves the above result in a different way.
Exercise. *Let Γ be a tree of finite diameter n. Show that all geodesics of length n have the same middle vertex (respectively, geometric edge) if n is even (respectively odd).*

The following result is an Euler-Poincare formula for a graph. To see that it is so, note that our edge set in a graph has twice the number of edges in familiar terminology. Thus, the assertion below is that the number of vertices minus the number of geometric edges is either 0 or 1 and it is 1 precisely when the graph is a tree.

Proposition 11. *Let $\Gamma = (X, Y)$ be a connected graph with X finite. Then, $|Y| \geq 2(|X|-1)$ with equality holding if, and only if, Γ is a tree.*

Proof. Let us start with a tree Γ first. If $|X| = 1$, then clearly $|Y| = 0$ and the equality $|Y| = 2(|X| - 1)$ holds. We can prove this for any finite tree Γ by induction on $|X|$ by passing from Γ to $\Gamma - P$ where P is a terminal vertex and noting that the latter has one less vertex and two less edges.

Now, let $\Gamma = (X, Y)$ be a general connected graph as in the statement. Choosing any maximal tree Λ, we have by 2.8 that $X(\Lambda) = X$. Evidently, all the edges in Λ are edges in Γ and so, $|Y(\Lambda)| \leq |Y|$ with equality precisely when $\Gamma = \Lambda$. Using the equality $Y(\Lambda) = 2(|X| - 1)$ for Λ, the proposition follows. $\square$

We saw that the realization of a tree is a contractible space. We finish this section with the topological structure of the realization of any connected graph. We introduce one notation for this purpose.

Let Γ be any connected graph and let Λ be a subgraph which is a disjoint union of a family $\Lambda_i, i \in I$ of trees. We shall define a new graph denoted by Γ/Λ as follows. Each vertex set $X(\Lambda_i)$ gives one vertex of Γ/Λ and each vertex of Γ outside Λ also gives one vertex. The edge set of Γ/Λ is defined as the set of edges of Γ which are not in Λ. Clearly, the map $e \mapsto \bar{e}$ in Γ defines also the inverse of any edge of Γ/Λ. Similarly, the origin and terminus of any edge of Γ/Λ is defined from the corresponding map on Γ by passing to quotients.

Proposition 12. *The realization of a connected graph Γ has the homotopy type of a bouquet of circles. Moreover, Γ is a tree if, and only if, the realization is contractible.*

Proof. Let Λ be a maximal subtree of Γ. Then, the graph Γ/Λ has a single vertex and, therefore, its realization, which is a CW complex of dimension 1 with a single 0-cell, it must be a bouquet of circles. Look at the pair (R_1, R_2)

where R_1 is the realization of Γ and, R_2 is the realization of Λ. Since R_2 is a subcomplex of the CW complex R_1, the pair has the homotopy extension property since it is a cofibration. Also, since R_2 is contractible, there is a homotopy $h_t : R_2 \to R_2$ $(0 \leq t \leq 1)$ so that h_0 is the identity and h_1 retracts to a point of Λ. So, there is a homotopy $H_t : R_1 \to R_1$ $(0 \leq t \leq 1)$ so that H_0 is the identity map, and each H_t agrees with h_t on R_2. Also, if R_0 denotes the realization of Γ/Λ, then one has the quotient map $p : R_1 \to R_0$ of R_1 by identification of Λ to a point. When $t = 1$, this gives a map H_1 which factors through the quotient map p. Thus, we have a map $f : R_0 \to R_1$ with $H_1 = f \circ p$. Thus, $f \circ p$ is homotopic to the identity map H_0.
Now, we show that $p \circ f$ is homotopic to the identity also. Since H_t leaves R_2 invariant for each t, it induces a homotopy $H'_t : R_0 \to R_0$. As we have $p \circ H_1 = H'_1 \circ p$ and $f \circ p = H_1$, we also have $p \circ f = H'_1$ as p is surjective. Thus, $p \circ f = H'_1$ is homotopic to H'_0, the identity map of R_0. Therefore, we have shown that we have a homotopy equivalence between R_0 and R_1. As the former is a bouquet of circles, so is the latter upto homotopy equivalence. Finally, R_1 is contractible if, and only if, R_0 is contractible and, this happens if, and only if, there are no circles i.e., R_0 is a point i.e., $\Gamma = \Lambda$. This proves the proposition. $\square$

Corollary 13. *Let Γ be a connected graph and let Ω be a disjoint union of subtrees of Γ. Then Γ is a tree if and only if Γ/Ω is a tree.*

5 Trees, Free groups and Schreier's Theorem

In this section, we look at graphs on which groups act and the idea is to deduce group-theoretic properties from the geometric properties of this action.

Definition 1. *A group G is said to* act on a graph *$\Gamma = (X, Y)$ if G acts on the set X in such a way that G takes edges to edges. In particular, G preserves an orientation of Γ if, and only if, it acts without inversion i.e., $ge \neq \bar{e}$ for any edge e and any $g \in G$.*
A group G acts freely *on Γ if it acts without inversion and a vertex can be fixed only by the identity element.*
If G acts without inversion on Γ, one can define the quotient graph *of Γ by G in a natural manner. It is defined to be the graph whose vertex set is the set $G\backslash X$ of orbits of vertices of Γ under the G-action and the edges are G-orbits of edges of Γ. It is extremely important to note that our discussion will always involve only group actions without inversions. This hypothesis is exactly what is needed to make the quotient graph to be actually a graph in our sense (namely, to have distinct edges $e, \bar{e}$). It is clear that when G is generated by a subset S, then G acts freely on the corresponding Cayley graph. We remark that the assumption that a group acts without inversions on a tree is not very serious; indeed, it always does so on the first barycentric subdivision.*

Now we can prove a very important characterisation of free groups in terms of its Cayley graph.

Proposition 2. *Let Γ be the Cayley graph corresponding to a group G and a subset S. Then, Γ is a tree if, and only if, G is a free group with S as a basis.*

Proof. First, suppose that G is free with a basis S. This means that each $g \in G$ is expressible uniquely in the form $g = s_1^{t_1} \cdots s_n^{t_n}$ where $s_i \in S$, $t_i = \pm 1$ for each i and $t_i = t_{i+1}$ if $s_i = s_{i+1}$. Call n to be the length $l(g)$ of G, and write G_n for the elements of length n in G. Now, if $g \in G_n$, then clearly in the Cayley graph Γ, the vertex g is adjacent to a unique element of G_{n-1}. This defines an inverse system $\cdots G_n \to G_{n-1} \cdots G_1 \to G_0 = \{1\}$. Evidently, their union Γ is a tree.

Conversely, suppose Γ is a tree. Then, $G =< S >$ and $S \cap S^{-1} = \emptyset$. Suppose the set S is not a basis for the group G. Look at the natural map θ from the free group $F(S)$ onto G. There is an element $\hat{g} \neq 1$ of minimal length in $F(S)$ with the property that it is in the kernel of θ. Write $l(\hat{g}) = n$ and $\hat{g} = s_1^{t_1} \cdots s_n^{t_n}$ for some $s_i \in S$. Note that since $S \cap S^{-1} = \emptyset$, the length $n \geq 3$. Call the vertices corresponding to the elements $s_1^{t_1} \cdots s_i^{t_i}$ of G as P_i, for $i = 1, \cdots, n$. Call P_0, the vertex corresponding to the identity. If P_i were not distinct, then we would get a word in $F(S)$ of smaller length in Ker θ. Since $P_0 = P_n$ and since $n \geq 3$, the geometric edges $\{P_0, P_1\}, \{P_1, P_2\}, \cdots, \{P_{n-1}, P_n\}$ and $\{P_n, P_0\}$ are all distinct. Thus, $P_0, \cdots, P_{n-1}$ form a circuit of length n, contradicting the assumption that Γ is a tree. Therefore, the proposition follows. $\square$

Theorem 3 (Schreier). *A group is free if, and only if, it acts freely on a tree. More precisely, suppose G acts freely on a tree $\Gamma = (X, Y)$. Then,*

(i) there is a tree T in Γ which maps injectively onto a maximal tree in $G \backslash \Gamma$.

(ii) For a choice of T as in (I) and a choice of an orientation Y_+ preserved by G, we have:

(a) G is free with a basis S comprising of elements $g \neq 1$ for which there is an edge $e \in Y_+$ starting in T and ending in gT and

(b) if $\Gamma^* = G \backslash \Gamma$ has only a finite number m of vertices, and a number a of edges, then $|S| - 1 = \frac{a}{2} - m$.

The statement (a) says for $G = F(x, y)$ acting on $\Gamma(G, \{x, y\})$ that $\{x, y\}$ is a basis of G. The quotient graph $G\backslash\Gamma$ is a bouquet of two circles. Note that the 'only if' part of the first assertion follows already from the previous proposition as one can take the tree to be the Cayley graph with respect to a basis. The 'if' part is proved by the other assertions which are stronger. Thus, we shall prove these other assertions. Before starting with that, let us draw important corollaries.

Corollary 4. *Every subgroup H of a free group G is free. Moreover, if $[G : H] < \infty$ and if the rank of G (denoted $\mathrm{rk}(G)$) is also finite then so is the rank of H which is given by: $(\mathrm{rk}(H) - 1) = [G : H](\mathrm{rk}(G) - 1)$.*

The last formula is called Schreier's index formula. It is an analogue of the classical Riemann-Hurwitz formula for coverings of Riemann surfaces.

Proof. Since G is free, it acts freely on its Cayley graph Γ with respect to a basis, which is a tree, as noted earlier. So, any subgroup H also acts freely on Γ and is, therefore, free.
For the Schreier formula, let us consider the graphs $\Gamma_G = G\backslash\Gamma$ and $\Gamma_H := H\backslash\Gamma$ where Γ is as above. Then, evidently Γ_G has only one vertex. Also, Γ_H has $[G : H]$ vertices and $[G : H]a$ edges where a is the number of edges of Γ_G. Therefore, by (II)b of 3.3, applied to both the graphs Γ_G and Γ_H, we get

$$\begin{aligned} \mathrm{rk}(G) - 1 &= \frac{a}{2} - 1 \\ \mathrm{rk}(H) - 1 &= \frac{[G : H]a}{2} - [G : H] = [G : H](\mathrm{rk}(G) - 1) \end{aligned}$$

This proves the corollary. □

The next result is of independent interest and will also be used in the proof of the main Theorem 3.

Lemma 5. *Let G act without inversion on a connected graph X. Then, every subtree T of the quotient graph $G\backslash X$ lifts to a subtree of X. One calls a lift of a maximal subtree of $G\backslash X$ a* <u>*tree of representatives.*</u>

Proof. The proof is existential and will use Zorn's lemma. We need to show that there is a subtree of X which maps injectively onto T. Let us look at the set Ω of all subtrees of X which map injectively into T. Clearly Ω is non-empty as it does have single points. Further, if T_i, $i \in I$ is a totally ordered (under inclusion) subset of Ω, then the union T_0 is again a tree and must map injectively to T because any two points of T_0 are in some $T_i, i \in I$ which does map injectively into T. Thus, $T_0 \in \Omega$ and therefore, every totally ordered family has an upper bound in Ω. Therefore, Ω has a maximal element $\tilde{T}$. Call T', the image of $\tilde{T}$ in $G\backslash X$. Now, $T' \subset T$. Suppose, if possible, that $T' \neq T$. Then, by the connectedness of T, there is an edge e of T which start at a vertex of T' and ends at a vertex of T which is not on T'. Let us take any lift $\tilde{e}$ of the edge e to an edge of X. Since $g\tilde{e}$ for any $g \in G$ also gives a lift, we may replace $\tilde{e}$ by a suitable $g\tilde{e}$ and assume that $\tilde{e}$ has its origin in $\tilde{T}$. Note that the terminus of $\tilde{e}$ is not a vertex of $\tilde{T}$ since its image in $G\backslash X$ is not a vertex of T'. But then the graph $\hat{T}$ formed by adjoining to the tree $\tilde{T}$, the vertex $t(\tilde{e})$ and the edges $\tilde{e}$ and $\bar{\tilde{e}}$ is a tree by statement (c) of Proposition 7. Moreover, it clearly injects into T under the quotient map. This contradicts the maximality of the choice of $\tilde{T}$. Thus, our assumption that $T' \neq T$ is false. $\square$

Proof. (Of Theorem 3.) It suffices to prove the statement (II) since (I) is given by Lemma 5. As G acts freely on Γ, and since T injects into the quotient graph $G\backslash\Gamma$, the translates gT are disjoint for different elements g of G. Therefore, the quotient graph $\Gamma' := \Gamma/(G.T)$ formed by contracting each tree gT to a single vertex, is a tree as seen in the proof of Proposition 12. Let us denote by (gT) the single vertex in Γ' that the tree gT in Γ corresponds to.

Then, the map $\alpha : (gT) \mapsto g$ is a bijection from the vertices of Γ' onto the vertices of $\Gamma(G, S)$. If this map can be extended to an isomorphism $\Gamma' \to \Gamma(G, S)$, then (a) of the theorem will follow by Proposition 2. Let us construct such an extension now.

Since the edges of Γ' are those in Γ which are not in $G.T$, the edge set of Γ' acquires an orientation $Y'_+ = Y_+ \cap Edge\Gamma'$. Thus, it suffices to define $\alpha : Y'_+ \to G \times S = Edge\Gamma(G, S)_+$.

Let e be an edge in Y'_+ which starts at gT and ends at some $g'T$. As this edge e is an edge of Γ itself, this means that $g^{-1}g' \in S$. Thus, we define

$$\alpha(e) = (g, g^{-1}g').$$

From the definition of S, it is clear that the above is a surjection onto $G \times S$. Injectivity is clear as remarked above. Thus, we do have an isomorphism α as asserted and (a) of the theorem follows.

To prove (b), we note that from (a), the elements of S are in bijection with the set W of those edges in Y which start in T and end outside T. Thus, $|W| = |S|$.

Now, the image T^* of T in Γ^* is a maximal tree. The orientation Y^*_+ of Γ^*, which is the image of the orientation Y_+ of Γ, is the disjoint union of $Y^*_+ \cap edgeT^*$ and $W^* =$ image of W in Γ^*.

Also, clearly $W \to W^*$ is bijective Thus, if Γ^* has finitely many vertices, say m, then

$$|Y^*_+| - m = |W^*| + |EdgeT^*| - |VertexT^*| = |W^*| - 1 = |S| - 1$$

by Proposition 11 noting that $m = |VertexT^*|$. This proves (b). $\square$

6 Trees and Amalgams

6.1 Groups which are amalgams–I

In this section we characterize groups which are amalgamated products of the form $G_1 \underset{A}{*} G_2$ *as those groups which act on trees with fundamental domain a segment. By a segment we mean a graph of the form:*

$$\overset{}{P} \quad \overset{y}{} \quad Q$$

Definition 1. *Let* G *be a group acting on a graph* Γ*. A* <u>*fundamental domain*</u> *for* $\Gamma \bmod G$ *is a subgraph* $\Delta \subset \Gamma$ *such that* $\Delta \simeq G\backslash\Gamma$*, the isomorphism being induced from the quotienting map from* Γ *to* $G\backslash\Gamma$*.*

The first point to observe is that fundamental domains may or may not exist. Let C_n *denote the* n*-cycle or the* n*-circuit. The group* $\mathbb{Z}/3\mathbb{Z}$ *acts on the graph* C_6 *by rotating by* 120^{o} *and the quotient is* C_2*. Since* C_6 *does not have a subgraph isomorphic to* C_2 *this action can not admit a fundamental domain. The following proposition, however, assures us of the existence of fundamental domains for a large class of actions of our interest.*

Proposition 2. *Let a group* G *act on a tree* T*. A fundamental domain for* $T \bmod G$ *exists if and only if* $G\backslash T$ *is a tree.*

Proof. Recall that the map $T \to G\backslash T$ has the tree lifting property of Lemma 5. Hence, if $G\backslash T$ is a tree, it admits a lift, and the image of any such lifting is a fundametal domain.

Conversely, if Δ is a fundamental domain, then since T has no circuits so also Δ has no circuits and so $G\backslash T$ can have no circuits. Since T is connected we get that $G\backslash T$ is also connected and hence it is a tree. $\square$

The reader is asked to construct an example of a tree T with an action of a group G for which there is no fundamental domain, or equivalently, when $G\backslash T$ is not a tree. Observes also that fundamental domains need not be unique. Indeed, if Δ is one then so is $g \cdot \Delta$ for any $g \in G$.

We are now in a position to state and prove the main theorems of this section which together characterize groups which are amalgams.

Theorem 3. *Let G be a group acting on a graph Γ. Let T a segment in Γ be a fundamental domain for $\Gamma \bmod G$. Let P, Q be the vertices of T and $e = \{y, \bar{y}\}$ be the geometirc edge of T. Let G_P, G_Q and $G_y = G_{\bar{y}}$ be the stabilizers of P, Q and y respectively. Then the following are equivalent:*

(i) Γ *is a tree.*

(ii) The canonical homomorphism $G_P \underset{G_y}{*} G_Q \to G$ is an isomorphism.

Theorem 4. *Let $G = G_1 \underset{A}{*} G_2$ be an amalgam. Then there exists a tree T on which G acts with a fundamental domain a segment such that if the vertices of this segment are $\{P, Q\}$ and the edges are $\{y, \bar{y}\}$ then $G_1 \simeq G_P$, $G_2 \simeq G_Q$ and $A \simeq G_y$.*

Proof. (Of Theorem 3 implies Theorem 4.)

Let $G = G_1 \underset{A}{*} G_2$. We define a graph Γ on which G acts as follows:

$$\begin{aligned} V(\Gamma) &= G/G_1 \amalg G/G_2 \\ E(\Gamma) &= G/A \amalg \overline{G/A} \end{aligned}$$

The map defining the extremities of an edge is given by

$$\begin{aligned} E(\Gamma) &\to V(\Gamma) \times V(\Gamma) \\ gA &\mapsto (gG_1, gG_2) \end{aligned}$$

With the obvious action of G on Γ it is clear that the stabilizer of the vertex $1 \cdot G_1$ is the group G_1 and similarly that of $1 \cdot G_2$ and the edge $1 \cdot A$ are G_2 and respectively A. Now by Theorem 3 - the part (ii) implies (i) - we get that Γ is a tree. $\square$

Proof. (Of Theorem 3.) Let G act on a graph Γ with a fundamental domain a segment T with vertices $\{P, Q\}$ and edges $\{y, \bar{y}\}$. The proof will follow from the following two lemmas.

Lemma 5. Γ *is connected if and only if* $G_P \cup G_Q$ *generate* G.

Proof. (Of Lemma 5.) Let Γ' be the connected component of Γ containing the segment T. Let the stabilizer of Γ' be G', i.e.,

$$G' = \{g \in G : g\Gamma' = \Gamma'\}.$$

Let G'' be the subgroup of G generated by $G_P \cup G_Q$.

Note that $G'' \subset G'$. If $g \in G_P \cup G_Q$ then $gT \cap T$ is non-empty hence the connected component containing gT which is $g\Gamma'$ is same as that containing T from which we get that $g\Gamma' = \Gamma'$,i.e., $g \in G'$. Since $G_P \cup G_Q \subset G'$ we get that $G'' \subset G'$.

Now if $G_P \cup G_Q$ generates G then $G = G' = G''$ and hence $G\Gamma' = \Gamma' \supset GT = \Gamma$, i.e., Γ is connected.

For the converse, suppose Γ is connected. Note that we can always write $\Gamma = G''T \amalg (G - G'')T$. (If the union is not disjoint then there exists $x \in G''$, $y \in G - G''$ such that either $y^{-1}x$ fixes P or Q or that $y^{-1}x$ takes P to Q or Q to P. The former contradicts $y \notin G''$ and the latter is ruled out since T is a fundamental domain.) We hence get that $\Gamma = G''T$. But Γ connected implies that $\Gamma' = \Gamma$ and so $G' = G$. Hence we have

$$G'T = GT = \Gamma = G''T.$$

This implies that $G'' \subset G'$ because if $x'' \in G''$ then $x''\Gamma' = x''\Gamma = x''G''T = G''T = \Gamma = \Gamma'$. Therefore $G'' = G' = G$,i.e., $G_P \cup G_Q$ generates G. $\square$

Lemma 6. Γ *has a circuit if and only if the canonical homomorphism*

$$G_P \underset{G_y}{*} G_Q \to G$$

is not injective.

Proof. (Of Lemma 6.) Let $c = (w_0, \ldots, w_n)$ be a circuit in Γ with $w_i \in E(\Gamma)$. Assume c has no backtracking, because if there is any backtracking then there is 'smaller' circuit without backtracking. This also implies that $n \geq 2$.

Let $w_i = h_i y_i$ where $h_i \in G$ and $y_i \in \{y, \bar{y}\}$. By projecting c down to Γ mod $G = T$ we get

$$o(y_i) = t(y_{i-1}) = P_i \in \{P, Q\}.$$

The same consideration gives that $\bar{y}_i = y_{i-1}$.

Note that

$$h_i P_i = h_i o(y_i) = o(h_i y_i) = o(w_i) = t(w_{i-1}) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i.$$

This gives for each i an element $g_i \in G_{P_i}$ such that $h_i = h_{i-1} g_i$. Further $g_i \notin G_y$ because if indeed $g_i \in G_y$ then

$$\bar{w_i} = \overline{h_i y_i} = \overline{h_{i-1} g_i y_i} = \overline{h_{i-1} y_i} = h_{i-1} \overline{y_i} = h_{i-1} y_{i-1} = w_{i-1}$$

contradicting that c has no backtracking. To summarize, for each i, $h_i = h_{i-1} g_i$ with $g_i \in G_{P_i} - G_y$.

Since c is a circuit, $o(w_0) = o(c) = t(c) = t(w_n)$. Which implies by going modulo G that $P_0 = o(y_0) = t(y_n)$. In particular,

$$h_0 P_0 = o(w_0) = t(w_n) = t(h_n y_n) = h_n t(y_n) = h_n P_0.$$

Successively using the definitions of the elements g_i we get

$$h_0 P_0 = h_n P_0 = h_{n-1} g_n P_0 = \cdots = h_0 g_1 \ldots g_n P_0.$$

Cancelling h_0 we get that there is an element $g_0 \in G_{P_0}$ such that $g_0 g_1 \ldots g_n = 1$.

Now we may start with such a sequence $g_0, g_1, \ldots g_n$ and construct a circuit c in Γ. To summarize the proof we have shown that the following are equivalent:

(i) There is a circuit $c = (w_0, \ldots, w_n)$ in Γ without backtracking.

(ii) There is a sequence $g_0, \ldots, g_n \in G_P \cup G_Q$ with $g_i \notin G_y$ for all $i \geq 1$ such that $g_0 g_1 \ldots g_n = 1$.

The second statement is of course another way to state that the canonical homomorphism from $G_P \underset{G_y}{*} G_Q \to G$ is not injective. $\square$

As mentioned before this finishes the proof of Theorem 3. $\square$

6.2 Applications to subgroups of amalgamated groups

In this section we use the characterization of amalgamated groups $G_1 \underset{A}{*} G_2$ *proved in the previous section and derive some consequences for subgroups of such amalgams.*

Proposition 7. *Let H be a subgroup of $G = G_1 \underset{A}{*} G_2$ such that $H - \{1\}$ does not intersect any conjugate of either G_1 or G_2. Then Γ is a free group.*

Proof. Let T be the tree on which G acts such that a fundamental domain is a segment as in Theorem 4. The hypothesis on the subgroup H can be restated as

$$\mathrm{Stab}_H(P) = H \cap \mathrm{Stab}_G(P) = \{1\}, \quad \forall P \in V(T),$$

i.e., that H acts freely on the tree T. Hence by Theorem 3 we get that H is free. $\square$

Recall Proposition 19 which states that any torsion element of $G = G_1 \underset{A}{*} G_2$ *can be conjugated inside either* G_1 *or* G_2*. This statement can be generalized to bounded subgroups.*

Definition 8. *A subset* Ω *of an amalgamated product* $G = \underset{A}{*} G_i$ *is said to be* <u>*bounded*</u> *if there exists* $M > 0$ *such that* $l(g) \leq M$ *for all elements of* $g \in \Omega$*. Here* $l(g)$ *is the length of the element* g *coming from its unique reduced expression. A subgroup is said to be a* <u>*bounded subgroup*</u> *if it is bounded as a subset of* G*.*

Proposition 9. *Let* H *be a bounded subgroup of an amalgam* $G = G_1 \underset{A}{*} G_2$*. Then* H *can be conjugated inside either* G_1 *or* G_2*.*

Proof. Let T be the tree on which G acts such that a fundamental domain Δ is a segment as in Theorem 4.

Let $V(\Delta) = \{P, Q\}$ be the vertices of Δ. Note that if $g \in G_1 \cup G_2$ then $gT \cap T$ is non-empty. Hence if Ω is a bounded subset of G then $\Omega \cdot P$ is a bounded subset of the metric space $V(T)$. In particular, $H \cdot P$ is a bounded subset of $V(T)$.

Let T' be the subtree of T generated by $H \cdot P$. The tree T' is simply the union of all geodesics in T joining all pairs of points in $H \cdot P$. In particular, T' is bounded and also H-stable.

In other words, the group H acts on a tree T' of finite diameter. By Corollary 10 there is either a vertex v or a geomteric edge $\{e, \bar{e}\}$ fixed by H. Since we have assumed that all our actions are without inversions, if H fixes $\{e, \bar{e}\}$ then H actually fixes both e and $\bar{e}$, hence it fixes the extremities of e.

So in all cases H fixes a vertex, i.e., H can be conjugated inside either G_1 or G_2. $\square$

Corollary 10 (To the proof of Proposition 9). *Let G be a group acting on a tree T. Suppose there is a vertex $v \in V(T)$ such that its G-orbit $G \cdot v$ is a bounded subset of $V(T)$ then there is a vertex in T which is fixed by G.*

This corollary resembles the famous Bruhat-Tits fixed point theorem of a bounded group of automorphisms of a building admitting a fixed point.

6.3 Groups which are amalgams–II

In this section we present a generalization of the results in Section 6.1 and characterize groups which are amalgamated products with any number of factors as groups G which act on trees T such that quotient $G\backslash T$ is also a tree. For this we need the following definition.

Definition 11. *A graph of groups $(\mathcal{G}, \Gamma)$ consists of*

(i) A graph Γ.

(ii) A collection of groups $\mathcal{G}$ consisting of

- *A group G_v for every vertex $v \in V(\Gamma)$*
- *A group G_e for every edge $e \in E(\Gamma)$ such that $G_e = G_{\bar{e}}$.*

(iii) For each edge $e \in E(\Gamma)$ a monomorphism $G_e \to G_{t(e)}$ denoted $x \mapsto x^e$.

If Γ is a tree then we call $(\mathcal{G}, \Gamma)$ a tree of groups.

The direct limit of the system of groups given by a graph of groups $(\mathcal{G}, \Gamma)$ *will be denoted*

$$G_\Gamma = \mathcal{G}_\Gamma = \varinjlim(\mathcal{G}, \Gamma).$$

Example 12. *The amalgamated product* $\underset{A}{*} G_i$ *is the direct limit of the following graph of groups:*

$$\begin{array}{ccc} & & G_1 \\ & A & \\ & \quad A & \\ A & & G_2 \\ & & \\ & A & \\ & & G_i \end{array}$$

Example 13. *Let* $(\mathcal{G}, T)$ *be a tree of groups. Let* v *be a terminal vertex of* T *and let* $T' = T - v$*. Suppose* $E(T) = E(T') \cup \{e, \bar{e}\}$*. Let* $\mathcal{G}'$ *be the restriction of* $\mathcal{G}$ *to* T'*. Then*

$$\mathcal{G}_T = \mathcal{G}'_{T'} \underset{G_e}{*} G_v.$$

We follow the convention that if $(\mathcal{G}, T)$ *is a tree of groups then every vertex group* G_v *and every edge group* G_e *is identified as a subgroup of* $G_T = \mathcal{G}_T = \varinjlim(\mathcal{G}, T)$*. We can now state the first main theorem of this section.*

Theorem 14. *Let* $(\mathcal{G}, T)$ *be a tree of groups. Then there exists a graph* Γ *containing* T *and an action of* G_T *on* Γ *characterized by:*

(i) T *is a fundamental domain for* $\Gamma \bmod G_T$.

(ii) $\mathrm{Stab}_{G_T}(v) = G_v$ *for all* $v \in V(T) \subset V(\Gamma)$.

(iii) $\mathrm{Stab}_{G_T}(e) = G_e$ *for all* $e \in E(T) \subset E(\Gamma)$.

Moreover the graph Γ *is a tree.*

Proof. The characterizing properties, in fact, force the vertex set and edge set of Γ to be given by:

$$V(\Gamma) := \amalg_{v \in V(T)} G_T/G_v = \amalg_{v \in V(T)} G_T \cdot v \tag{15}$$

$$E(\Gamma) := \amalg_{e \in E(T)} G_T/G_e = \amalg_{e \in E(T)} G_T \cdot e \tag{16}$$

The extremities of an edge in Γ, namely, the map $E(\Gamma) \to V(\Gamma) \times V(\Gamma)$ is given by $xG_e \mapsto (xG_{\mathrm{o}(e)}, xG_{\mathrm{t}(e)})$ for any $x \in G_T$. This is well defined since G_e is a subgroup of $G_{\mathrm{o}(e)}$ and $G_{\mathrm{t}(e)}$. Clearly, Γ is a graph containing T and comes with a canonical action of G_T such that T is a fundamental domain. It suffices now to show that Γ is a tree.

Since T is the direct limit of its finite subtrees and *everything in sight* commutes with direct limits, we may assume that T is itself finite. Let w be a terminal vertex of T. Let $T' = T - w$. Let $\{y, \bar{y}\}$ be the edges of T connecting w with T'. Let Γ' be the graph associated to T' by the theorem and by induction Γ' is a tree. By Example 13 we get that $G_T = G'_{T'} \underset{G_y}{*} G_w$. Also Γ' is a subgraph of Γ by construction and in fact $\cup_{g \in G_T} g \cdot \Gamma'$ is a disjoint union of trees inside Γ. Let

$$\widetilde{\Gamma} = \frac{\Gamma}{\cup_{g \in G_T} g \cdot \Gamma'}.$$

It is clear that G_T acts on $\widetilde{\Gamma}$ with fundamental domain T/T' which is a segment with one vertex as T' and the other vertex being w. Since $G_T = G'_{T'} \underset{G_y}{*} G_w$ we get by Theorem 3 that $\widetilde{\Gamma}$ is a tree. By Corollary 13 we get that Γ is a tree since we obtained $\widetilde{\Gamma}$ by quotienting out a disjoint union of trees and so did no change the homotopy type. □

We now prove the converse. For the converse, we begin with a group G acting on a graph Γ such that a fundamental domain is a tree T. Let $(\mathcal{G}, T)$

be the tree of groups determined by the stabilizers for the action of G on T, i.e.,

$$\begin{aligned} \forall v \in V(T),\ G_v &:= \mathrm{Stab}_G(v) \\ \forall e \in E(T),\ G_e &:= \mathrm{Stab}_G(e) \end{aligned}$$

Let G_T be the direct limit of the system $(\mathcal{G}, T)$. Since by definition G_v and G_e are subgroups of G we get a canonical map $G_T \to G$. Note that if Γ is connected then this map is surjective.

Let $\widetilde{\Gamma}$ be the tree associated to $(\mathcal{G}, T)$ by Theorem 14. By the hypothesis that T is a fundamental domain for G-action on Γ we get

$$V(\Gamma) := \amalg_{v \in V(T)}\, G \cdot v \tag{17}$$

$$E(\Gamma) := \amalg_{e \in E(T)}\, G \cdot e \tag{18}$$

Comparing with Equations (15) and (16) we get that there is a canonical map $\widetilde{\Gamma} \to \Gamma$ which is $G_T \to G$ equivariant. We are now in a position to state the converse.

Theorem 19. *With the notations as above, the following are equivalent:*

(i) Γ *is a tree.*

(ii) $\widetilde{\Gamma} \to \Gamma$ *is an isomorphism.*

(iii) $G_T \to G$ *is an isomorphism.*

Proof. That (2) is equivalent to (3) follows from Equations (15), (16), (17) and (18). That (2) implies (1) is a tautology since $\widetilde{\Gamma}$ is a tree. The only implication which needs a proof is (1) implies (2).

Note that the map $\widetilde{\Gamma} \to \Gamma$ is locally injective, i.e., it is injective on the set of edges with a given origin. (See Exercise 24.) Now the proof follows from the following lemma.

Lemma 20. *Let $f : \widetilde{X} \to X$ be a morphism of graphs where $\widetilde{X}$ is a connected graph and X is a tree. If f is locally injective then it is actually injective.*

Proof. (Of Lemma 20.) It is enough to show that f is injective on paths without backtracking. Let $\widetilde{c}$ be a path in $\widetilde{X}$ without backtracking such that f restricted to $\widetilde{c}$ is not injective. The image c of $\widetilde{c}$ under f will have to either a circuit or a backtracking. Since X is a tree it has to have a backtracking. But the image c having a backtracking contradicts local injectivity. □

This also concludes the proof of Theorem 19 □

6.4 $\mathrm{PSL}_2(\mathbb{Z})$

In this section we show that the group $\mathrm{PSL}_2(\mathbb{Z})$ is a certain free product as in Example 10.

For this we need a little bit of preliminaries. Let $\mathfrak{h}$ denote the upper half plane of all complex numbers z with $\mathrm{Im}(z) > 0$. Let $\mathrm{SL}_2(\mathbb{R})$ denote the group of all two-by-two matrices with real entries and of determinant one.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{h}$ via linear fractional transformations given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

This action is transitive as can be seen by:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \cdot i = x + iy$$

for any x and $y > 0$. We leave it to the reader to check that the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ of the point i is the subgroup $\mathrm{SO}(2)$.

For the action of $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathfrak{h}$ *it is a classical fact that a fundamental domain is given by the region*

$$\{z \in \mathfrak{h} : |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1\}.$$

This region is depicted in the following diagram.

In this diagram, the point i *and the point* $\rho = 1/2 + i\sqrt{3}/2$ *are rather special. We ask the reader to verify that*

$$\begin{aligned} \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(i) &= \mathbb{Z}/4\mathbb{Z} \\ \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\rho) &= \mathbb{Z}/6\mathbb{Z} \end{aligned}$$

Consider the segment of the circle $|z| = 1$ *which connects the points* i *and* ρ *then if we take all the* $\mathrm{SL}_2(\mathbb{Z})$ *translates of this segment it turns out that this geometric object is in fact the geometric realization of a tree. By construction a fundamental domain for the action of* $\mathrm{SL}_2(\mathbb{Z})$ *on this tree is the 'segment' joining the points* i *and* ρ*. If* e *is the edge denoting this segment then the stabilizer of this edge is the kernel of the action, namely* $\mathbb{Z}/2\mathbb{Z}$ *(because any linear fractional transformation which fixes three distinct points necessarily fixes every point in* $\mathfrak{h}$*). We hence get that*

$$\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}.$$

The action of $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathfrak{h}$ *factors through* $\mathrm{PSL}_2(\mathbb{R})$ *and computing the* PSL_2 *stabilizers of* i*,* ρ *and* e *we get*

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

6.5 Exercises

Exercise 21. Show that any torsion-free subgroup of $PSL(2, \mathbb{Z})$ is free. Explicitly describe an index 6 free subgroup of $PSL(2, \mathbb{Z})$.

Exercise 22. Let $(\mathcal{G}, T)$ be a a tree of groups. Let v be a terminal vertex of T and let $T' = T - v$. Suppose $E(T) = E(T') \cup \{e, \bar{e}\}$. Let $\mathcal{G}'$ be the restriction of $\mathcal{G}$ to T'. Then prove that

$$\mathcal{G}_T = \mathcal{G}'_{T'} \underset{G_e}{*} G_v.$$

Exercise 23. Justify the convention that if $(\mathcal{G}, T)$ is a tree of groups then every vertex group G_v and every edge group G_e can indeed be identified as a subgroup of $G_T = \mathcal{G}_T = \varinjlim(\mathcal{G}, T)$.

Exercise 24. With the notations of Theorem 19 prove that the map $\widetilde{\Gamma} \to \Gamma$ is locally injective.

Exercise 25. *Show using the amalgamated product structure that*

(i) The abelianization of $\mathrm{PSL}_2(\mathbb{Z})$ *is* $\mathbb{Z}/6\mathbb{Z}$.

(ii) The abelianization of $\mathrm{SL}_2(\mathbb{Z})$ *is* $\mathbb{Z}/12\mathbb{Z}$.

Exercise 26. *Let* $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ *be the dihedral group of order* $2n$. *Show that*

$$\mathrm{GL}_2(\mathbb{Z}) = D_4 \underset{D_2}{*} D_6.$$

7 Structure of groups acting on trees

We saw earlier that when a group G acts freely without inversion on a tree X, then G is a free group. When G acts (not necessarily freely but) without inversion on a tree X such that the quotient graph $G \backslash X$ is a tree, then G is an amalgam of the vertex stabilisers for vertices of a tree of representatives of $G \backslash X$. A notion due to H.Bass, which generalises the notion of amalgams is that of the fundamental group of a graph of groups. We shall see examples *shortly.*

The motivation comes from basic algebraic topology. If a group G acts properly discontinuously on a simply-connected, path-connected and locally path-connected topological space $\tilde{X}$, then one can recover G back as the fundamental group of the quotient space $G\backslash\tilde{X}$. Analogously, if a group G acts without inversions on a tree X, then we shall define a suitable structure of a graph of groups on the quotient $G\backslash X$ such that G can be identified with the fundamental group (to be defined) of such an object. Recall from section 4, that a graph of groups (G, Y) entails providing a connected graph Y, vertex stabilisers G_v and edge stabilisers $G_e = G_{\bar{e}}$ such that, there are monomorphism $g \mapsto g^e$ from G_e into $G_{t(e)}$ where $t(e)$ is the terminal vertex of e. In the notation (G, Y), there is no group G but one ought to think of G as a kind of functor which associates groups to vertices and to edges.

Definition 1. *For a graph of groups (G, Y), let $F(G, Y)$ be the group generated by the vertex stabilisers G_v and edges e subject to the relations:*

$$\bar{e} = e^{-1} \ , \ eg^e e^{-1} = g^{\bar{e}} \quad \forall \ g \in G_e.$$

Let T be a maximal tree in Y. Then, the <u>fundamental group</u> $\pi_1(G, Y, T)$ of the graph of groups (G, Y) at T is defined as the quotient of $F(G, Y)$ by the normal subgroup generated by the edges of T.

This definition is quite similar to the usual definition of the fundamental group as an edge path group.

Equivalently, if, for each edge e of Y, the image in $\pi_1(G, Y, T)$ is denoted by g_e, then one can see that $\pi_1(G, Y, T)$ is generated by the groups G_P as P runs over vertices of Y and the elements g_e as e runs over edges, subject to the relations

$$g_e a^e g_e^{-1} = a^{\bar{e}} \ , \ g_{\bar{e}} = g_e^{-1} \quad \forall \quad a \in G_e \ ,$$

$$g_e = 1 \ \ \forall \ \ y \in edge(T).$$

Remark 2. Let R be the normal subgroup of $\pi_1(G, Y, T)$ generated by the images of G_P. Then, the quotient group $\pi_1(G, Y, T)/R$ is a free group with a basis $\{g_e : e \in E_+ \setminus (T \cap E_+)\}$ where E_+ is an orientation of Y.
As a matter of fact, the above quotient is the (usual) fundamental group of the graph Y relative to the maximal tree T.

Example 3. Suppose that edge stabilisers are all trivial. Then,

$$\pi_1(G, Y, T) = (*_P G_P) * F$$

where F is a free group with a basis as in the remark above.
In particular, if all vertex stabilisers are trivial, then the fundamental group is a free group of rank $|Edge(Y) \setminus Edge(T)|$.

Example 4. If Y is a segment with vertices P, Q and edge e from P to Q, then $\pi_1(G, Y, Y) = G_P *_{G_e} G_Q$.
More generally, if Y is a tree, then

$$\pi_1(G, Y, Y) = lim_{\rightarrow}(G, Y).$$

It is the amalgam of the vertex groups amalgamated along the edge groups.

Example 5. Let Y be a loop at a point P. Let us call the edges as e and $\bar{e}$. We have then two injective homomorphisms $a \mapsto a^e$ and $\theta : a \mapsto a^{\bar{e}}$ from G_e to G_P. Then, the maximal tree is the single point P and $\pi_1(G, Y, P) = F(G, Y)$. So, it is generated by G_P and an element $g = g_e$, modulo the relations $ga^e g^{-1} = a^{\bar{e}}$ for all $a \in G_e$.
If we identify G_e with a subgroup of G_P by means of $a \mapsto a^e$, then $\pi_1(G, Y, P)$ is just the group obtained from (G_e, G_P, θ) as an HNN extension. Thus, $\pi_1(G, Y, P)$ is the semi-direct product of the cyclic group $< g >$ with the normal subgroup R generated by all the conjugates $g^n G_P g^{-n}$ for $n \in \mathbb{Z}$.

Example 6. *The fundamental group of any graph of groups (G, Y) with respect to a maximal subtree T can be constructed successively as a free product with amalgamation for each edge in T followed by an HNN construction for each edge not in T.*

Our main goal is a general structure theorem for groups acting on trees. The theorem will tell us that such a group is the fundamental group of a suitable graph of groups. A crucial ingredient in constructing this suitable graph of groups is the notion and the existence of the universal covering of a graph of groups on which the fundamental group acts. We proceed to introduce it now.

Definition 7. *Let (G, Y) be a graph of groups with Y connected. Let T be a maximal subtree and let E_+ be an orientation of Y. For any edge e, let us write $|e|$ for the edge e or $\bar{e}$ which is in E_+ and write π for the fundamental group $\pi_1(G, Y, T)$. Recall that the image of G_e in $G_{t(e)}$ is denoted by G_e^e and that π is generated by the various vertex stabilisers G_P along with elements g_e corresponding to edges e modulo certain relations.*
Then, the graph $\tilde{X}$ is defined as follows.
Define Vert $\tilde{X} = \sqcup_{P \in VertY} \pi/G_P$ where π/G_P denotes the set of left cosets of G_P in π.
Define Edge $\tilde{X} = \sqcup_{e \in EdgeY} \pi/G_w^w$ where $w = \overline{|e|}$.
If we call the coset corresponding to 1 in π/G_P as $\tilde{P}$ and the coset corresponding to 1 in π/G_w^w with $w = \overline{|e|}$ as $\tilde{e}$, we have sections Vert $Y \to$ Vert $\tilde{X}$; $P \mapsto \tilde{P}$ and Edge $Y \to$ Edge $\tilde{X}$; $e \mapsto \tilde{e}$.
Now, the vertices of $\tilde{X}$ are $g\tilde{P}$ and the edges are $g\tilde{e}$ for $g \in \pi$, $P \in$ Vert Y and $e \in$ Edge Y.
We must define the inverse of each edge and the origin and the terminus of

each edge of $\tilde{X}$ now. We use the notation χ for the characteristic function of E_+ i.e., for each edge e of Y, we have $\chi(e) = 1$ or 0 according as $e \in E_+$ or not. Then, we define

$$\overline{g\tilde{e}} = g\tilde{\bar{e}}$$

$$o(g\tilde{e}) = gg_e^{\chi(e)-1}\widetilde{o(e)}$$

$$t(g\tilde{e}) = gg_e^{\chi(e)}\widetilde{t(e)}$$

In the three definitions, we note that the left hand sides depend only on the coset of g in π/G_z^z where $z = \overline{|e|}$ and, we need to check that the right hand sides also remain the same. This is the contention of the following result.

Lemma 8. *The above three expressions are well-defined.*

Proof. For the first expression, note that the right hand side is a coset of π/G_w^w where $w = \overline{|\bar{e}|}$. Since the left hand side depends only on the coset of g in π/G_z^z where $z = \overline{|e|}$, we need to check that the right hand side also remains the same coset in π/G_w^w when g is replaced by any other element gx where $x \in G_z^z$. Since $z = w$, this is clear.
Let us prove that the second definition is also meaningful. We need to prove that

$$xg_e^{\chi(e)-1}\widetilde{o(e)} = g_e^{\chi(e)-1}\widetilde{o(e)}.$$

First, let us look at the case when $e \in E_+$ i.e., when $\chi(e) = 1$. Then, $z = \bar{e}$ and $x \in G_z^z \leq G_{t(\bar{e})} = G_{o(e)}$. Thus, $x\widetilde{o(e)} = \widetilde{o(e)}$ as asserted.
Now, let us look at the other case when $e \notin E_+$ i.e., $\chi(e) = 0$. Then, $z = e$ and $x \in G_e^e$.
But, in π, we have the relation $g_e a^e g_e^{-1} = a^{\bar{e}}$ for all $ainG_e$.
In other words, $g_e x g_e^{-1} \in G_{\bar{e}}^{\bar{e}} \leq G_{t(\bar{e})} = G_{o(e)}$ which proves that

$$xg_e^{-1}\widetilde{o(e)} = g_e^{-1}\widetilde{o(e)}.$$

Thus, we have shown that the second expression is also well-defined. The third expression for e is just the same as the second expression for $\bar{e}$. The lemma is proved. $\square$

The most important property of the universal covering constructed is that it is actually a tree. The proof that we discuss is due to H.Bass. However, we need an alternative definition of the fundamental group of a graph of groups. This definition will depend on paths (analogous to the usual edge path group in topology) and will show that the fundamental group does not really depend on the choice of a maximal subtree.

Let (G, Y) be a graph of groups and $F(G, Y)$ be the group in Definition 1. Recall what this means. This is the group generated by the vertex stabilisers G_v and edges e subject to the relations:

$$\bar{e} = e^{-1} \ , \ eg^e e^{-1} = g^{\bar{e}} \ \ \forall \ g \in G_e.$$

For any path $c = e_1 \cdots e_n$ in Y where e_i starts at a vertex P_{i-1} and ends at P_i, and for any sequence of elements $r_i \in G_{P_i}$, $(i = 0, \cdots, n)$, write $\mu = (r_0, \cdots, r_n)$. Then, the pair (c, μ) is called a word of type c in $F(G, Y)$. To such a word, one associates the element $r_0 e_1 r_1 e_2 \cdots e_n r_n$ of $F(G, Y)$ and denotes it by $|c, \mu|$. For $n = 0$, one defines $|c, \mu| = r_0$. Note that we have identified elements of the vertex stabilisers with their canonical images in $F(G, Y)$.

For a vertex P_0, the fundamental group $\pi_1(G, Y, P_0)$ of (G, Y) at P_0 is defined to be the elements of $F(G, Y)$ of the form $|c, \mu|$, where c is a path starting and ending at P_0. Then, we have:

Proposition 9. *Let (G, Y) be a graph of groups, let P_0 be a vertex and let T be a maximal subtree. Then, under the natural map $p : F(G, Y) \rightarrow$*

$\pi_1(G,Y,T)$, the subgroup $\pi_1(G,Y,P_0)$ maps isomorphically onto $\pi_1(G,Y,T)$. In particular, the fundamental group is independent of the choice of P_0 as well as independent of the choice of T.

Proof. The argument is similar to the usual one for the edge path group. For any vertex P of Y, call the geodesic joining P_0 to P as c_P for simplicity. If c_P is the concatenation of edges $e_1, \cdots, e_n$ in that order, then look at the corresponding element $\gamma_P = e_1 \cdots e_n$ of $F(G,Y)$.

Then, each element x of G_P gives another new element of $F(G,Y)$ viz., $\tilde{x} = \gamma_P x \gamma_P^{-1}$.

Similarly, any edge e of Y gives an element $\tilde{e} = \gamma_{o(e)} e \gamma_{t(e)}^{-1}$ of $F(G,Y)$.

Of course, these elements are in the subgroup $\pi_1(G,Y,P_0)$ of $F(G,Y)$ and we shall show that the maps $x \mapsto \tilde{x}$ and $e \mapsto \tilde{e}$ factor through to a homomorphism from $\pi_1(G,Y,T)$ to $\pi_1(G,Y,P_0)$.

For an edge e in T, either the geodesic $c_{o(e)}$ from P_0 to $o(e)$ comes via $t(e)$ or the geodesic $c_{t(e)}$ from P_0 to $t(e)$ comes via $o(e)$ since there are no circuits in T. In the first case, $\gamma_{t(e)}\bar{e} = \gamma_{o(e)}$ and in the second case, $\gamma_{o(e)}e = \gamma_{t(e)}$.

In either case, we have the element $\tilde{e} = 1$ in $F(G,Y)$ since $\bar{e} = e^{-1}$ in this group.

Also, clearly for any edge e, we have $\tilde{e}\tilde{\bar{e}} = 1$.

Finally, if $a \in G_e$, then

$$\begin{aligned}\tilde{e}\widetilde{a^e}\tilde{e}^{-1} &= \gamma_{o(e)} e \gamma_{t(e)}^{-1} \gamma_{t(e)} a^e \gamma_{t(e)}^{-1} \gamma_{t(e)} e^{-1} \gamma_{o(e)}^{-1} \\ &= \gamma_{o(e)} e a^e e^{-1} \gamma_{o(e)}^{-1} = \gamma_{o(e)} a^{\bar{e}} \gamma_{o(e)}^{-1} = \widetilde{(a^{\bar{e}})}.\end{aligned}$$

Thus, we have shown that for any element x of a vertex stabiliser and any edge e, the corresponding elements $\tilde{x}$ and $\tilde{e}$ satisfy the relations which we quotient out by to go from $F(G,Y)$ to $\pi_1(G,Y,T)$. Therefore, there is a homomorphism $f : \pi_1(G,Y,T) \to \pi_1(G,Y,P_0)$ which maps x to $\tilde{x}$ and g_e to

$\tilde{e}$.

Under the projection p, the elements γ_P map to the trivial element. In other words, $p \circ f = Id$.

We shall show that $f \circ p = Id$. For this, look at a closed path c starting at P_0, and having edges $e_1, \cdots, e_n$ say. Then, denoting by P_i the terminating vertices for the edges e_1, and for any word (c, μ) of type c where $\mu = (r_0, \cdots, r_n)$, the element $r_0 e_1 r_1 e_2 \cdots e_n r_n$ of $F(G, Y)$ is actually in $\pi_1(G, Y, P_0)$.

Note that $\gamma_{P_0} = 1$, $\tilde{r_i} = \gamma_{P_i} r_i \gamma_{P_i}^{-1}$ and $\tilde{e_i} = \gamma_{P_i} e_i \gamma_{P_{i+1}}^{-1}$.

So

$$\tilde{r_0}\tilde{e_1} \cdots \tilde{e_n}\tilde{r_n} = \gamma_{P_0} r_0 e_1 r_1 e_2 \cdots e_n r_n \gamma_{P_0}^{-1} = r_0 e_1 r_1 e_2 \cdots e_n r_n.$$

Thus, $f \circ p = Id$ as well and the proposition is proved. □

If we know when an element of $F(G, Y)$ associated to a path in Y can be trivial, we can use it to show later the crucial property of the "universal covering" $\tilde{X}$ of a graph of groups that it has no circuits. This is the following technical theorem akin to Britton's lemma and its proof is also due to Bass. To state it, we need one notion.

Definition 10. *Let (c, μ) be a word of type c in $F(G, Y)$. One says that (c, μ) is reduced if, whenever $e_{i+1} = \bar{e}_i$, we have $r_i \notin G_{e_i}^{e_i}$. For $n = 0$, the definition says $r_0 \neq 1$. Therefore, we note that* every word whose type is a path of non-zero length which does not backtrack, is reduced.

Theorem 11. *If (c, μ) is a reduced word, then corresponding element $|c, \mu|$ of $F(G, Y)$ is nontrivial.*

Before discussing the rather technical proof, we draw corollaries and use them.

Corollary 12. *(a) The homomorphisms $G_P \to F(G, Y)$ are injective.*

(b) If (c, μ) is reduced and, if $l(c) \geq 1$, then $|c, \mu| \notin G_{P_0}$.

(c) If T is a maximal subtree and if (c, μ) is a reduced word whose type c is a closed path, then the image of $|c, \mu|$ in $\pi_1(G, Y, T)$ is nontrivial.

Proof. The assertion (a) is just the statement of the theorem when $l(c) = 0$. For (b), we just observe that if $|c, \mu|$ were in G_{P_0}, then we would have a reduced word (c, μ') where $\mu' = (|c, \mu|^{-1} r_0, r_1, \cdots, r_n)$ with $|c, \mu'| = 1$ contradicting the theorem.
To prove (c), notice that $|c, \mu| \in \pi_1(G, Y, P_0)$. We proved above that the natural map from $F(G, Y)$ to $\pi_1(G, Y, T)$ maps the subgroup $\pi_1(G, Y, P_0)$ isomorphically onto $\pi_1(G, Y, T)$.
Thus, the corollary follows. $\square$

Now, we can prove that the universal covering is actually a tree and will use it in the proof of the main structure Theorem 14.

Theorem 13. *Let (G, Y) be a graph of groups with the graph Y being connected. Let $\tilde{X} = \tilde{X}(G, Y, T)$ be the "universal covering" graph constructed in 5.7 corresponding to a maximal subtree T of Y and an orientation E_+. Then, $\tilde{X}$ is a tree.*

Proof. Let us first show that $\tilde{X}$ is connected.
Note that $\pi = \pi_1(G, Y, T)$ acts on $\tilde{X}$ and that Y can be identified with the quotient graph $\pi \ \tilde{X}$. Recall that T contains all vertices of Y and $g_e = 1$ for all edges e of T. Thus, $o(\tilde{e}) = \widetilde{o(e)}$ and $t(\tilde{e}) = \widetilde{t(e)}$ for all e of T. In other words, $P \mapsto \tilde{P}$, $e \mapsto \tilde{e}$ is a lift $T \hookrightarrow \tilde{T}$ of T to a tree.
Now, for each edge e of Y, the corresponding edge $\tilde{e}$ has $o(\tilde{e}) = \widetilde{o(e)}$ or $t(\tilde{e}) = \widetilde{t(e)}$ according as whether $\chi(e) = 1$ or 0. In other words, either the origin or the terminus of any edge of $\tilde{X}$ is on $\tilde{T}$. Therefore, the subgraph W

of $\tilde{X}$ generated by the edges $\tilde{e}$, e an edge of Y, is connected.
As the edges of $\tilde{X}$ are given by the π-orbits of $\tilde{e}$, where e runs over edges of Y, we have that $\pi.W = \tilde{X}$. Thus, it suffices to produce a finite subset S of π which generates it and check that $W \cup sW$ is connected for each $s \in S$.
It will then follow by induction on n that each $W \cup s_1W \cup s_1s_2W \cup \cdots \cup s_1 \cdots s_nW$ is connected for all $s_i \in S \cup S^{-1}$.
Take $S = \bigcup_P G_P \cup \bigcup_e g_e$. Clearly, S generates π. Let $s \in S$.
If $s \in G_P$, then evidently, W and sW have $\tilde{P}$ as a common vertex because elements of G_P fix $\tilde{P}$. Thus, $W \cup sW$ is connected.
If $s = g_e$, then again W and sW have a common vertex viz., $o(\tilde{e})$ or $t(\tilde{e})$ according as whether $\chi(e) = 0$ or 1.
Hence, we have shown that $\tilde{X}$ is connected. We need to check now that there are no circuits.
Suppose, if possible, that there is a path $\tilde{c}$ in $\tilde{X}$ starting and ending at the same vertex P_0 and has no backtracking (i.e., if $s_1\tilde{e}_1, \cdots, s_n\tilde{e}_n$ is the sequence of edges of $\tilde{c}$, then $s_{i+1}\tilde{e}_{i+1} \neq \overline{s_i\tilde{e}_i}$).
Let $P_1, \cdots, P_n = P_0$ be vertices of Y such that the edges $s_i\tilde{e}_i$ ends at $\tilde{P}_i$. We shall produce a reduced word in $F(G, Y)$ whose corresponding element in π is actually trivial.
Let us write χ_i in place of $\chi(e_i)$ and g_i in place of g_{e_i} for simplicity. Then, we note :

$$t(s_n\tilde{e}_n) = s_n g_n^{\chi_n}\tilde{P}_n = o(s_1\tilde{e}_1) = s_1 g_1^{\chi_1 - 1}\tilde{P}_0$$

$$t(s_1\tilde{e}_1) = s_1 g_1^{\chi_1}\tilde{P}_1 = o(s_1\tilde{e}_1) = s_2 g_2^{\chi_2 - 1}\tilde{P}_1$$

$$\cdots\cdots$$

$$t(s_{n-1}\tilde{e}_{n-1}) = s_{n-1} g_{n-1}^{\chi_{n-1}}\tilde{P}_{n-1} = o(s_n\tilde{e}_n) = s_n g_n^{\chi_n - 1}\tilde{P}_{n-1}.$$

Therefore, writing $q_i = s_i g_i^{\chi_i - 1}$, we have

$$q_n g_n \tilde{P}_0 = q_1 \tilde{P}_0$$

$$q_1 g_1 \tilde{P}_1 = q_2 \tilde{P}_1$$

$$\cdots\cdots$$

$$q_{n-1} g_{n-1} \tilde{P}_{n-1} = q_n \tilde{P}_{n-1}$$

Since the stabiliser of $\tilde{P}_i$ is precisely the subgroup G_{P_i}, we have elements $r_i \in G_{P_i}$ such that

$$q_n g_n r_n = q_1$$

$$q_1 g_1 r_1 = q_2$$

$$\cdots\cdots$$

$$q_{n-1} g_{n-1} r_{n-1} = q_n$$

This obviously gives the relation

$$g_1 r_1 g_2 r_2 \cdots g_n r_n = 1.$$

In other words, the word (c, μ) of type c in $F(G, Y)$, where c is the image of $\tilde{c}$ in Y and $\mu = (1, r_1, \cdots, r_n)$, gives rise to the trivial element of $F(G, Y)$. This means by theorem 5.11 that (c, μ) cannot be a reduced word.

However, let $e_{i+1} = \bar{e}_i$. Then, $g_{i+1} = g_i^{-1}$ and evidently $\chi_{i+1} = 1 - \chi_i$.

We saw above that

$$s_i g_i^{\chi_i - 1} g_i r_i = s_{i+1} g_{i+1}^{\chi_{i+1} - 1}.$$

So, $r_i \in G_{e_i}^{e_i}$ if, and only if, $s_i^{-1} s_{i+1} \in g_i^{\chi_i} G_{e_i}^{e_i} g_i^{-\chi_i}$.

Since there is no backtracking,

$$\overline{s_i \tilde{e}_i} \neq s_{i+1} \tilde{e}_{i+1} = \overline{s_{i+1} \tilde{e}_i}.$$

Therefore, $s_i^{-1}s_{i+1} \notin g_i^{\chi_i} G_{e_i}^{e_i} g_i^{-\chi_i}$.
This means that the word (c, μ) is reduced, producing a contradiction and proving the theorem. $\square$

Proof. (Of Theorem 13.) The idea is to reduce the proof to two cases which can be checked individually. These are the case when Y is a segment and the case when Y is a loop. Before making the reduction, let us verify the theorem for these two cases. The first case will be verified directly but the second case will use the truth of the theorem for trees.

When Y is a segment :

Here Y has two vertices P_{-1} and P_1 joined by an edge e from P_{-1} to P_1 and its inverse edge.
The element $|c, \mu|$ looks like $r_0 e^{t_1} r_1 e^{t_2} \cdots r_n$ with $r_0 \in G_{P_{-t_1}}, r_i \in G_{P_{t_i}} \setminus G_e^{e^{t_i}}$ where $t_i = -t_{i+1} = \pm 1$.
Now, Y itself is a tree and $\pi_1(G, Y, Y) = G_{P_{-1}} *_{G_e} G_{P_1}$.
The homomorphism $\phi : F(G, Y) \to \pi_1(G, Y, Y)$ takes $|c, \mu|$ to $r_0 r_1 \cdots r_n$. The latter is not trivial as seen in the very first section.

When Y is a loop at a vertex 0 :

As we determined above, the group $F(G, T)$ is the semi-direct product of the infinite cyclic group generated by the loop e and the normal subgroup generated by G_0.
Further, it was seen there that R is the free product of the groups $G_n = e^n G_0 e^{-n}$ amalgamated along the group $A = G_e$ according to the homomorphisms

$$A \to G_{n-1} \ ; \ a \mapsto e^{n-1} a^{\bar{e}} e^{1-n}$$

$$A \to G_n \ ; \ a \mapsto e^n a^e e^{-n}.$$

Then, the element $|c, \mu|$ looks like

$r_0 e^{t_1} r_1 e^{t_2} \cdots r_n$ with $r_i \in G_0$, $t_i = \pm 1$ and whenever $t_{i+1} = -t_i$, we have $r_i \notin A^{e_{t_i}}$.

Now, if $t_1 + \cdots + t_n \neq 0$, then clearly $|c, \mu| \notin R$ which, a fortiori, shows that $|c, \mu| \neq 1$.

Suppose that $t_1 + \cdots + t_n \neq 0$. Call $d_i = t_1 + \cdots + t_i$. Then, the element can be rewritten as

$s_0 s_1 \cdots s_n$ where $s_i = e^{d_i} r_i e^{-d_i}$.

Note that $s_i \in G_{d_i}$ and $d_0 = d_n = 0$. Also, if $e_{i+1} + e_i = 0$ i.e., if $d_{i+1} = d_{i-1}$, then the fact that (c, μ) is reduced means that $s_i \notin e^{d_i} A^{e_{t_i}} e^{-d_i}$.

We shall view this element $s_0 \cdots s_n$ as an element associated to a reduced word whose type is a closed path in an appropriate tree of groups (K, T) i.e., a graph of groups where T is a tree.

Consider T to be the tree whose vertices are integers and edges join consecutive integers. Then, the groups G_n and the homomorphisms $A \to G_{n-1}$, $A \to G_n$ define a graph of groups (K, T). Then, $R = \pi_1(K, T, T)$ and the element $s_0 \cdots s_n$ is indeed associated to a reduced word of (K, T) whose type is a closed path since $d_0 = d_n = 0$.

Applying corollary (c) of the theorem to the case (K, T) (we are assuming the theorem for trees which will be reduced to the first case later), we conclude that the element $s_0 \cdots s_n$ in $\pi_1(K, T, T)$ is not trivial. Therefore, $|c, \mu|$ itself is nontrivial.

Finally, we come to the general case (G, Y) and show that it can be reduced to the two cases above. *This reduction is the most nontrivial part of the proof.* To make it as transparent as possible, we break it up into easier steps.

Step I : What is required ?

First, notice that given a graph (G, Y) of groups, and a connected subgraph, there is an obvious 'restriction' of (G, Y) to a graph of groups (G, Z). The idea is to:

• Choose Z such that (G, Z) satisfies the theorem,

• Define a suitable graph of groups (H, W) on the contracted graph (also called the quotient graph) $W = Y/Z$ and,

• Associate to each word (c, μ) of (G, Y), a word (c', μ') of (H, W) such that a reduced word is associated to a reduced word and the corresponding element $|c', \mu'|$ is trivial if, and only if, $|c, \mu|$ is.

If we are able to make these choices, an induction argument on the number of edges would prove the theorem since it reduces ultimately to a segment.

Step II : Construction of (H, W)

Note that, in $W = Y/Z$, the subgraph Z of Y corresponds to a vertex (Z) and that the set of its vertices is Vertex $W =$ (Vertex $Y-$ Vertex $Z) \cup \{(Z)\}$. Also, the edges of W are, by definition, Edge $W =$ Edge $Y-$ Edge Z. Moreover, the origin and the terminus of each edge of W are defined as follows.

If e is an edge of W starting at a vertex outside Z, then its origin $o_W(e) = o(e)$ and its terminus $t_W(e) = t(e)$ or (Z) according as whether e ends outside Z or inside Z.

If e is an edge of W starting at a vertex of Z, then it must end outside Z and then $o_W(e) = (Z)$, $t_W(e) = t(e)$.

Now, we have assumed that Z has been so chosen that (G, Z) satisfies the theorem and, a fortiori, the corollary 1. This means that for each vertes P of Z, there is an injective homomorphism from G_P to $F(G, Z)$. With this in mind, let us define the graph of groups (H, W) as follows.

For each $P \in$ vertex W, let $H_P = G_P$ or $F(G, Z)$ according as whether $P \neq (Z)$ or $P = (Z)$.

For each $e \in$ Edge W, define $H_e = G_e$.

Notice that we have injections $H_e \to H_{t_W(e)}$.
Here, we have used the truth of corollary for (G, Z). Now, there is a homomorphism from $F(G, Z)$ to $F(G, Y)$ defined by mapping each G_P (for each vertex P of Z) to itself and each edge e of Z to itself, regarded as an edge of Y. The reason is that, evidently, the relations $\bar{e} = e^{-1}$ and $ea^e e^{-1} = a^{\bar{e}}$ defining $F(G, Z)$ are relations in $F(G, Y)$. Now, the projection $(G, Y) \to (H, W)$ induces a homomorphism θ from $F(G, Y)$ into $F(H, W)$.
We claim that θ is an isomorphism.
Define $\theta' : F(H, W) \to F(G, Y)$ as follows. On the vertex stabilisers H_P with $P \neq (Z)$, define θ' as the identity map. Define θ' on $H_{(Z)} = F(G, Z)$ by the above homomorphism to $F(G, Y)$. On edges of W also, define θ' as the identity mapping. Then, the relations defining $F(H, W)$ also clearly hold for their corresponding images under θ'. Thus, θ' is well- defined. Further, $\theta \circ \theta' = Id$ and $\theta' \circ \theta = Id$. This proves the claimed isomorphism.

Step III : Associating (c', μ')

The association is a natural one. We give an example to illustrate it. If c is a concatenation $e_1 \cdots e_4$ of paths starting at P_0 and going to P_4 and, if P_0, P_1, P_2 are the only vertices among these in Z, and e_2 is the only edge among these in Z, then $c' = (e_1, e_3, e_4)$ and $\mu' = (r_0, r_1 e_2 r_2, r_3, r_4)$. This makes sense because the element $r_1 e_2 r_2$ is in $F(G, Z) = H_{(Z)}$.
In general, we describe it now. Let c be a concatenation $e_1 \cdots e_n$ of paths starting at P_0 and going to P_n, and $i < j$, consider the subpath c_{ij} which is the conatenation of the edges $e_i \cdots e_{j-1}$. If, for some $i < j$, the subpath c_{ij} is contained in Z, then we shall denote by r_{ij}, the element $|c_{ij}, \mu_{ij}|$ of $F(G, Z) = H_{(Z)}$. Here, we have written μ_{ij} for $(r_i, \cdots, r_j)$. In other words, $r_{ij} = r_{i-1} e_i \cdots e_{j-1} r_{j-1}$.
Therefore, let us break the path into subpaths corresponding to paths in Z.

Let $0 \leq i_0 \leq j_0 < i_1 \leq j_1 < \cdots < i_m \leq j_m \leq n$ with the properties that: each c_{i_t,j_t} is contained in Z and each vertex/each edge of c which is in Z is inside c_{i_t,j_t} for some t.
So, the paths $c_{j_t,i_{t+1}}$ are paths of non-zero length whose vertices other than the extremities, are all outside Z; these, therefore, give paths in W. Hence, we define

$$c' = (\cdots, c_{j_{t-1},i_t}, \cdots)$$

$$\mu' = (\cdots, \mu_{j_{t-1}+1,i_t-1,r_{i_t,j_t}}, \mu_{j_t+1,i_{t+1}-1}, \cdots)$$

Step IV : (c', μ') is reduced if (c, μ) is

If c' is the vertex $P = (Z)$ of W, then it is contained in Z and, by the truth of the theorem for (G, Z), we have that $|c, \mu| \neq 1$ and, so $|c', \mu'| \neq 1$.
If c' is a vertex P of W different from (Z), then $c = P$ and $\mu = r_0 \neq 1$. Since $H_P = G_P$ in this case, it follows that (c', μ') is reduced.
Let us assume that $l(c') \geq 1$. Suppose it is the concatenation $w_1 w_2 \cdots w_m$ of edges of W.
We need to show that if $w_{i+1} = \bar{w}_i$, then we must have $r'_i \notin H^{w_i}_{w_i}$ where $\mu' = (r'_0, \cdots, r'_m)$.
If $t_W(w_i) \neq (Z)$, then since (c, μ) is reduced, our contention is true. We are left with the case when $t_W(w_i) = (Z)$. There are two possibilities. If (w_i, r'_i, w_{i+1}) is of the form (e_j, r_j, e_{j+1}) where $e_{j+1} = \bar{e}_j$. Then $r_j \notin G^{e_j}_{e_j}$. Since r'_i is the image of r_j in $H_{(Z)}$, since $G_{t(e_j)} \to H_{(Z)}$, and since under this homomorphism, $G^{e_j}_{e_j}$ transforms into $H^{w_i}_{w_i}$, it follows that $r'_i \notin H^{w_i}_{w_i}$.
The other possibility is that (w_i, r'_i, w_{i+1}) is of the form $(e_{j_t}, r_{j_t,k_t}, e_{k_t+1})$ where $j_t < k_t$ and $r_{j_t,k_t} = |c_{j_t,k_t}, \mu_{j_t,k_t}|$ as defined earlier. Look at the subpath c_{j_t,k_t} which has non-zero length. Applying the corollary 2 to the theorem for the graph of groups (G, Z), we have that $r_{j_t,k_t} \notin G_Q$ where $Q = o(c_{j_t,k_t}) = t(e_{j_t})$.

This means, a fortiori, that r_{j_t,k_t} is not contained in the subgroup $H_{w_i}^{w_i}$ of G_Q. This proves the last step and, thus, the theorem as well. $\square$

Finally, we come to the Bass-Serre structure theorem for a group G acting without inversion on a tree X. The idea is to make the quotient graph $Y = G\backslash X$ a graph of groups in such a way that its fundamental group is naturally isomorphic to G.

In fact, we shall look at any connected graph X on which G acts without inversion and produce a suitable structure of graph of groups on the quotient graph $Y = G\backslash X$.

Let T be a maximal tree in Y and let $j : T \to X$ be a lift of T to a tree in X. As before, we fix an orientation E_+ of Y and write χ for the characteristic function of E_+.

We would like to define a map (extending j and denoted by j again) from Edge Y to Edge X such that $j(\bar{e}) = \overline{j(e)}$. It suffices to define $j(e)$ for edges $e \in E_+$ which are not on T. For such edges e, we define its origin $o(j(e))$ to be a vertex in $j(T)$ i.e., $o(j(e)) = j(o(e))$.

Since $t(j(e))$ and $j(t(e))$ project to the same edge $t(e)$ in Y, we must have some $\gamma_e \in G$ so that $t(j(e)) = \gamma_e j(t(e))$. We have defined $j(e)$, γ_e etc. for edges $e \in E_+$ which are not on T. To extend γ to all edges of Y, we put $\gamma_e = 1$ for edges e of T and we put $\gamma_{\bar{e}} = \gamma_e^{-1}$ for all edges e of Y. Then, we have for each edge e,

$$o(j(e)) = \gamma_e^{\delta e - 1} j(o(e))$$

$$t(j(e)) = \gamma_e^{\delta e} j(t(e))$$

The vertex stabilisers and the edge stabilisers for the G-action on the graph X are denoted by G_P, G_e etc. Let us now define the graph of groups (G, Y).

For each vertex P and each edge e of Y, define

$$G_P = G_{j(P)} \quad , \quad G_e = G_{j(e)}$$

The homomorphism $a \mapsto a^e$ from G_e to $G_{t(e)}$ is defined by $a^e = \gamma_e^{-\delta e} a \gamma_e^{\delta e}$.
Associated to this graph of groups and the maximal tree T is its fundamental group $\pi_1(G, Y, T)$.
Since this fundamental group is generated by the vertex stabilisers G_P and the symbols g_e for edges e in Y, we have a group homomorphism $\phi : \pi_1(G, Y, T) \to G$ given by the inclusions $G_P \leq G$ and $g_e \mapsto \gamma_e$.
Recall the universal covering $\tilde{X} = \tilde{X}(G, Y, T)$ of (G, Y) defined in 5.7. We have a map $\psi : \tilde{X}(G, Y, T) \to X$ defined by $g\tilde{P} \mapsto \phi(g)j(P)$ and $g\tilde{e} \mapsto \phi(g)j(e)$.
It is easy to see that ψ is a ϕ-equivariant graph morphism. With these notations, the main theorem asserts:

Theorem 14 (Bass-Serre). *The following three properties are equivalent:*

(i) X is a tree.

(ii) $\psi : \tilde{X} \to X$ is a graph isomorphism.

(iii) $\phi : \pi_1(G, Y, T) \to G$ is a group isomorphism.

Note that the interesting part is the implication (I) $\Rightarrow$ (III) which means that: If a group G acts on a tree X without inversion, then G is generated by the vertex stabilisers G_P (P vertex of $G\backslash X$) along with symbols γ_e indexed by edges e of $G\backslash X$ with the defining relations

$$\gamma_e a^e \gamma_e^{-1} = a^{\bar{e}} \ , \ \gamma_{\bar{e}} = \gamma_e^{-1} \ , \ \gamma_e = 1 \ \ \forall \text{ edge } T.$$

Proof. Evidently, (II) implies (I) directly from theorem 5.11.
Also, (III) clearly implies (II).
To show (II) and (III) are equivalent, we assume (II) holds. Let N denote the kernel of ϕ. Then, for every vertex P of Y, since ϕ gives an isomorphism from $G_{\tilde{P}}$ to $G_{j(P)}$, we have $N \cap G_P = \{1\}$. But, if N is nontrivial, then, for $1 \neq n \in N$, the vertices $\tilde{P}$ and $n\tilde{P}$ are distinct but have the same image $j(P)$ in X. This contradicts (II) and shows that $N = \{1\}$. Thus (III) follows.
Finally, to show (I) implies (II) as well, look at the smallest subgraph W of X containing all $j(e)$, as e varies over all edges of Y. Then, each edge of W has atleast one extremity in $j(T)$ and we have $G.W = X$.
Now $W \subset \psi(\tilde{X})$ and ϕ gives isomorphisms between corresponding vertex stabilisers of $\tilde{X}$ and X as well as between corresponding edge stabilisers of $\tilde{X}$ and X. Now, we appeal to the result 20 from the previous section to show that the maps ϕ and ψ are surjective and ψ is locally injective (i.e., injective on the set of edges with a given origin). The result appealed to is the following one. If a group acts on a tree with a segment as fundamental domain, then it is an amalgam of the two vertex stabilisers amalgamated along the edge stabiliser. This (or, rather, the subfact that G is generated by the vertex stabilisers) can be generalized without difficulty as follows :
Let G be a group acting on a connected graph X and let T be a tree of representatives of $G\backslash X$. Let Y be a subgraph of X containing T and suppose that each edge of Y either starts or ends in T. Suppose that $G.Y = X$ and that for every edge e of Y which starts at T has a corresponding element g_e of G such that $g_e t(e) \in$ Vertex T. Then, G is generated by the elements g_e and the vertex stabilisers G_P for vertices P of T.
Now, 20 proves the assertion that (I) implies (II).
The proof of the structure theorm is complete. □

We shall now derive some applications of the structure theorem. In particular, we shall prove Kurosh's theorem which determines the structure of a subgroup of a free product or, more generally, of a free product with amalgamation. Roughly, Kurosh's theorem asserts that the subgroup of a free product of groups $G_i, i \in I$ is also a free product of conjugates of the G_i's and a free group. But, let us first note some immediate applications of the structure theorem.

Corollary 15. *Let G act without inversion on a tree X. Then*

> *(a) If N denotes the subgroup of G generated by the vertex stabilisers, then N is normal in G and, G/N is a free group.*
>
> *(b) If H is a subgroup whose intersection with any vertex stabiliser is trivial, then H is free.*
>
> *(c) If G is finite, then it fixes a vertex.*

Proof. To prove (a), note that normality of N is evident from its definition and the structure theorem gives us that $G/N \cong \pi_1(G\backslash X, T)$ for some maximal tree T of $G\backslash X$. Since vertex stabilisers in G coincide with those in N, we have that N is generated by its vertex stabilisers and, we have $\pi_1(N\backslash X, T_0) = \{1\}$ for a maximal subtree T_0 in $N\backslash X$. This means that $N\backslash X$ must be a tree. Moreover, G/N acts freely on this tree since the stabiliser in G/N of any vertex Nx is the subgroup NG_x/N which is trivial. Thus, G/N is a free group.
(b) follows by observing that, under the hypothesis, H acts freely on X.
For (c), start with any vertex x of X. Let N be the maximum of the lengths of the geodesics from x to gx as g varies over G. Look at the subtree T of X generated by the orbit Gx. Clearly, T is G-invariant and every reduced

path is of length $\leq 2N$. If T has at most edge, then each of its vertices (≤ 2 in number) is G-fixed, since G acts without inversion. Therefore, let us suppose that there is a vertex of T with atleast 2 edges emanating from it. If we remove from T each vertex which has a unique edge starting from it, and the corresponding edge, we get a G-invariant subtree T' in which every reduced path has length $\leq 2N-2$. An induction argument, gives us (c) now. $\square$

Theorem 16. *Let $G = *_A G_i$ be a free product of a family $G_i, i \in I$ of groups amalgamated along a subgroup A. Suppose H is a subgroup which intersects every conjugate subgroup of every G_i only trivially. Then, there exists a free subgroup F of G and a set $X_i \subset G/G_i$ which is a system of coset representatives for $H\backslash G/G_i$ such that*

$$H = (*_{i \in I, x \in X_i} H \cap x G_i x^{-1}) * F.$$

The particular case where A is trivial, is known as the Kurosh subgroup theorem.

Proof. The main idea is to consider the graph of groups defined by the G_i's and by A in section 4. Recall that one constructed a tree T whose vertices are the elements of I along with an extra vertex 0 (not in I) and whose edges are $(0,i),(i,0)$ for $i \in I$. Then, a graph of groups (G,X) was constructed by putting $G_0 = A$, and putting G_e for each edge e to be also A. This is a tree of groups i.e., T is a tree. Then, the amalgam $G_T = lim_{\rightarrow}(G,T)$ acts on a tree X which contains T and has the property that T is a fundamental domain and the vertex and edge stabilisers for the action are G_P, G_e respectively, for vertices P and edges e of T. Thus, we have $G = G_T$ above.
Thus, the stabliser of any edge ge in X is the conjugate gAg^{-1} of A where e

is an edge of T. Similarly, the stabilisers of vertices of X are conjugates of A and of the G_i's.
Applying the structure theorem for the action of the subgroup H on X, we have that $H = \pi_1(H, Y, T_0)$ where $Y = H\backslash X$ and T_0 is a maximal subtree of Y.
The hypothesis that H intersects conjugates of A only trivially, shows that $H_e = \{1\}$ for each edge e of Y. We know the structure of the fundamental group in this case; we have

$$H = \pi_1(H, Y, T_0) \cong (*_P H_P) * F$$

where F is a free group and P runs through the vertices of T_0.
Notice that the vertices of X are parametrized by the set $G/A \sqcup_{i \in I} G/G_i$ and the vertices of T_0 are, therefore, parametrized by the set $H\backslash G/A \sqcup_{i \in I} H\backslash G/G_i$. Choosing a lift of the tree T_0 to a tree in X, one has the systems of representatives $X_A \subset G/A$ and $X_i \subset G/G_i$ of $H\backslash G/A$ and $H\backslash G/G_i$ respectively. The proof is finished by observing that a vertex stabiliser H_P with $P = xA$ is $H \cap xAx^{-1} = \{1\}$ and one with $P = xG_i$ is $H \cap xG_ix^{-1}$. $\square$

Theorem 17. *Let G be a free group with finite basis S and let σ be an automorphism of G. Then, the subgroup $H := Fix(\sigma)$ of G is free of finite rank.*

Proof. One knows that the Cayley graph T of G with respect to S is a tree on which G acts freely. Note that σ acts on the vertex set. Let us consider the set of edges $[g, gs]$ for which the geodesic joining $\sigma(g)$ and $\sigma(gs)$ contains the edge. We claim that $H = Fix(\sigma)$ acts on the set F of these edges. Indeed, for any such edge and for any $h \in H$, look at the edge $[hg, hgs]$. Now $g = \sigma(g)s_1 \cdots s_k$ and $\sigma(gs) = gst_1 \cdots t_l$ for some $s_i, t_j \in S$. Then,

$$hg = h\sigma(g)s_1 \cdots s_k = \sigma(hg)s_1 \cdots s_k$$

and

$$\sigma(hgs) = h\sigma(gs) = hgst_1 \cdots t_l.$$

Therefore, clearly the geodesic from $\sigma(hg)$ to $\sigma(hgs)$ contains the edge $[hg, hgs]$. The main observation will be that the quotient set $H\backslash F$ is finite; that is, we claim that there are only finitely many edges in F upto the left H-action. To see this, notice that if an edge $[g, gs]$ is in F, then the geodesic from 1 to $\sigma(s)$ contains the edge $[\sigma(g)^{-1}g, \sigma(g)^{-1}gs]$. Note that an element of the form $\sigma(g)^{-1}g$ determines the right coset Hg. In other words, the map

$$[g, gs] \mapsto \{(Hg, s) : [1, \sigma(s)] \supset [\sigma(g)^{-1}g, \sigma(g)^{-1}gs]\}$$

is a bijection from $H\backslash F$ onto the set on the right. Here $[1, \sigma(s)]$ denotes the geodesic from 1 to $\sigma(s)$. Note that the set on the right is finite as S is finite. Thus, we have proved that $H\backslash F$ has only finitely many edges and is, hence, a finite graph. Recall the procedure of collapsing the edges of a disjoint union of trees to form a graph of the same topological type as the original one. Here, if we collapse all the edges on F to points and look at the graph T/F, then $H\backslash(T/F) = H\backslash T \setminus H\backslash F$. This has finitely many connected components as $H\backslash F$ is a finite graph (with at most $|S|$ edges). Since, by the main structure theorem, H can be identified with the fundamental group of $H\backslash T$, it suffices to show that each component of $H\backslash(T/F)$ corresponds to a free group of rank at most 1. To see this, look at any edge $[g, gs]$ which is *not* in F. Evidently, the graph $T \setminus [g, gs]$ has two connected components and exactly one of g and gs is in the component which contains the geodesic from $\sigma(g)$ to $\sigma(gs)$. We can re-orient T so that for edges $[g, gs]$ not in F, the vertex gs is in the component containing the geodesic from $\sigma(g)$ to $\sigma(gs)$. Clearly, H preserves this orientation. Therefore, for any edge $[g, gs]$ not in F, the geodesic from g to $\sigma(g)$ starts with the edge $[g, gs]$; in other words this geodesic determines the edge $[g, gs]$. Thus, for every $g \in G$, there is exactly one edge in T/F starting at g (all other edges have collapsed to one vertex in T/F). Thus, each component of $H\backslash(T/F)$ $\square$

8 Ihara's Theorem

8.1 A primer on non-Archimedean local fields

Let F be a field. To begin with we define the notion of a valuation on the field F.

Definition 1. *A valuation on a field F is a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that*

(i) $|x| = 0$ *if and only if* $x = 0$.

(ii) $|xy| = |x||y|$.

(iii) There is a constant C such that if $|x| \leq 1$ then $|1 + x| \leq C$.

Remark 2. We define two valuations $|\cdot|_1$ and $|\cdot|_2$ equivalent if there is an $\alpha > 0$ such that $|x|_1 = |x|_2^{\alpha}$ for all $x \in F$. Clearly every valuation is equivalent to one with the contant C being equal to 2. In fact, if $C = 2$ then one can show (see Exercise 28) that the valuation satisfies the usual triangle inequality, namely,

$$|x + y| \leq |x| + |y|.$$

Definition 3. *A valuation $|\cdot|$ on a field F is said to be a non-Archimedean valuation if it satisfies one (and hence any) of the following equivalent conditions:*

(i) $C = 1$.

(ii) The valuation satisfies the ultra-metric inequality, namely,

$$|x + y| \leq \max\{|x|, |y|\}.$$

(iii) $|n| \leq 1$ *for all* n *in the ring generated by* $1 \in F$.

We heneceforth deal with a field F *equipped with a non-Archimedean valuation. The definition* $|x| = 1$ *for all* $x \in F^*$ *evidently gives an example of a valuation. We call such a valuation trivial and shall study only non-trivial valuations. Further, it is convenient to sometimes think in terms of an additive valuation. An additive non-Archimedean valuation on* F *is a map* $v : F \to \mathbb{R} \cup \{\infty\}$ *such that*

(i) $v(x) = \infty$ *if and only if* $x = 0$.

(ii) $v(xy) = v(x) + v(y)$.

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

The relation between v *and* $|\cdot|$ *is given by the existence of a number* c *with* $0 < c < 1$ *and such that* $|x| = c^{v(x)}$. *Note that our assumption that* $|F^*|$ *is nontrivial is equivalent to the assumption that the corresponding additive valuation* v *is non-trivial; that is,* $v(F^*) \neq \{0\}$. *We shall be concerned only with discrete valuations, i.e.,* $v(F^*)$ *(resp.* $|F^*|$*) is a discrete subgroup of* $\mathbb{R}$ *(resp.* $\mathbb{R}_{>0}$*). We may and shall normalize* v *such that* $v(F^*) = \mathbb{Z}$.

Associated to a field F *and a non-Archimedean valuation* v *or* $|\cdot|$ *is its ring of integers* $\mathcal{O}$ *defined by*

$$\mathcal{O} := \{x \in F : |x| \leq 1\} = \{x \in F : v(x) \geq 0\}$$

and an ideal $\mathfrak{P}$ *of* $\mathcal{O}$ *defined by*

$$\mathfrak{P} := \{x \in F : |x| < 1\} = \{x \in F : v(x) \geq 1\}.$$

We leave it to the reader to check that the group of units $\mathcal{O}^\times$ is given by:

$$\mathcal{O}^\times = \mathcal{O} - \mathfrak{P} = \{x \in F : |x| = 1\} = \{x \in F : v(x) = 0\}$$

and so $\mathcal{O}$ is a local ring with a unique maximal ideal $\mathfrak{P}$. The quotient $k_F = \mathcal{O}/\mathfrak{P}$ is a field and will be called the residue field of F.

Let ϖ be a uniformizer for F, i.e., an element such that $v(\varpi) = 1$. So a uniformizer is well defined up to a unit element in the ring of integers. We ask the reader to check that every (fractional) ideal of F is a power of the maximal ideal $\mathfrak{P}$ and so looks like $\mathfrak{P}^m$ for some integer m and every ideal is principal, indeed, we have $\mathfrak{P}^m = \varpi^m\mathcal{O}$. Hence $\mathcal{O}$ is a local principal ideal domain and such rings are also sometimes called discrete valuation rings.

The valuation $|\cdot|$ on F makes F into a metric space. The distance function is defined by $d(x, y) = |x - y|$. Hence we may apply topological adjectives to F, for example, the assertion that F is locally compact makes sense.

Proposition 4. *Let F be a field endowed with a non-Archimedean discrete valuation v or $|\cdot|$. Then the following are equivalent:*

(i) F is a locally compact topological field.

(ii) F is complete and the residue field k_F is finite.

Proof. Let Ω be a set of representatives for $\mathcal{O}/\mathfrak{P}$, i.e., $\mathcal{O} = \amalg_{x\in\Omega}\, x + \mathfrak{P}$. We begin with the proof of (2) implies (1). To begin with, since F is complete we have

$$\mathcal{O} = \left\{\sum_{i=0}^{\infty} a_i\varpi^i : a_i \in \Omega\right\}.$$

To see this, consider a series $x = \sum_{i=0}^{\infty} a_i\varpi^i$ as in the right hand side. Let $x_n = \sum_{i=0}^{n} a_i\varpi^i$. For $n \geq m$ we have $|x_n - x_m| \leq |\varpi|^{m+1}$ and since $|\varpi| < 1$ we get that the sequence $\{x_n\}$ is a Cauchy sequence of elements in $\mathcal{O}$. Since

F is complete and $\mathcal{O}$ is closed we get that sequence indeed converges to x and that $x \in \mathcal{O}$.

For the reverse inclusion, let $x \in \mathcal{O}$. There is a unique $a_0 \in \Omega$ such that $x \in a_0 + \mathfrak{P}$. Again there is unique $a_2 \in \Omega$ such that $x \in a_0 + a_1\varpi + \mathfrak{P}^2$. Continuing this way we get a unique sequence $\{a_i\}$ such that $x \in a_0 + a_1\varpi + \cdots + a_i\varpi^i + \mathfrak{P}^{i+1}$ and hence x is in the right hand side.

Once we have this description of $\mathcal{O}$ we can show that it is actually compact. Let $\{U_\alpha\}$ be an open cover of $\mathcal{O}$. Suppose it has no finite subcover. Since k_F and hence Ω is finite we get that there is some $a_0 \in \Omega$ such that $a_0 + \mathfrak{P}$ admits no finite subcover. By the same token, we get that there is an element $a_1 \in \Omega$ such that $a_0 + a_1\varpi + \mathfrak{P}^2$ admits no finite subcover. Continuing this way, we get a sequence $\{a_n\}$ of elements in Ω such that for all i, $a_0 + a_1\varpi + \cdots + a_i\varpi^i + \mathfrak{P}^{i+1}$ admits no finite subcover. Let $x = \sum_{i\geq 0} a_i\varpi^i$. We have seen that $x \in \mathcal{O}$ and hence there is some β such that $x \in U_\beta$ and since U_β is open there is some $r \gg 0$ such that $x + \mathfrak{P}^r \subset U_\beta$. This contradicts the fact that $a_0 + a_1\varpi + \cdots + a_{r-1}\varpi^{r-1} + \mathfrak{P}^r$ admits no finte subcover. Hence $\mathcal{O}$ is compact. Now any $x \in F$ has a compact neighbourhood, namely, $x + \mathcal{O}$ and so F is locally compact.

Now we prove (1) implies (2). Since F is locally compact, let C be a compact neighbourhood of $0 \in F$. Choose $r \gg 0$ such that $\mathfrak{P}^r \subset C$. Since $\mathfrak{P}^r$ is a closed subset of a compact set it is itself compact. Hence $\mathcal{O} = \varpi^{-r}\mathfrak{P}^r$ is compact.

Since $\mathcal{O} = \amalg_{x\in\Omega} x + \mathfrak{P}$ is a disjoint union of open sets, we get that Ω must be finite, i.e., k_F is a finite field.

Let $\{x_n\}$ be a Cauchy sequence in F. Since $|\cdot|$ satisfies the ultra-metric inequality, it also satisfies the triangle inequality and hence we get for all

$x, y \in F$ that

$$||x| - |y|| \leq |x - y|.$$

(This shows that the valuation map is a continuous map.) In particular we get that the sequence $\{|x_n|\}$ is a Cauchy sequence of real numbers. The valuation being discrete implies that the value $|x_n|$ is eventually constant, i.e., there is some m and some n_0 such that $|x_n| = |\varpi|^m$ for all $n \geq n_0$. Put $y_n = \varpi^{-m} x_n$ for $n \geq n_0$. Then $\{y_n\}_{n \geq n_0}$ is a Cauchy sequence of elements in $\mathcal{O}^\times \subset \mathcal{O}$ both of which are compact and so y_n admits a limit point and hence so does x_n which proves that F is complete. $\square$

Definition 5. *By a non-Archimedean local field we mean a field F equipped with a non-trivial discrete non-Archimedean valuation v (or $|\cdot|$) such that F is locally compact or equivalently that F is compelete and the residue field is finite. We will let q_F denote the cardinality of the residue field.*

Remark 6. In the above definition there is some redundancy as there is a theorem of Gelfand and Tornheim which states that any Archimedean local field necessarily is a subfield of Complex numbers with the valuation induced from the usual absolute value. In particular such a field contains the rational field $\mathbb{Q}$ and so the valuation can not be discrete.

The reader is urged to go through the exercises at the end of this chapter dealing with the specific example of the non-Archimedean local field $\mathbb{Q}_p$, also called the field of p-adic rational numbers.

8.2 GL(2) and SL(2)

In this section we consider the two groups $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ for a non-Archimedean local field F. They are defined as follows:

$$\mathrm{GL}_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in F;\ \det(g) = ad - bc \neq 0 \right\} \quad (7)$$

$$\mathrm{SL}_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in F;\ \det(g) = ad - bc = 1 \right\} \quad (8)$$

We give $\mathrm{GL}_2(F)$ *the* p*-adic topology, namely, the topology it inherits from the topology on* F*. One may think of* $\mathrm{GL}_2(F)$ *as either an open subset of* F^4 *via the non-vanishing of the determinant homomorphism or a closed subset of* F^5 *via the zero locus of the polynomial* $(AD - BC)Y - 1$ *in five variables. The topology induced from either embedding is the same and from now on we will use only this topology on* $\mathrm{GL}_2(F)$*. We equip* $\mathrm{SL}_2(F)$ *with the topology induced from* $\mathrm{GL}_2(F)$*. This way both of them are locally compact totally disconnected topological groups. Both these groups have the following important open compact subgroups:*

$$\mathrm{GL}_2(\mathcal{O}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathcal{O};\ \det(g) \in \mathcal{O}^\times \right\} \quad (9)$$

$$\mathrm{SL}_2(\mathcal{O}) = \mathrm{GL}_2(\mathcal{O}) \cap \mathrm{SL}_2(F) \quad (10)$$

We begin with the following proposition.

Proposition 11. *The group* $\mathrm{GL}_2(\mathcal{O})$ *is a compact open subgroup of* $\mathrm{GL}_2(F)$ *and any compact subgroup can be conjugated inside it.*

Proof. That $\mathrm{GL}_2(\mathcal{O})$ is compact open follows from the above mentioned embeddings in F^4 and F^5 and the fact that $\mathcal{O}$ is a compact open subring of F. To prove the assertion that any compact subgroup may be conjugated inside it, we introduce the very important notion of lattices in p-adic vector spaces.

We let $V = F^2 = F^{2\times 1}$ be the two dimensional F-vector space consisting of column vectors. If

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then $V = Fe_1 \oplus Fe_2$. By a *lattice in* V we mean a rank two free $\mathcal{O}$ submodule of V. For example, $L_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ is a lattice in V that we sometimes refer to as the standard lattice.

Note that $\mathrm{GL}_2(F)$ acts on V via the so-called standard representation and this action is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

With respect to this action we ask the reader to check that

$$\mathrm{Stab}_{\mathrm{GL}_2(F)}(L_0) = \mathrm{GL}_2(\mathcal{O}).$$

The above equation brings out the connection between compact subgroups of $\mathrm{GL}_2(F)$ and lattices in V. Let L be any lattice and let $\{v_1, v_2\}$ be an $\mathcal{O}$-basis of L. Let $g \in \mathrm{GL}_2(F)$ such that $g(e_1) = v_1$ and $g(e_2) = v_2$. Then since the action is linear we get $g(L_0) = L$ and hence

$$\mathrm{Stab}_{\mathrm{GL}_2(F)}(L) = g\mathrm{Stab}_{\mathrm{GL}_2(F)}(L_0)g^{-1}.$$

Let C be any compact subgroup of $\mathrm{GL}_2(F)$. We can *average* L_0 over C and get a C stable lattice. Put

$$L = \sum_{c \in C} c \cdot L_0.$$

Actually the above summation is finite since c runs over cosets $C/C \cap \mathrm{Stab}_{\mathrm{GL}_2(F)}(L_0)$ which is a finite set by compactness of C. By the above remarks we have

$$C \subset \mathrm{Stab}_{\mathrm{GL}_2(F)}(L) = g\mathrm{Stab}_{\mathrm{GL}_2(F)}(L_0)g^{-1} = g\mathrm{GL}_2(\mathcal{O})g^{-1}.$$

□

In fact, $\mathrm{GL}_2(\mathcal{O})$ *is a maximal compact subgroup and alongwith the above proposition we get that there is only one conjugacy class maximal compact subgroups for* $\mathrm{GL}_2(F)$. *To prove maximality we prove the Cartan decomposition.*

Proposition 12 (Cartan). *Let* $G = \mathrm{GL}_2(F)$ *and* $K = \mathrm{GL}_2(\mathcal{O})$. *Let*

$$A = \left\{ \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} : n, m \in \mathbb{Z} \ \text{ and } \ n \geq m \right\}.$$

Then we have $G = K \cdot A \cdot K = \amalg_{a \in A} KaK$.

Proof. Let $g \in G$. Let L_0 be the standard lattice in V. Let $L = gL_0$. Choose $r \gg 0$ such that $\varpi^r L \subset L_0$. Applying the structure theory of modules over PIDs to the $\mathcal{O}$-module $L_0/\varpi^r L$ we get that there is an $\mathcal{O}$-basis $\{v_1, v_2\}$ of L_0 and positive (since $r \gg 0$) integers a_1, a_2 such that $\{\varpi^{a_1} v_1, \varpi^{a_2} v_2\}$ is an $\mathcal{O}$-basis for $\varpi^r L$.

Let $k \in G$ be the element such that $k(v_i) = e_i$ for $i = 1, 2$. Since k stabilizes L_0 we get that $k \in K$. Note that we have

$$\begin{pmatrix} \varpi^{a_1} & 0 \\ 0 & \varpi^{a_2} \end{pmatrix} \cdot k \cdot L_0 = k \cdot \varpi^r \cdot g \cdot L_0 = \mathcal{O}\varpi^{a_1} e_1 \oplus \mathcal{O}\varpi^{a_2} e_2.$$

This implies that there are integers b_1 and b_2 (in fact $b_i = a_i - r$) such that

$$g = k_1 \cdot \begin{pmatrix} \varpi^{b_1} & 0 \\ 0 & \varpi^{b_2} \end{pmatrix} \cdot k_2.$$

If $b_1 < b_2$ then we may rewrite the above equation as

$$g = (k_1 w^{-1}) \cdot \begin{pmatrix} \varpi^{b_2} & 0 \\ 0 & \varpi^{b_1} \end{pmatrix} \cdot (w^{-1} k_2)$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in K$. This proves that $G = KAK$.

We now prove that the union $\cup_{a \in A} KaK$ is a disjoint union. Suppose for integers a_1, a_2, b_1, b_2 with $a_1 \geq a_2$ and $b_1 \geq b_2$ we have

$$K \begin{pmatrix} \varpi^{a_1} & 0 \\ 0 & \varpi^{a_2} \end{pmatrix} K = K \begin{pmatrix} \varpi^{b_1} & 0 \\ 0 & \varpi^{b_2} \end{pmatrix} K$$

Considering the absolute value of determinants of elements on both sides gives that $a_1 + a_2 = b_1 + b_2$. Now for any $g \in G$ let $I(g)$ be the ideal of F generated by the entris of g. It is easy to see that $I(g)$ depends only on the double coset KgK. Applying this to above equality fo double cosets while using $a_1 \geq a_2$ and $b_1 \geq b_2$ gives that $a_2 = b_2$ and hence $a_1 = b_1$. $\square$

Corollary 13. $\mathrm{GL}_2(\mathcal{O})$ *is a maximal compact subgroup of* $\mathrm{GL}_2(F)$.

Proof. Exercise! $\square$

Corollary 14 (Cartan). *Let* $G = \mathrm{SL}_2(F)$ *and* $K = \mathrm{SL}_2(\mathcal{O})$. *Let*

$$A = \left\{ \begin{pmatrix} \varpi^{n} & 0 \\ 0 & \varpi^{-n} \end{pmatrix} : n \in \mathbb{Z}_{\geq 0} \right\}.$$

Then we have $G = K \cdot A \cdot K = \amalg_{a \in A} KaK$.

Proof. Let $g \in \mathrm{SL}_2(F)$. By Cartan decomposition for $\mathrm{GL}_2(F)$ we get

$$g = k_1 \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} k_2$$

with $k_1, k_2 \in \mathrm{GL}_2(\mathcal{O})$ and integers $n \geq m$. Since $\det(g) = 1$ we get $m = -n$ and $\det(k_1)\det(k_2) = 1$. Hence we may rewrite g as

$$g = \left(k_1 \begin{pmatrix} \det(k_1)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix} \left(\begin{pmatrix} \det(k_2)^{-1} & 0 \\ 0 & 1 \end{pmatrix} k_2 \right)$$

giving us the Cartan decomposition. The disjointness of the union follows from the disjointness assertion of Proposition 12. □

Corollary 15. $\mathrm{SL}_2(\mathcal{O})$ *is a maximal compact subgroup of* $\mathrm{SL}_2(F)$.

Proof. Imitate the proof of Corollary 13! □

Corollary 16. *There are two conjugacy classes of maximal compact subgroups of* $\mathrm{SL}_2(F)$ *and they are represented by* $\mathrm{SL}_2(\mathcal{O})$ *and* $\left(\begin{smallmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{SL}_2(\mathcal{O}) \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. Let C be a compact subgroup of $\mathrm{SL}_2(F)$. Then since it is also a compact subgroup of $\mathrm{GL}_2(F)$ there exists $g \in \mathrm{GL}_2(F)$ such that $C \subset g\mathrm{GL}_2(\mathcal{O})g^{-1}$. Use Cartan decomposition for $\mathrm{GL}_2(F)$ and write

$$g = k_1 \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} k_2 = k_1 a k_2$$

with $k_i \in \mathrm{GL}_2(\mathcal{O})$ as in Proposition 12. Let $k_i' = k_i \left(\begin{smallmatrix} \det(k_i)^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O})$.

Now since diagonal matrices commute with each other we have

$$\begin{aligned} C \subset g\mathrm{GL}_2(\mathcal{O})g^{-1} &= k_1 a k_2 \mathrm{GL}_2(\mathcal{O}) k_2^{-1} a^{-1} k_1^{-1} \\ &= k_1' a \mathrm{GL}_2(\mathcal{O}) a^{-1} (k_1')^{-1} \end{aligned}$$

Replacing C by the conjugate $C_1 = (k_1')^{-1}Ck_1'$, we get $C_1 \subset a\mathrm{GL}_2(\mathcal{O})a^{-1}$ and since C_1 is a subgroup of $\mathrm{SL}_2(F)$ we actually get $C_1 \subset a\mathrm{SL}_2(\mathcal{O})a^{-1}$.

The case when $n \equiv m \pmod 2$*:* Note that inner conjugation by a may be replaced by inner conjugation by an element of $\mathrm{SL}_2(F)$ as:

$$a\mathrm{SL}_2(\mathcal{O})a^{-1} = \begin{pmatrix} \varpi^{(n-m)/2} & 0 \\ 0 & \varpi^{(m-n)/2} \end{pmatrix} \mathrm{SL}_2(\mathcal{O}) \begin{pmatrix} \varpi^{(n-m)/2} & 0 \\ 0 & \varpi^{(m-n)/2} \end{pmatrix}^{-1}.$$

Hence C can be conugated inside $\mathrm{SL}_2(\mathcal{O})$.

The case when $n \equiv m+1 \pmod 2$*:* We may now write

$$a\mathrm{SL}_2(\mathcal{O})a^{-1} = \begin{pmatrix} \varpi^{(n-m)/2} & 0 \\ 0 & \varpi^{(m-n)/2} \end{pmatrix} \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathcal{O}) \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \varpi^{(n-m)/2} & 0 \\ 0 & \varpi^{(m-n)/2} \end{pmatrix}^{-1}$$

and hence, in this case, C can be conjugated inside $\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathcal{O}) \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. $\square$

Remark.

There are analogues for GL_n and SL_n for a general n, of all the results 6.11 through 6.16. While the analogues of 6.11 until 6.15 are easy to guess and similar to proof, the analogue of 6.16 is that $SL_n(F)$ has exactly n maximal compact subgroups upto conjugacy. Of course, these are all conjugate in $GL_n(F)$. The proof of the above analogue of 6.16 is most naturally/conceptually given by using the so-called Bruhat-Tits building associated to SL_n over F; the next section discusses this for the case $n = 2$ where this is a tree.

8.3 The tree associated to SL_2 over a non-Archimedean local field

We continue with our notation of a non-Archimedean local field F and ite related paraphernalia like $\mathcal{O}$, $\mathfrak{P} = \varpi\mathcal{O}$, the valuation v or $|\cdot|$ on F etc.

In this section we construct a tree on which $\mathrm{SL}_2(F)$ *acts such that a fundamental domain is a segment. Towards this end, we recall the definition of a lattice in the two dimensional* F*-vector space* $V = F^2 = F^{2\times 1}$.

Definition 17. *A lattice* L *in* V *is an* $\mathcal{O}$*-submodule of maximal rank or equivalently a rank two free* $\mathcal{O}$*-submodule of* V*. Two lattices* L_1, L_2 *are said to be equivalent if there is some* $x \in F^*$ *such that* $L_1 = xL_2$*. This is an equivalence relation and the equivalence classes will be called lattice classes.*

Given two lattice classes Λ_1 *and* Λ_2 *we define a notion of distance between them as follows. Let* L_1 *and* L_2 *be lattices in* Λ_1 *and* Λ_2 *respectively. By the structure theory of modules over PIDs we get that there are vectors* v, w *and integers* a, b *such that*

$$L_1 = \mathcal{O}v \oplus \mathcal{O}w, \quad \text{and} \quad L_2 = \mathcal{O}\varpi^a v \oplus \mathcal{O}\varpi^b w.$$

We define

$$d(\Lambda_1, \Lambda_2) = |a - b| \tag{18}$$

where the absolute value on the right hand side is the usual absolute value of real numbers. Let us check that this definition is indeed valid, i.e., the right hand side depends only on the lattice classes and not the individual lattices. To see this, consider the lattices a_1L_1 *and* a_2L_2 *for some* $a_1, a_2 \in F^*$*. Then,* a_1L_1 *has a basis* $\{a_1v, a_1w\}$ *and* a_2L_2 *has a basis* $\{a_2\varpi^a v, a_2\varpi^b w\}$*. Note that* $a_2a_1^{-1} = u\varpi^n$ *where* $v(a_2a_1^{-1}) = n$ *and* u *is a unit. Putting* $a_2 = a_1u\varpi^n$*, we see that* a_2L_2 *has a basis* $\{a_1\varpi^{a+n}v, a_1\varpi^{b+n}w\}$*. Since* $|(a+n) - (b+n)| = |a - b|$*, the above definition is valid.*
We are now in a position to define a graph. It will be proved that this graph is indeed a tree.

Definition 19. *Let X be a graph whose vertex set $V(X)$ and edge set $E(X)$ are defined by:*

(i) $V(X)$: This is the set of all lattice classes Λ of V.

(ii) $E(X)$: This is the set of all edges $\Lambda_1.\Lambda_2$ where there is an edge joining the vertices Λ_1 and Λ_2 if $d(\Lambda_1, \Lambda_2) = 1$.

For each such edge $\Lambda_1.\Lambda_2$, wed define its origin $o(\Lambda_1.\Lambda_2)$ and its terminus $t(\Lambda_1.\Lambda_2)$ to be, respectively, Λ_1 and Λ_2. Finally, we define the opposite edge $\overline{\Lambda_1.\Lambda_2} = \Lambda_2.\Lambda_1$.

Theorem 20. *The graph X in Definition 19 is a tree.*

Proof. We begin the proof with a simple observation on lattices and lattice classes. Give a lattice L and a lattice class Λ' there is a unique lattice $L' \in \Lambda'$ such that one (and hence any) of the following equivalent conditions hold:

(i) $L' \subset L$ and L' is maximal with respect to this property.

(ii) $L' \subset L$ and $L' \not\subseteq \varpi L$.

(iii) $L' \subset L$ and L/L' is monogenic, i.e., cyclic as an $\mathcal{O}$-module.

In this case we have $L/L' \simeq \mathcal{O}/\mathfrak{P}^n$ where $n = d(\Lambda, \Lambda')$.

We begin by showing that X is connected. Let $\Lambda, \Lambda' \in V(X)$. Choose lattices L, L' in the respective lattice classes satisfying the above properties. Now choose a Jordan-Hölder sequence

$$L' = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

such that the succesive quotients satisfy $L_i/L_{i+1} \simeq k = \mathcal{O}/\mathfrak{P}$. If Λ_i is the lattice class of L_i then again by the above properties we get that there is edge joining Λ_i with Λ_{i+1} and so a path joining Λ and Λ'.

We now show that X is a tree. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ be a path without backtracking in X. We will show that $d(\Lambda_0, \Lambda_n) = 1$ which will show in particular that this path cannot be a circuit. We will prove this by induction. The assertion holds by definition for $n = 1$. Assume from now on that $n \geq 2$.

Choose lattices $L_i \in \Lambda_i$ such that

(i) $L_0 \supset L_1 \supset \cdots \supset L_n$.

(ii) $L_i / L_{i+1} \simeq k$.

(iii) $L_n \not\subseteq \varpi L_{n-1}$ and $L_{n-1} \not\subseteq \varpi L_0$ (the latter by induction hypothesis).

These properties actually imply that $L_n \not\subseteq \varpi L_0$ which gives $d(\Lambda_0, \Lambda_n) = 1$.

Note that both L_n and ϖL_{n-2} are inverse images of lines in $L_{n-1}/\varpi L_{n-1}$. These lines are distinct, because, if not then

$$L_n = \varpi L_{n-2} + \varpi L_{n-1} = \varpi L_{n-2}$$

which gives $\Lambda_n = \Lambda_{n-2}$ which is a backtracking. Since V is two dimensional these two distinct lines span the k-vector space $L_{n-1}/\varpi L_{n-1}$, i.e., $L_{n-1} = L_n + \varpi L_{n-2} + \varpi L_{n-1} = L_n + \varpi L_{n-2}$. Now if $L_n \subset \varpi L_0$ then we would get $L_{n-1} \subset \varpi L_0 + \varpi L_{n-2} = \varpi L_0$ which contradicts the second part of (iii) above. □

Corollary 21. *The quantity $d(\Lambda_1, \Lambda_2)$ coincides with the distance function on the vertex set of the tree X.*

Proof. Follows trivially from the proof of Theorem 20. □

8.4 The action of $\mathrm{SL}_2(F)$ on the tree X

We now study the action of the group $\mathrm{SL}_2(F)$ on the tree X. The action itself is naturally defined since $\mathrm{GL}_2(F)$ acts (via the standard representation) on

V and hence on the set of lattices. The action being linear, we get an action on the set of lattice classes. It turns out that, unfortunately, the action of $\mathrm{GL}_2(F)$ itself has inversions; however, the subgroup $\mathrm{SL}_2(F)$ acts without inversions. The first task is to verify this.

Towards this, we introduce a number $\chi(L_1, L_2)$ which depends on two lattices L_1, L_2 in V and is defined by :

$$\chi(L_1, L_2) := l(L_1/L_3) - l(L_2/L_3) \quad \textit{for any lattice } L_3 \subset L_1 \cap L_2 \tag{22}$$

where $l(M)$ denotes the length of a finite $\mathcal{O}$-module M.

Lemma 23. *Let L, L_1, L_2 be lattices respectively in the lattice classes Λ, Λ_1 and Λ_2. Then for any $g \in \mathrm{GL}_2(F)$ we have*

(i) $\chi(gL_1, gL_2) = \chi(L_1, L_2)$.

(ii) $d(g\Lambda_1, g\Lambda_2) = d(\Lambda_1, \Lambda_2)$.

(iii) $\chi(L, gL) = v(\det(g))$.

(iv) $d(\Lambda, g\Lambda) \equiv v(\det(g)) \pmod 2$.

Proof. We leave the proofs of (1),(2) and (4) as an exercise to the reader and give details of the proof of (3).

Let L_0 be the standard lattice and let $h \in \mathrm{GL}_2(F)$ be such that $L = hL_0$. We have

$$\chi(L, gL) = \chi(hL_0, ghL_0) = \chi(L_0, h^{-1}ghL_0).$$

Applying the Cartan decomposition for $\mathrm{GL}_2(F)$ we may write

$$h^{-1}gh = k_1 \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} k_2$$

where $k_i \in \mathrm{GL}_2(\mathcal{O}) = \mathrm{Stab}(L_0)$. We have

$$\chi(L_0, h^{-1}ghL_0) = \chi\left(L_0, k_1\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} k_2 L_0\right) = \chi\left(L_0, \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} L_0\right).$$

It is easily checked that that

$$\chi\left(L_0, \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix} L_0\right) = n + m = v(\det(g))$$

by noting that $l(\mathcal{O}/\mathfrak{P}^n) = n$. □

Proposition 24. $\mathrm{SL}_2(F)$ *acts on the tree* X *without inversions.*

Proof. Let $g \in \mathrm{SL}_2(F)$ and let $\Lambda \in V(X)$ be a lattice class. From Lemma 23 we have

$$d(\Lambda, g\Lambda) \equiv v(\det(g)) = v(1) = 0 \pmod 2$$

which implies that if g does not fix the vertex $V(X)$ then it moves it by a distance of at least 2. In particular, g can not take an edge e to its inverse, because, if it did, it would move any of the extremities of e by a distance 1. □

Proposition 25. *If* L *is a lattice in a lattice class* $\Lambda \in V(X)$ *and* G *is any subgroup of* $\mathrm{SL}_2(F)$ *then we have*

$$\mathrm{Stab}_G(\Lambda) = \mathrm{Stab}_G(L).$$

Proof. Clearly $\mathrm{Stab}_G(L) \subset \mathrm{Stab}_G(\Lambda)$. For the reverse inclusion, let $g \in \mathrm{Stab}_G(\Lambda)$. By definition, this means that there is an element $x \in F^*$ such that $gL = xL$. Hence using Lemma 23

$$\chi(L, xL) = \chi(L, gL) = v(\det(g)) = v(1) = 0$$

which implies that $x \in \mathcal{O}^\times$ and so $xL = L$. Hence we get $gL = xL = L$, i.e., $g \in \mathrm{Stab}_G(L)$. □

Proposition 26. *For the action of* $\mathrm{SL}_2(F)$ *on the tree* X *a fundamental domain is a segment.*

Proof. Fix a vertex $\Lambda_0 \in V(X)$. Define

$$\begin{aligned} V(X)^+ &= \{\Lambda \in V(X) : d(\Lambda_0, \Lambda) \equiv 0 \pmod 2\} \\ V(X)^- &= \{\Lambda \in V(X) : d(\Lambda_0, \Lambda) \equiv 1 \pmod 2\} \end{aligned}$$

Using the Cartan decomposition for $\mathrm{SL}_2(F)$ it can be checked that $\mathrm{SL}_2(F)$ acts transitively on $V(X)^+$ and $V(X)^-$. (We ask the reader to fill in the details here.)

The proposition will follow if one shows that the set $\{e \in E(X) : \mathrm{o}(e) = \Lambda_0\}$ of edges is contained in one $\mathrm{SL}_2(F)$-orbit. Equaivalently, if we show that set $\{\Lambda \in V(X) : d(\Lambda_0, \Lambda) = 1\}$ is in one $\mathrm{SL}_2(F)$-orbit. Fix a lattice $L_0 \in \Lambda_0$ and take lattices L in Λ such that $L \subset L_0$ and $L \not\subseteq \varpi L_0$. Then L corresponds to a line in the two dimensional k-vector space in $L_0/\varpi L_0$ and vice-versa. Hence the set $\{\Lambda \in V(X) : d(\Lambda_0, \Lambda) = 1\}$ is in bijection with $\mathbb{P}^1(k)$. In other words the three sets

$$\{e \in E(X) : \mathrm{o}(e) = \Lambda_0\},\ \{\Lambda \in V(X) : d(\Lambda_0, \Lambda) = 1\} \text{ and } \mathbb{P}^1(k)$$

are in bijection. The fact that $\mathrm{SL}_2(k)$ acts transitively on $\mathbb{P}^1(k)$ now finishes the proof. $\square$

Theorem 27. *Let* $G = \mathrm{SL}_2(F)$ *for a non-Archimedean local field* F*. Let* $K = \mathrm{SL}_2(\mathcal{O})$ *and let* I *be the subgroup of matrices in* K *which are upper triangular modulo* $\mathcal{P}$*. With respect to the injective maps* $I \to K$ *given by inclusion and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \varpi b \\ \varpi^{-1} c & d \end{pmatrix}$$

we get that $G = K \underset{I}{*} K$.

8.5 Exercises

Exercise 28. Let $|\cdot|$ be a valuation on a field F. Show that the constant $C = 2$ if and only if the valuation satisfies the triangle inequality.

Exercise 29. Let $|\cdot|$ be a valuation on a field F. Show that the three statements in Definition 3 are indeed equivalent to each other, i.e., $C = 1$ if and only if the ultra-mteric inequality holds if and only if the valuation is bounded by 1 on any element in the ring generated by $1 \in F$.

Exercise 30. Let p be a prime number.

(i) Consider the following map on $\mathbb{Q}^*$:

$$\left|\frac{p^m a}{b}\right|_p = p^{-m}$$

whenever $(p, ab) = (a, b) = 1$. Show that this gives a non-Archimedean valuation on $\mathbb{Q}$. Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to this valuation. Show that $\mathbb{Q}_p$ is a non-Archimedean local field with the respect to the valuation extending the given one.

(ii) Show that $\mathbb{Q}_p$ is not isomorphic to $\mathbb{R}$.

(iii) If p and q are two distinct primes then show that $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic to each other.

Exercise 31. Complete the proof of Lemma 23.

Exercise 32. Show that the action of $\mathrm{GL}_2(F)$ on the tree X of Section 8.3 is an action with inversions, by showing that for every edge e of X there is a $g \in \mathrm{GL}_2(F)$ such that $ge = \bar{e}$.

Exercise 33. Let $\mathbb{F}_q$ be the finite field with q elements.

(i) Show that the projective line $\mathbb{P}^1(\mathbb{F}_q)$ over $\mathbb{F}_q$ has $q+1$ elements.

(ii) Show that $\mathrm{SL}_2(\mathbb{F}_q)$ acts transitively on $\mathbb{P}^1(\mathbb{F}_q)$.

(iii) Show that the tree X of Section 8.3 associated to $\mathrm{SL}_2(F)$ has the property that every vertex has $q+1$ edges with that vertex as the origin.

Exercise 34. Draw the geometric realizations of the trees associated to $\mathrm{SL}_2(\mathbb{Q}_2)$ and $\mathrm{SL}_2(\mathbb{Q}_3)$.

Exercise 35. A subset Ω of $\mathrm{GL}_2(F)$ is said to be bounded if there is a number M such that

$$|g_{ij}| \leq M, \quad \forall g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \Omega.$$

Let H be a subgroup of $\mathrm{SL}_2(F)$. The show that the following are equivalent:

(i) H is a bounded subgroup. (Bounded in the above sense.)

(ii) There is a lattice L in V which H stabilizes.

(iii) There is a vertex in X which is fixed by H.

(iv) There is a vertex Λ in X such that its H-orbit is bounded as a subset of the vertex set $V(X)$.

Show that this gives another proof of Corollary 16.

Exercise 36 (Ihara). Let F be a non-Archimedean local field. Show that a subgroup of $\mathrm{SL}_2(F)$ which does not contain any bounded subgroup is a free group.

9 Serre's property (FA)

Earlier, we characterized free groups as those groups which act freely on trees. On the other hand, an amalgam $G = G_1 *_A G_2$ with $G_1 \neq A \neq G_2$ acts on a segment PQ where $G_P = G_1, G_Q = G_2$ and A is the stabilizer of the edge. However, this action also does not fix any vertex. In general, one has the following definition :

Definition 9.1. *(Property (FA)*
A group G has property (FA) if it fixes a vertex of any tree on which it acts.

The acronym (FA) stands for the fixing property on arbres (trees). Thus, free groups or groups which are amalgams do not have the property (FA).

Lemma 9.2. *Let N be a normal subgroup of a group G.*
(a) If G has property (FA), then so does G/N.
(b) If N and G/N have property (FA), then so does G.
(c) If H has property (FA) and H is of finite index in G, then G also has property (FA).

Proof: (a) is obvious.
For (b), let X be a tree on which G acts. Then, G/N acts on the tree X^N and, therefore, has a fixed vertex. Evidently, this is, by definition, a vertex of X fixed by G.
For (c), consider the normal subgroup $M \stackrel{d}{=} \bigcap_{g \in G} gHg^{-1}$ of G.
Now $X^M \neq \phi$ since $X^H \neq \phi$.
Also, G/M acts on X^M; since G/M is finite, it must fix a vertex. Thus, once again $X^G \neq \phi$.

Remark: We shall see later that the converse of lemma (c) is false, in general.

Lemma 9.3. *If G has property (FA) and is contained in an amalgam $G_1 *_A G_2$, then G is contained in a conjugate of G_1 or of G_2.*

Proof: The action of G on the tree associated with $G_1 *_A G_2$ has a fixed vertex. The stabiliser of any vertex of this tree is a conjugate of either G_1 or G_2.

Now, we can prove the following characterization theorem.

Theorem 9.4. *G has property (FA) if, and only if, it satisfies:*
(a) G is not an amalgam,
(b) there is no nontrivial homomorphism from G to $\mathbb{Z}$ and,
(c) G is not the union of a strictly increasing sequence of subgroups.

Proof: Suppose G has property (FA).
Clearly (a) holds as observed above since amalgams act without fixed points on appropriate segments. Since $\mathbb{Z}$ acts without fixed points on the doubly infinite line by translations, we have that $\mathbb{Z}$ does not have property (FA) and, therefore, G cannot have (FA) if $\mathbb{Z}$ were a quotient of G. Thus, (b) holds good.
To see that (c) is true, let us suppose, if possible, that $G_1 \subseteq G_2 \subseteq \ldots$ be a strictly increasing sequence of subgroups such that $G = \bigcup\limits_{n \geq 1} G_n$. Then, we have the tree X whose vertices are the cosets of G_n $\forall\ n \geq 1$ and, two vertices are joined precisely when they are of the form aG_n and bG_{n+1} with $aG_{n+1} = bG_{n+1}$.

Since G has a fixed vertex aG_n of X_1 we have $G = G_n$, a manifest contradiction. Thus, (c) also holds.

Conversely, suppose G has the properties (a), (b) and (c). Let X be a tree on which G acts. Look at the quotient graph $T = G/X$. Note that $\pi_1(T)$ is a free group. By the main structure theorem, G is a quotient of $\pi_1(T)$. But, this contradicts (b) unless $\pi_1(T)$ is trivial i.e., T is a tree. By lifting it to a tree in X, we can write G as the direct limit $\varinjlim(G, T)$ of the tree of groups where the vertex stabilizers are G_P for vertices P of T and edge stabilizers are G_e for edges e of T. Moreover, if T_1 runs through finite subtrees of T, the union of the groups $\varinjlim(G, T_1)$ equals G. By (c), this cannot be a strictly increasing union.
In other words, $G = \varinjlim(G, T_1)$ for some finite subtree T_1 of T. We may assume that T_1 is minimal with respect to this property. If T_1 is a single point P, then obviously $G = G_P$ i.e. G fixes P and this would prove that G has property (FA). If T_1 were not a single vertex, then there is a terminal vertex Q of T_1 such that $T_1 - \{Q\}$ is a tree as proved earlier. But then if e is the (unique) edge from Q to $T_1 - \{Q\}$, then $G = G_Q *_{G_e} G_{T_1 - \{Q\}}$. This is a nontrivial amalgam as the minimality of T_1 implies $G_{T_1 - \{Q\}} \neq G$ and also $G_Q \neq G$.
The resultant contradiction of the assumption (a) shows that this case is impossible; that is, T_1 must be a point.

Corollary: A finitely generated torsion group G has property (FA).

Proof: Suffices to check that such a group is not an amalgam (property (b) is obvious as G is a torsion group; property (c) holds because G is finitely generated.) Suppose, if possible, that $G = G_1 *_A G_2$.

Then, $\forall\ g_1 \in G_1 \backslash A, g_2 \in G_2 \backslash A$, the element $g_1 g_2$ is cyclically reduced; it must thus have infinite order, an impossibility.

Remark:
The above theorem is used in the following manner. If it is possible to show

somehow that a certain group G has the property (FA), then it follows that G cannot be an amalgam. Indeed, we shall do this for $SL_3(\mathbb{Z})$ but it is also valid for arithmetic groups in Q-algebraic groups of rank atleast 2. Now, it may not be easy to directly check whether a group acting on a tree has a fixed point or not. However, it is often to check that specific elements have fixed points. To use this, we need to know that if G is generated by elements which have fixed points, then G itself has fixed points. This is what we try to do is what follows.

Proposition 9.5. *: Let σ be an automorphism of a tree X. Let $P \in X$. Let $n = Min.\{\ell(P,Q) : Q \in X^\sigma\}$. Then, there is a unique point $Q \in X^\sigma$ such that $\ell(P,Q) = n$. Moreover $\ell(P,\sigma(P)) = 2n$ and $Q \in$ is the midpoint of the geodesic joining P and $\sigma(P)$.*

Proof: First, we note more generally that for any two disjoint subtrees T_1, T_2 of a tree X, there are unique points $P_1 \in T_1, P_2 \in T_2$ satisfying $\ell(P_1,P_2) = \mathrm{Min}\{\ell(A,B) : A \in T_1, B \in T_2\}$. The reason is that once we have located P_1, P_2 taking on this minimal value, for any $A \in T_1, B \in T_2$ the juxtaposition of the geodesics AP_1, P_1P_2 and P_2B is a path without back tracking and, hence, its length is strictly bigger than $\ell(P_1,P_2)$. Applying this to the trees X^σ and P, we get a unique $Q \in X^\sigma$ with $\ell(P,Q) = n$. By the definition of n, all the points of the geodesic from P to Q, apart from Q, are outside X^σ.

We claim that the geodesic from P to $\sigma(P)$ is obtained by juxtaposing the geodesic joining P to Q and that joining Q to $\sigma(P)$. If not, then the path $PQ\sigma(P)$ will have some back tracking. Since PQ and $Q\sigma(P)$ are geodesics, this means that the last edge P_1Q of PQ is the first edge of $Q\sigma(P)$. As the geodesic from Q to $\sigma(P)$ is simply the σ-translate of the geodesic from Q to

P, we get $\sigma(P_1) = P_1$ i.e. $P_1 \in X^\sigma$. This contradicts the definition of n as noted earlier.

Hence $\ell(P, \sigma(P)) = \ell(P, Q) + \ell(Q, \sigma(P)) = 2n$ since the geodesic from Q to $\sigma(P)$ is the σ-translate of the geodesic from Q to P. It is obvious that Q is the midpoint of the geodesic joining P and $\sigma(P)$.

Our aim is to prove a result of Tits which characterises an automorphism of a tree which acts without fixed points.

Proposition 9.6. *(Tits). Let σ be an automorphism of a tree X. Then, σ has no fixed points if, and only if, there exists an infinite path on which σ acts by translation of some non-zero amplitude.*
(Such an automorphism is called hyperbolic).

Proof: Suppose T is an infinite path in X on which σ acts as a translation of amplitude $d > 0$.

If σ had fixed points on X, then $\forall\ Q \in T$, look at the unique geodesic QP between Q and X^σ. Evidently $P \notin T$; otherwise $\ell(P, \sigma)$ would be $d > 0$ by hypothesis.

We proved that the geodesic joining Q and Q^σ is obtained by juxtaposing QP and $P \cdot Q^\sigma$. But the whole geodesic joining Q and Q^σ is in T; so $P \in T$ which again gives a contradiction. So, σ has no fixed points on X.

Conversely, suppose σ has no fixed points on X.

Look at the number $d = \inf\{\ell(P, P^\sigma) : P \in \ Vert(X)\}$. Consider $T \overset{d}{=} \{P \in X : \ell(P, P^\sigma) = d\}$. We shall show that T is an infinite path on which σ acts by a translation of amplitude d.

Let $P \in T$. Look at the geodesic $p = P_1, P_2, \dots P_d$ joining $P = P_0$ and $P_d = P^\sigma$. Consider the geodesic p^σ which joins P^σ and P^{σ^2}. It is $P^\sigma P_1^\sigma P_2^\sigma \dots$. If the juxtaposition pp^σ had backtracking, then $P_{d-1} = P_1^\sigma$.

This is impossible if $d = 1$ since σ acts without inverting the edge P_0P_1. If $d > 1$, then again we would get that $\ell(P_1, P_1^d) = d - 2$ which is impossible by the definition of d.

Hence, the geodesics p and p^σ can be juxtaposed without any backtracking. Similarly, by induction, one can show that $\forall\ n \in \mathbb{Z}$, the geodesic p^{σ^n} can be juxtaposed without backtracking to yield a (doubly) infinite path, say T_P. In fact, we will show that $T = T_P$.

If $Q \in X$ is at distance ℓ from T_P, then $\exists\ P' \in T_P$ for which $\ell(Q, P') = \ell$. As we observed in the previous proof, the juxtaposition $QP', P'(P')^\sigma$ and $(P')^\sigma Q^\sigma$ has no backtracking and $\ell(Q, Q^\sigma) = \ell + d + \ell = d + 2\ell$. Therefore $Q \in T$, we would have $\ell(Q, Q^d) = d$ i.e., $\ell = 0$. So $Q \in T_P$.

One of the main aims of this section is to show that $SL_3(\mathbb{Z})$ is not an amalgam. This will be proved by showing that $SL_3(\mathbb{Z})$ has property (FA). To do this, it will turn out that it suffices to show that under an action of $SL_3(\mathbb{Z})$ on a tree, certain generating elements must have fixed points. Towards this, we first prove:

Proposition 9.7. *: Let $A, B \leq G$ and suppose G acts on a tree X so that $X^A \neq \phi \neq X^B$. If $X^{ab} \neq \phi\ \forall\ a \in A_1 b \in B$ then the subgroup $H = \langle A, B\rangle$ has fixed points on X.*

Proof: Since $X^H = X^A \cap X^B$, we need to show that X^A and X^B cannot be disjoint under the given hypotheses. Suppose they are. Let $P \in X^A, Q \in X^B$ be such that PQ is the shortest geodesic between X^A and X^B. If PQ is $PP_1P_2 \ldots P_{n-1}Q$, then $P_1 \notin X^A$. So $\exists\ a \in A$ such that $P_1^a \neq P_1$. Therefore, the juxtaposition of the geodesics $QP_{n-1} \ldots P_1P$ and $p \cdot P_1^a P_2^a \ldots Q^a$ gives the geodesic from Q to $Q^a = Q^b$ (as $Q \in X^B, Q^b = Q$). By a previous proposition, the element ab fixes the midpoint P of the geodesic from Q

to Q^{ab}. Thus $P^{ab} = P$ i.e., $P^b = P^{a^{-1}} = P$ i.e., $P \in X^B$, which is a contradiction. Hence $X^A \cap X^B \neq \phi$.

Corollary: Let $G = \langle g_1, \ldots, g_n \rangle$ act on a tree X so that each g_i and each $g_i g_j$ have fixed points. Then $X^G \neq \phi$.

Proof: Take $A = \langle g_1, \ldots, g_{n-1} \rangle$ and $B = \langle g_n \rangle$. Applying the proposition, it follows by induction that $X^G \neq \phi$.

Remarks: We shall see in the next section while discussing group actions on so-called **R**-trees that the above corollary generalises to this more general situation also. The following result, although of independent interest, will also be useful in the proof that $SL_3(\mathbf{Z})$ has the property (FA).

Combining the previous corollary with the first corollary of this section, we get :

Corollary : If G is a group generated by a finite set of torsion elements, then G has property (FA). In particular, triangle groups of the form

$$< x, y | x^a, y^b, (xy)^c >$$

for $a, b, c \geq 1$ cannot be amalgams.

Proposition 9.8. *: Let G be a finitely generated nilpotent group acting on a tree X. Then, exactly one of the following two possibilities occurs:*
(a) G has a fixed point. (b) $\exists$ a nontrivial homomorphism $\theta : G \to \mathbb{Z}$ and a straight path T on which each $g \in G$ acts as translation by $\theta(g)$. In particular if $S \subseteq G$ is a subset where each $s \in S$ has a fixed point, then $\langle S \rangle_a$ itself has a fixed point.

Proof: Note first that if G satisfies (b) and if $\theta(g) \neq 0$, then Tit's proposition shows that g cannot have a fixed point. Thus, (a) and (b) are mutually exclusive.

We shall agrue by induction on the length n of a composition series

$$\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$$

where G_i/G_{i-1} is cyclic.

Clearly, if $G = \{1\}$ (i.e. $n = 0$), (a) holds. Suppose $n \geq 1$ and that G_{n-1} satisfies either (a) or (b). If G_{n-1} satisfies (a), then one can consider the action of the cyclic group G/G_{n-1} on the tree $X^{G_{n-1}}$ and the last proposition implies that G itself has a fixed point.

If G_{n-1} has no fixed point, then consider the straight path T on which G_{n-1} acts by translations given by a nontrivial homomorphism α from G_{n-1} to $\mathbb{Z}$. As $G_{n-1} \trianglelefteq G, T$ is left stable by G. Thus, we have a homomorphism $G \to Aut(T)$ whose image contains a group of translations $\alpha(G_{n-1}) \cong \mathbb{Z}$. Hence, this image either be $\cong$ $\mathbb{Z}$ or $\cong$ D_∞, the infinite dihedral group. However, D_∞ is not nilpotent. Thus G does act on T by translations via a nontrivial homomorphism from G to $\mathbb{Z}$. Thus G satisfies (b).

Corollary: If G is as above and if $g \in G$ satisfies $g^n \in [G, G]$ for some $n \geq 1$, then g has a fixed point on any tree on which G acts.

Proof: If G itself has a fixed point, there is nothing to prove. If not, then G satisfies (b) of Proposition. Under a homomorphism $\theta G \to \mathbb{Z}$, the element g evidently maps to 0. Therefore, g leaves the tree T fixed.

Theorem 9.9. *: $SL_3(\mathbb{Z})$ has property (FA). In particular, it is not an amalgam.*

Proof: It is very easy to see that $SL_3(\mathbb{Z})$ is generated by the matrices $X_{ij} = I + e_{ij}$ for $i + j$. This is proved by using the Euclidean algorithm. In fact, using the properties $e_{ij}e_{kl} = \delta_{jk}$, it follows that

$$\begin{aligned} X_{23} &= [X_{21}, X_{13}], X_{31} = [X_{32}, X_{21}], \\ X_{12} &= [X_{13}, X_{32}]. \end{aligned}$$

Note that the matrices X_{13}, X_{21}, X_{32} are also commutators:

$$X_{13} = [X_{12}, X_{23}], X_{21} = [X_{23}, X_{31}], X_{32} = [X_{31}, X_{12}].$$

Notice that the group generated by $\{X_{21}, X_{13}\}$ is nilpotent and X_{23} is in the commutator subgroup. Similarly, it is true that each of these six elements in the commutator subgroup of a finitely generated nilpotent group.

Using the previous corollary, it follows that when $SL_3(\mathbb{Z})$ acts on a tree X, each of these six matrices has a fixed point. Consequently, each of the corresponding six nilpotent subgroups has fixed points on X. In particular, the products $X_{21}X_{13}, X_{32}X_{21}$ and $X_{13}X_{32}$ have fixed points.

By an earlier corollary, the group generated by X_{13}, X_{21} and X_{32} has a fixed point on X. However this is the whole group $SL_3(\mathbb{Z})$.

As mentioned in the first section, this theorem has been generalized to all Chevalley groups of rank at least 2 by Margulis and Tits.

10 R-trees

One may think of our (simplicial) tree as a set of vertices provided with a distance function which takes only integer values - we did so in section 2. It turns out that this point of view can be profitably generalized as follows. Let Λ be a totally ordered abelian group. Then, a Λ-metric space is a set X along with a distance function $d : X \times X \to \Lambda$ which satisfies :
$d(x, y) = 0$ if, and only if, $x = y$;
$d(x, y) = d(y, x)$ and;

$d(x,z) \leq d(x,y) + d(y,z)$.

One calls any subset S of X which is isometric to an interval $[\alpha, \beta]$ in Λ, a segment. Given a segment, obviously the two points which map onto α and β are called its endpoints.

One defines a Λ-tree to be a Λ-metric space X in which :

(a) any two points x, y are the endpoints of a unique segment (which we denote by $[x,y]$);

(b) for $x, y, z \in X$, $[x,y] \cap [x,z] = [x,w]$ for some w;

(c) if $[x,y] \cap [x,z]$ is $\{x\}$, then $[x,y] \cup [x,z] = [y,z]$.

Note that our trees defined earlier are **Z**-trees.

We shall be concerned with groups acting by isometries on Λ-trees. The analogue of 'action without inversions' (again referred to by this same phrase) is the following :

whenever an element g leaves a segment $[x,y]$ invariant, either g fixes both x and y or, the distance $d(x,y)$ is a multiple of 2 in Λ (in which case g fixes the midpoint of $[x,y]$).

Then, our earlier observation that 'the free group acts (without inversions and) freely on its Cayley graph which is a tree', generalizes easily. It gives us the fact that a free product of copies of Λ acts freely and without inversions on a Λ-tree. In fact, the underlying set of this free product can be made into a Λ-tree naturally and the action is by the (left) regular representation.

If $\Lambda = \mathbf{R}$, then obviously the latter condition always holds, and so every action is without inversions. In simple language, a metric space which contains no homeomorphic image of a circle and in which any two points are the extremities of a segment, is an **R**-tree. Unlike **Z**-trees, it turns out that there are groups acting freely on **R**-trees which are not free products of copies of **R**. Thus, the action of a group on an **R**-tree gives a new structure on the group. In fact, the fundamental groups of most Riemann surfaces admit free actions on **R**-trees. These more general trees have come up naturally in the work of Culler, Morgan, Shalen, Voigtmann etc. on Teichmuller spaces and, are therefore, very useful to study.

Here are some generalities on $\mathbb{R}$-trees which will be used in a result due to Culler & Voigtmann that we will discuss in the last chapter.

If (X, d) is an $\mathbb{R}$-tree, a subtree is a subset Y which is convex i.e, if $a, b \in Y$ then $[a, b] \subseteq Y$. Clearly, then (Y, d) is an $\mathbb{R}$-tree.
A subtree Y is said to be closed if Y intersects each segment of X either is a segment of X or not at all. Clearly, segments are closed subtrees.

It is not hard to prove (see, for example, page 31 of Chiswell's book):

Lemma 10.1. *: Let $Y, Z \subseteq X$ be closed subtrees of an $\mathbb{R}$-tree X and let $Y \cap Z \neq \phi$. Then, there are points $y \in Y, z \in Z$ such that $[y, z] \cap Y = \{y\}, [y, z] \cap Z = \{z\}$. Moreover, $[y, z] \subseteq [y_0, z_0]$ for any $y_0 \in Y, z_0 \in Z$.*

Definition 1. *The segment $[y, z]$ above is called the bridge between Y and Z.*

One defines $d(Y, Z)$ to be $d(y, z)$ in this case.
If g is an automorphism of an $\mathbb{R}$-tree X, then either $X^g \neq \phi$ (in which case g is called elliptic) or $C_g \overset{d}{=} \{P \in X : [gP, P] \cap [P, gP] = \{P\}\}$ is a non-empty, closed, $\langle g \rangle$-invariant subtree on which g acts as a translation by some positive real number $\ell(g)$. Page 82 of Chiswell's book may be consulted for a proof. In the latter case, g is called hyperbolic and C_g is called the characteristic subtree corresponding to g. For uniformity of notation, let us also put $C_g = X^g$ and $\ell(g) = 0$ if g is elliptic. This is, in a sense, analogous to the trace function of a representation. We have:

Lemma 10.2. *Let $g, h \in Aut(X,d)$. Then,*
(a) $C_{ghg^{-1}} = gC_h$ *and* $\ell(ghg^{-1}) = \ell(h)$,
(b) $C_{g^{-1}} = C_g$, *and*
(c) $\ell(g^n) = |n|\ell(g), C_g \subseteq C_{g^n}\ \forall\ n \in \mathbb{Z}$.
If $\ell(g) > 0$, *then* $C_g = C_{g^n}\ \forall\ n \neq 0$.

For more details on $\mathbf{R}$-trees, the reader may consult Chiswell's book ([C]) or the surveys by Culler & Shalen ([$CS1$], [$CS2$]). In the last chapter, we discuss a criterion due to Culler & Voigtmann for the property (FA) to hold. This would prove (FA) at one stroke for many groups including the automorphism group Aut F of a free group of rank at least 3 and the group $SL_n(\mathbf{Z})$ for $n \geq 3$.

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