ON THE TOTAL CHARACTER OF FINITE GROUPS

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Abstract. For a finite group $G$, we study the total character $\tau_G$ afforded by the direct sum of all the non-isomorphic irreducible complex representations of $G$. We resolve for several classes of groups (the Camina $p$-groups, the generalized Camina $p$-groups, the groups which admit $(G, Z(G))$ as a generalized Camina pair), the problem of existence of a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\chi) = \tau_G$ for some irreducible character $\chi$ of $G$. As a consequence, we completely determine the $p$-groups of order at most $p^5$ (with $p$ odd) which admit such a polynomial. We deduce the characterization that these are the groups $G$ for which $Z(G)$ is cyclic and $(G, Z(G))$ is a generalized Camina pair and, we conjecture that this holds good for $p$-groups of any order.

1. Introduction

In this paper, $G$ denotes a finite group. Let $\text{Irr}(G)$ and $\text{nl}(G)$ be the set of all irreducible characters of $G$ and the set of all nonlinear irreducible characters of $G$ respectively. Then $\text{lin}(G) = \text{Irr}(G) \setminus \text{nl}(G)$ is the set of linear characters of $G$. Suppose $\rho$ is the direct sum of all the non-isomorphic irreducible complex representations of $G$. The character $\tau_G$ afforded by $\rho$ is called the total character of $G$, that is, $\tau_G = \sum_{\chi \in \text{Irr}(G)} \chi$. Since $\tau_G$ is stable under the action of the Galois group of the splitting field of $G$, $\tau_G(g) \in \mathbb{Z}$ for all $g \in G$.

The dimension $\tau_G(1)$ of $\rho$ seems to have remarkable connections with the geometry of the group. For instance, in the case of the symmetric group $G = S_n$, $\tau_G(1)$ is the number of involutions of $S_n$ \cite{[10]} and, in the case of $G = GL(n, q)$, $\tau_G(1)$ is the number of symmetric matrices in $GL(n, q)$ \cite{[5]}.\[MSC(2010): Primary: 20C15; Secondary: 20D15.
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It is a consequence of a well known theorem due to Burnside and Brauer ([17] Theorem 4.3) that, the total character of the group $G$ is a constituent of $1 + \chi + \cdots + \chi^{m-1}$ if $\chi$ is a faithful character which takes exactly $m$ distinct values on $G$. S. M. Gagola, Jr. & M. L. Lewis classified (in [13]) all the solvable groups for which $\tau_G$ equals $\chi^2$, for some $\chi \in \text{Irr}(G)$. A. Mann also studied the decomposition of $\chi^2$ and proved:

"A nonabelian group $G$ has a faithful irreducible character $\chi$ such that $\text{Irr}(\chi^2) \subseteq \text{lin}(G)$ if and only if $|G'| = 2$ and $Z(G)$ is cyclic".

Here, $\text{Irr}(\chi^2)$ is the set of all irreducible constituents of $\chi^2$ ([11] Theorem 22.7)).

Motivated by this, K. W. Johnson raised the following question:

Does there exist an irreducible character $\chi$ of $G$ and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$? (see [18]).

The aim of the article is to answer a weaker version of this question for several classes of $p$-groups including all $p$-groups of order at the most $p^5$; we examine the existence of a polynomial $f(x) \in \mathbb{Q}[x]$ and $\chi \in \text{Irr}(G)$ such that $f(\chi) = \tau_G$. We call such a polynomial $f(x) \in \mathbb{Q}[x]$, if it exists, a Johnson polynomial of $G$. This problem has been studied for dihedral groups $D_{2n}$ in [18] where it is proved that $D_{2n}$ has a Johnson polynomial if and only if $8 \nmid n$. To describe the classes of groups to which our results apply, we recall some definitions

A pair $(G, N)$ is said to be a generalized Camina pair (abbreviated GCP) if $N$ is normal in $G$ and, all nonlinear irreducible characters of $G$ vanish outside $N$ ([12]). There are a number of equivalent conditions for $(G, Z(G))$ to be a GCP. An equivalent condition we will refer to is:

A pair $(G, Z(G))$ is a GCP if and only if for all $g \in G \setminus Z(G)$, the conjugacy class of $g$ in $G$ is $gG'$. In this case, one can easily observe that $G' \subseteq Z(G)$ and $\chi(1) = |G/Z(G)|^{1/2}$ for all $\chi \in \text{nl}(G)$. For such types of groups, the first author and R. Sarma investigated (in [19]) the existence of a Johnson polynomial. The following theorem was proved in [19].

**Theorem 1.1.** ([19] Theorem 3.2] Let $(G, Z(G))$ be a GCP. Then $G$ has a Johnson polynomial if and only if $Z(G)$ is cyclic. In fact, if $Z(G)$ is cyclic then a Johnson polynomial of $G$ is given by

$$f(x) = d^2 \sum_{j=1}^{r} (x/d)^{lj} + d \sum_{j=1}^{m} (x/d)^{lj},$$

where $d = |G/Z(G)|^{1/2}$, $r = |Z(G)/G'|$, $m = |Z(G)|$ and $l = |G'|$. In particular, $f(x) = d^2(x/d)^m + d \sum_{j=1}^{m-1} (x/d)^j$ when $Z(G)$ is cyclic and $Z(G) = G'$.

Further, the above theorem was used by the authors in [19] to classify all the nonabelian $p$-groups of order $p^4$ ($p$ an odd prime) which have a Johnson polynomial. The purpose of this article is to examine the existence of a Johnson polynomial for $p$-groups of order greater than $p^4$. In this direction, we examine the family of Camina $p$-groups and generalized Camina groups. As a consequence, we are able to obtain a complete classification of groups of order $p^5$ which admit a Johnson polynomial.

A pair $(G, N)$ is said to be a Camina pair if $1 < N < G$ is a normal subgroup of $G$ and for every element $g \in G \setminus N$, $gN \subseteq \text{Cl}_G(g)$, the conjugacy class of $g$. In the special case $N = G'$, the group $G$
is said to be a Camina group. More generally, a group $G$ is said to be a generalized Camina group if $Cl_G(g) = gG'$ for every element $g \in G \setminus G'Z(G)$. It is known (see [13]) that a nilpotent, generalized Camina group $G$ is isoclinic to Camina group which is a $p$-group; the prime $p$ is said to be associated to $G$.

Then, our main results can be stated as follows:

**Theorem A.** Let $G$ be a Camina $p$-group. Then $G$ has a Johnson polynomial if and only if the nilpotency class of $G$ is 2 and $Z(G)$ is cyclic.

**Theorem B.** Let $(G, Z(G))$ be a Camina pair and let $(G/Z(G), Z(G/Z(G)))$ be a generalized Camina pair. Then $G$ does not possess a Johnson polynomial.

**Theorem C.** Let $G$ be a nilpotent, generalized Camina group with associated prime $p$. Then $G$ has a Johnson polynomial if and only if the nilpotency class of $G$ is 2 and $Z(G)$ is cyclic.

In the last section, we apply the above theorems to obtain the complete list of all groups of order $p^5$ (with $p$ odd) which admit a Johnson polynomial. This is proved using case-by-case considerations (using a description of all groups of order $p^5$ by R. James ([8, Section 4.5])) but, in particular, we deduce the following:

**Theorem D.** Let $G$ be a nonabelian $p$-group of order $p^5$ with $p$ odd. Then $G$ has a Johnson polynomial if $Z(G)$ is cyclic and $(G, Z(G))$ is a GCP.

In view of Theorem [13] and the above theorems, it seems reasonable to pose the following conjecture for $p$-groups:

**Conjecture:** A nonabelian $p$-group (with $p$ odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

2. Notations and Preliminaries

Throughout, $C_n$ denotes the cyclic group of order $n$. Suppose $G$ is a finite group. Then $Z(G)$, $G' = G_2$ and $cd(G)$ denote respectively the center, the commutator subgroup and the set of irreducible character degrees of $G$. If $a, b \in G$, then $b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. For $g \in G$, $Cl_G(g)$ denotes its conjugacy class $\{x^{-1}gx : x \in G\}$. The nilpotency class of a nilpotent group $G$ is the number $n$ such that $G_n \neq 1$ and $G_{n+1} = 1$, where $G_2 = [G, G] = G'$ and $G_{i+1} = [G_i, G]$ for $i \geq 2$. Further, if $H$ is a subgroup of $G$ and $\chi$ a character of $G$, $\chi|_H$ denotes the restriction of $\chi$ to $H$. Suppose $N$ is a normal subgroup of $G$. Then we denote by $\text{Irr}(G/N) = \text{Irr}(G) \setminus \text{Irr}(G/N)$.

We start by recalling some basic results that we will need later.

**Lemma 2.1.** [7, Theorem 2.32]

(1) If $G$ has a faithful irreducible character, then $Z(G)$ is cyclic.

(2) If $G$ is a $p$-group and $Z(G)$ is cyclic, then $G$ has a faithful irreducible character.
Proposition 2.2. An abelian group has a Johnson polynomial if and only if it is cyclic. In fact, if $G$ is a cyclic group of order $n$ then $f(x) = 1 + x + \cdots + x^{n-1}$ is a Johnson polynomial of $G$ and $f(\lambda) = \tau_G$ for every faithful irreducible character of $G$.

Proof. Let $f(x)$ be a Johnson polynomial of $G$. Suppose, to the contrary, $G$ is non-cyclic. Then by Lemma 2.1 $\ker(\chi) \neq \{1\}$ for all $\chi \in \text{Irr}(G)$. Since $G$ is an abelian group, $\tau_G$ is the regular character of $G$. Hence

$$\tau_G(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $g \neq 1 \in \ker(\chi)$, we have $\tau_G(g) = f(\chi(g)) = f(\chi(1)) = \tau_G(1)$, which is a contradiction.

Conversely, let $G = \langle a \rangle$ be the cyclic group of order $n$. Set $\zeta_n = e^{2\pi i/n}$ and $f(x) = \sum_{i=0}^{n-1} x^i$. Consider the linear character $\lambda : G \to \mathbb{C}^*$ defined by $a \mapsto \zeta_n$. Then $\lambda$ is a faithful irreducible character and $f(\lambda) = \sum_{i=0}^{n-1} \lambda^i = \tau_G$. \hfill $\square$

Lemma 2.3. Let $G$ be a non-abelian group. Then $\sum_{\chi \in \text{lin}(G)} \chi(g) = 0$ for each $g \in G \setminus G_2$.

In this article, whenever we prove a certain group $G$ does not possess a Johnson polynomial, we use the following simple observation.

Proposition 2.4. Let $\chi$ be an irreducible character of $G$. If $g_1, g_2 \in G$ are such that $\chi(g_1) = \chi(g_2)$ but $\tau_G(g_1) \neq \tau_G(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau_G$.

Proposition 2.5. Let $G$ be a non-abelian group. Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of $G$ and $\chi \in \text{Irr}(G)$ is such that $f(\chi) = \tau_G$. Then $\chi$ is a nonlinear faithful character.

Proof. Suppose $f(x) \in \mathbb{Q}[x]$ and $\chi \in \text{lin}(G)$. Since $G$ is non-abelian, $\text{nl}(G)$ is non-empty. Pick $\psi \in \text{nl}(G)$. Then the inner product of $\psi$ with $f(\chi)$ is zero but with $\tau_G$ is 1. Hence $f(\chi) \neq \tau_G$.

Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of $G$ and $\chi \in \text{nl}(G)$ is such that $f(\chi) = \tau_G$ with $\ker(\chi) \neq \{1\}$. Since $\cap_{\chi \in \text{Irr}(G)} \ker(\chi) = \{1\}$, $\tau_G(1) = \tau_G(g)$ for all $g \neq 1 \in G$. Take $g \neq 1 \in \ker(\chi)$. Then $\tau_G(1) = f(\chi(1)) = f(\chi(g)) = \tau_G(g)$, which is a contradiction. \hfill $\square$

3. Camina $p$-Groups

In this section, we investigate the existence question of a Johnson polynomial for Camina $p$-groups. A. R. Camina in [2] initiated the study of these groups. We start by recalling the definition.

Definition 3.1. (2) Suppose $N$ is a normal subgroup of $G$. A pair $(G, N)$ is a Camina pair if $1 < N < G$ is a normal subgroup of $G$ and for every element $g \in G \setminus N$, $gN \subseteq \text{Cl}_G(g)$.

It is clear that if $(G, N)$ is a Camina pair and if $H$ is normal in $G$ and $H \leq N$ then $(G/H, N/H)$ is also a Camina pair. The following lemma gives a number of equivalent condition for a pair $(G, N)$ to be a Camina pair.

Lemma 3.2. [17] Lemma 3] Let $N$ be a normal subgroup of $G$ and let $g \in G \setminus N$. Then following are equivalent:
\( (1) \) \( \chi(g) = 0 \) for all \( \chi \in \text{Irr}(G|N) \),

\( (2) \) \( |C_G(g)| = |C_{G/N}(gN)| \),

\( (3) \) \( gN \subseteq \text{Cl}_G(g) \).

It is easy to see that if \((G,N)\) is a Camina pair, then \( Z(G) \leq N \leq G' \).

Camina groups have been studied by many authors \[3, 15, 16\]. By Lemma 3.2, it is clear that if \( G \) is Camina group, then \( \chi(g) = 0 \) for all \( \chi \in \text{nl}(G) \) and \( g \in G \setminus G' \). In \[3\], Dark and Scoppola proved:

**Theorem 3.3.** \((3)\) If \( G \) is a finite Camina \( p \)-group, then the nilpotency class of \( G \) is at most 3, i.e., \( G_4 = \{1\} \).

**Lemma 3.4.** \[15\] Corollary 2.3] Let \( G \) be a \( p \)-group of nilpotency class \( r \). If \((G,G_k)\) is a Camina pair, then \( G_i/G_{i+1} \) has exponent \( p \) for \( k-1 \leq i \leq r \).

**Theorem 3.5.** \[15\] Theorem 5.2] Let \( G \) be a Camina \( p \)-group of nilpotency class 3 and let \( |G|/G_2| = p^n, |G_2/G_3| = p^n \). Then

\( (1) \) \( (G,G_3) \) is a Camina pair,

\( (2) \) \( m = 2n \) and \( n \) is even.

**Corollary 3.6.** \[15\] Corollary 5.3] If \( G \) is a Camina \( p \)-group of nilpotency class 3, then \( Z_2(G) = G_2 \) and \( Z(G) = G_3 \).

**Remarks on Camina \( p \)-groups of class 3.**

Suppose \( G \) is a Camina \( p \)-group of nilpotency class 3. Then by Lemma 3.4, \( G/G_2, G_2/G_3, \) and \( G_3 \) are elementary abelian \( p \)-groups and by Corollary 3.6 we have \( G_3 = Z(G) \). Now by Theorem 3.5, we have \((G,G_3)\) is a Camina pair, \( |G|/G_2| = p^{2n}, |G_2/G_3| = p^n \) and \( |G/G_3| = p^{3n} \) where \( n \) is even. We will show that \( \text{nl}(G) = \text{Irr}(G|G_3) \sqcup \text{nl}(G/G_3) \) and \( cd(G) = \{1, p^n, p^{3n/2}\} \).

Take \( \chi \in \text{Irr}(G|G_3) \). Now \( \chi\upharpoonright_{G_3} = \chi(1)\lambda \) for some \( \lambda \in \text{Irr}(G_3) \). Thus

\[
|G| = \sum_{g \in G} |\chi(g)|^2 = \sum_{g \in G_3} |\chi(g)|^2 \quad \text{(since \((G,G_3)\) is a Camina pair)}
\]

\[
= \sum_{g \in G_3} |\chi(1)\lambda(g)|^2
\]

\[
= |\chi(1)^2|G_3|.
\]

Hence \( \chi(1)^2 = |G/G_3| = p^{3n} \) for all \( \chi \in \text{Irr}(G|G_3) \). Thus we have a bijection

\[
\Phi : \text{Irr}(G_3) \setminus \{1_{G_3}\} \rightarrow \text{Irr}(G|G_3) \text{ defined by}
\]

\[
\Phi(\lambda)(g) := \begin{cases} 
p^{3n/2}\lambda(g) & \text{if } g \in G_3, \\
0 & \text{otherwise},
\end{cases}
\]

(3.1)
Proof.
By Theorem 3.5, we have (3.3)

Therefore we have

Hence \( \chi(1)^2 = |G/G_2| = p^{2n} \) for all \( \chi \in \text{nl}(G/G_3) \). Thus we have a bijection

\[
\Psi : \text{Irr}(G/G_2) \setminus \{1_{G_2/G_3}\} \longrightarrow \text{nl}(G/G_3)
\]

\[
\Psi(\lambda)(g) := \begin{cases} 
  p^{3n/2}(\lambda \circ \eta)(g) & \text{if } g \in G_2, \\
  0 & \text{otherwise,}
\end{cases}
\]

where \( \eta : G \longrightarrow G/G_3 \) is the natural homomorphism and \( 1_{G_2/G_3} \) is the trivial character of \( G_2/G_3 \).

Therefore we have \( |\text{nl}(G/G_3)| = |G_2/G_3| - 1 = p^n - 1 \). Now

\[
|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G/G_2| + (|G_3| - 1)|G/G_3| + (|G_2/G_3| - 1)|G/G_2|.
\]

This shows that \( \text{nl}(G) = \text{Irr}(G/G_3) \sqcup \text{nl}(G/G_3) \) as a disjoint union and \( cd(G) = \{1, p^n, p^{3n/2}\} \).

Now, we can compute the total character of a Camina \( p \)-group of nilpotency class 3.

**Proposition 3.7.** Let \( G \) be a Camina \( p \)-group of nilpotency class 3. Then the total character \( \tau_G \) is given by,

\[
\tau_G(g) = \begin{cases} 
  p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n & \text{if } g = 1, \\
  p^{2n} - p^{3n/2} + (p^n - 1)p^n & \text{if } g \in G_3 \setminus \{1\}, \\
  p^{2n} - p^n & \text{if } g \in G_2 \setminus G_3, \\
  0 & \text{otherwise.}
\end{cases}
\]

*Proof.* By Theorem 3.5, we have \( |G/G_2| = p^{2n} \), \( |G_2/G_3| = p^n \) and \( |G/G_3| = p^{3n} \) where \( n \) is even. In view of (3.1) and (3.2), we have all the nonlinear irreducible character of \( G \). Hence, if \( g = 1 \), then

\[
\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{nl}(G)} \chi(1) = p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n.
\]

If \( g \in G \setminus G_2 \), then by Lemma 2.3 and (3.1), (3.2), we get

\[
\tau_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) = 0.
\]
If \( g \neq 1 \in G_3 \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) \\
= |G/G_2| + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g) \\
= p^{2n} - p^{3n/2} + (p^n - 1)p^n \quad \text{(by (3.1) and (3.2)).}
\]

Finally, if \( g \in G_2 \setminus G_3 \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) \\
= |G/G_2| + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g) \\
= p^{2n} - p^n \quad \text{(by (3.1) and (3.2)).}
\]

This completes the proof. \( \square \)

Now, we are ready to characterize Camina \( p \)-groups which admit a Johnson polynomial (Theorem A).

**Proof of Theorem A.** By Theorem 3.3, the nilpotency class of \( G \) is at most 3, i.e., \( G_4 = 1 \). Suppose \( G \) is of nilpotency class equal to 3. If \( Z(G) \) is not cyclic then by Lemma 2.1 \( G \) has no faithful irreducible character. Therefore, from Proposition 2.5 \( G \) has no Johnson polynomial. Now suppose \( Z(G) \) is cyclic and \( \chi \) is a faithful irreducible character of \( G \). Let \( f(x) \in \mathbb{C}[x] \) with \( f(\chi) = \tau_G \). From (3.1) and (3.2), it is clear that \( \chi \in \text{Irr}(G|G_3) \) and \( \chi(g) = 0 \) for all \( g \in G \setminus G_3 \). Now take \( h \in G_2 \setminus G_3 \). Then from (3.3), we have \( f(\chi(h)) = f(0) = \tau_G(h) = p^{2n} - p^n \). If \( g \in G \setminus G_2 \), then from (3.3), we get \( f(\chi(g)) = f(0) = \tau_G(g) = 0 \). Therefore, we have a contradiction to the existence of a Johnson polynomial.

Next suppose that nilpotency class of \( G \) is 2 i.e., \( 1 < G_2 \leq Z(G) \). Since \( G \) is a Camina group, each nonlinear irreducible character of \( G \) vanishes outside \( G_2 \). Therefore, \( G_2 = Z(G) \). Thus \( (G, Z(G)) \) is a generalized Camina pair and hence from Theorem 1.1 the proof is complete. \( \square \)

4. **Groups for which \((G, Z(G))\) is a Camina pair**

In [14], M. L. Lewis began the study of those groups \( G \) for which \((G, Z(G))\) is a Camina pair and, proved that such a group \( G \) must be a \( p \)-group for some prime \( p \). The next lemma ([15, Lemma 2.1]) was proved by Macdonald in a more general setting where \( G \) is a \( p \)-group with \((G, N)\) as a Camina pair. In the case \( N = Z(G) \), this reduces to the following.

**Lemma 4.1.** ([15]) Let \( G \) be a \( p \)-group of nilpotency class \( r \) and let \((G, Z(G))\) be a Camina pair. Then \( Z(G) = G_r \).

**Remarks on \( \text{Irr}(G|Z(G)) \) when \((G, Z(G))\) is a Camina pair.**
Suppose \((G, Z(G))\) is a Camina pair. Then by Lemma 3.2 \(\chi(g) = 0\) for all \(\chi \in \text{Irr}(G/Z(G))\) and for all \(g \in G \setminus Z(G)\). Let \(1_{Z(G)}\) be the trivial character of \(Z(G)\). Now take any \(\chi \in \text{Irr}(G/Z(G))\). Then,

\[
|G| = \sum_{g \in G} |\chi(g)|^2 = \sum_{g \in Z(G)} |\chi(1)\lambda(g)|^2,
\]

where \(\lambda \in \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\}\). Therefore, \(\chi(1)^2 = |G/Z(G)|\). Hence we have a bijection

\[
\Phi : \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \rightarrow \text{Irr}(G/Z(G)) \quad \text{such that}
\]

\[
(4.1) \quad \Phi(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}
\]

**Proposition 4.2.** Let \((G, Z(G))\) be a Camina pair and let \((G/Z(G), Z(G/Z(G)))\) be a generalized Camina pair. Then the total character \(\tau_G\) is given by the following expressions:

\[
\tau_G(1) = |G/G_2| + (|Z(G)| - 1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2}, \quad \text{where } m = |Z(G/Z(G))| - |Z_2(G)/G_2|;
\]

\[
\tau_G(g) = |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2} \quad \text{when } 1 \neq g \in Z(G);
\]

\[
\tau_G(g) = |G/G_2| - |Z_2(G)/G_2||G/Z_2(G)|^{1/2} \quad \text{if } g \in G_2 \setminus Z(G);
\]

\[
\tau_G(g) = 0 \quad \text{if } g \in G \setminus G_2.
\]

**Proof.** Since \((G, Z(G))\) is Camina pair, \(\text{Irr}(G/Z(G))\) is given by (4.1). Therefore, there are \(|Z(G)| - 1\) nonlinear irreducible characters of degree \(|G/Z(G)|^{1/2}\). It is given that \((G/Z(G), Z(G/Z(G)))\) is a generalized Camina pair. So,

\[
|G/Z(G), G/Z(G)| = G_2Z(G)/Z(G) \subseteq Z(G/Z(G)) = Z_2(G)/Z(G).
\]

Since \((G, Z(G))\) is a Camina pair, \(Z(G) \subseteq G_2\). Hence \(G_2Z(G)/Z(G) = G_2/Z(G)\). There is a bijection

\[
\Psi : \text{Irr}(Z(G/Z(G))|G_2/Z(G)) \rightarrow \text{nl}(G/Z(G)) \quad \text{such that}
\]

\[
(4.2) \quad \Psi(\lambda)(g) := \begin{cases} |G/Z_2(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise} \end{cases}
\]

(see [19 Theorem 3.1]). Thus \(G\) has \(|Z(G/Z(G))| - |Z_2(G)/G_2|\) nonlinear irreducible characters with \(Z(G)\) in their kernels and, degree of each such character is \(|G/Z_2(G)|^{1/2}\). Now

\[
\sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2 + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(1)^2 = |G/G_2| + (|Z(G)| - 1)|G/Z(G)| + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)| = |G|.
\]

This shows that \(\text{nl}(G) = \text{Irr}(G/Z(G)) \cup \text{nl}(G/Z(G))\).

Since \((G/Z(G), Z(G/Z(G)))\) is a generalized Camina pair, use [19 Proposition 3.1] to get,

\[
(4.3) \quad \tau_{G/Z(G)}(g) = \begin{cases} |G/G_2| + m|G/Z_2(G)|^{1/2} & \text{if } g \in Z(G) \\ |G/G_2| - |Z_2(G)/G_2||G/Z_2(G)|^{1/2} & \text{if } g \in G_2 \setminus Z(G) \\ 0 & \text{otherwise,} \end{cases}
\]
where \( m = |Z(G/Z(G))| - |Z_2(G)/G_2| \). We use \( \tau_{G/Z(G)} \) to calculate \( \tau_G \).

Next, if \( g = 1 \), then

\[
\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]

(4.4)

\[
= |G/G_2| + (|Z(G)| - 1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2},
\]

where \( m = |Z(G/Z(G))| - |Z_2(G)/G_2| \).

If \( g \neq 1 \in Z(G) \), then by (4.1) and (4.2) we have

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]

(4.5)

\[
= |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G)/Z(G)| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2}.
\]

If \( g \in G_2 \setminus Z(G) \), then then by (4.1), (4.2) and (4.3), we have

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]

(4.6)

\[
= |G/G_2| + 0 - |Z_2(G)/G_2||G/Z_2(G)|^{1/2}.
\]

If \( g \in G \setminus G_2 \), then then by (4.1), (4.2) and (4.3), one can easily get that

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]

(4.7)

\[
= 0.
\]

This completes the proof. \( \square \)

**Proof of Theorem B.** In view of Propositions 2.4 and 4.2, \( G \) has no Johnson polynomial. \( \square \)

## 5. Generalized Camina groups

In this section, we study the total character for a generalized Camina group and characterize those groups which admit a Johnson polynomial. We begin by recalling the important notion of isoclinism introduced by Philip Hall.

**Definition 5.1.** Let \( G, H \) be finite groups. \( G \) and \( H \) are said to be isoclinic if there exist isomorphisms \( \theta : G/Z(G) \rightarrow H/Z(H) \) and \( \phi : G_2 \rightarrow H_2 \) such that

\[
[\theta(g_1 Z(G)), \theta(g_2 Z(G))] = \phi([g_1 Z(G), g_2 Z(G)]) \quad \text{for all} \quad g_1, g_2 \in G.
\]

The notion of isoclinism was first introduced by P. Hall \[6\] who proved that two isoclinic nilpotent groups have the same nilpotency class. It is also known that isoclinic groups of the same order have the same character degrees. Recall:

**Definition 5.2.** \([13]\) A group \( G \) is said to be a Generalized Camina group if \( Cl_G(g) = gG_2 \) for every \( g \in G \setminus G_2 Z(G) \).
This generalization of a Camina group was introduced by M. L. Lewis in [13]. It is clear from the definition that if $G$ is a generalized Camina group, then either $G$ has nilpotence class 2 or $G/Z(G)$ is a Camina group. The author proved that $G$ is a nilpotent generalized Camina group if and only if $G$ is isoclinic to a nilpotent Camina group $H$ and $H$ must be $p$-group ([13]). Lewis also pointed out that a Camina group which is isoclinic to $G$ will be a $p$-group for the same prime $p$; one calls $p$, the prime associated to $G$.

**Definition 5.3.** Let $N$ be a normal subgroup of $G$ and let $\chi \in \text{Irr}(G)$. We say that $\chi$ is fully ramified with respect to $G/N$ if $\chi\downarrow_N = e\theta$ and $\theta\uparrow_G = e\chi$ for some $\theta \in \text{Irr}(N)$ and some integer $e$.

In [13], Lewis proved the following theorem:

**Theorem 5.4.** [13, Theorem 3] Let $G$ be a nilpotent, generalized Camina group of nilpotency class 3. Then following are true:

1. $G/G_2Z(G)$, $G_2Z(G)/Z(G)$, and $G_3 = G_2 \cap Z(G)$ are elementary abelian $p$-groups for some prime $p$.
2. $|G/G_2Z(G)| = p^{2n}$ and $|G_2Z(G)/Z(G)| = |G_2/G_3| = p^n$ for some even integer $n$.
3. $cd(G) = \{1, p^n, p^{3n/2}\}$.
4. $Z(G/G_3) = G_2Z(G)/G_3$ and $G_2Z(G)/G_3 = G_2/G_3 \times Z(G)/G_3$.
5. Every character in $\text{nl}(G/G_3)$ is fully ramified with respect to $G/G_2Z(G)$ and every character in $\text{Irr}(G|G_3)$ is fully ramified with respect to $G/Z(G)$.

**Remarks on Generalized Camina groups of nilpotency class 3.**

Suppose $G$ is a nilpotent, generalized Camina group of nilpotency class 3. Then from the above theorem, we have $|G/Z(G)| = p^{3n}$ and one can observe that there are two bijections namely,

$$\Phi_1 : \text{Irr}(Z(G)|G_3) \rightarrow \text{Irr}(G|G_3)$$

such that

$$\Phi_1(\lambda)(g) = \begin{cases} 
p^{3n/2}\lambda(g) & \text{if } g \in Z(G), \\
0 & \text{otherwise,} \end{cases}$$

and

$$\Psi_1 : \text{Irr}(G_2Z(G)/G_3 | G_2/G_3) \rightarrow \text{nl}(G/G_3)$$

such that

$$\Psi_1(\lambda)(g) = \begin{cases} 
p^n(\lambda \circ \eta)(g) & \text{if } g \in G_2Z(G), \\
0 & \text{otherwise,} \end{cases}$$

where $\eta : G \rightarrow G/G_3$ is the natural homomorphism. Therefore $G$ has $|Z(G)| - |Z(G)/G_3|$ nonlinear irreducible characters of degree $p^{3n/2}$ and $(|G_2/G_3| - 1)|Z(G)/G_3|$ nonlinear irreducible characters of degree $p^n$, and $\text{nl}(G) = \text{Irr}(G|G_3) \sqcup \text{nl}(G/G_3)$.
Lemma 5.5. Let $G$ be a generalized Camina group of nilpotency class 3. Then

\begin{equation}
\sum_{\lambda \in \text{Irr}(Z(G)/G_3)} \lambda(g) = \begin{cases} 
-|Z(G)/G_3| & \text{if } g \in G_3, \\
0 & \text{if } g \in Z(G) \setminus G_3 
\end{cases}
\end{equation}

and

\begin{equation}
\sum_{\lambda \in \text{Irr}(G_2Z(G)/G_3 | G_2/G_3)} \lambda(g) = \begin{cases} 
(p^n - 1)|Z(G)/G_3| & \text{if } g \in G_3, \\
-|Z(G)/G_3| & \text{if } g \in G_2 \setminus Z(G), \\
0 & \text{otherwise,}
\end{cases}
\end{equation}

where $|G_2/G_3| = p^n$.

Proposition 5.6. Let $G$ be a generalized Camina group of nilpotency class 3. Then the total character $\tau_G$ is given by,

\begin{equation}
\tau_G(g) = \begin{cases} 
|G/G_2| + rp^{3n/2} + (p^n - 1)|Z(G)/G_3|p^n & \text{if } g = 1, \\
|G/G_2| - |Z(G)/G_3|p^{3n/2} + (p^n - 1)|Z(G)/G_3|p^n & \text{if } g \neq 1 \in G_3, \\
|G/G_2| - |Z(G)/G_3|p^n & \text{if } g \in G_2 \setminus Z(G), \\
0 & \text{otherwise,}
\end{cases}
\end{equation}

where $r = |Z(G)| - |Z(G)/G_3|$.

Proof. If $g = 1$, then

\[
\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{nil}(G)} \chi(1) = |G/G_2| + (|Z(G)| - |Z(G)/G_3|)p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n
\]

If $g \neq 1 \in G_3$, then

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) = |G/G_2| + \sum_{\chi \in \text{Irr}(G/G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g).
\]

Now use (5.1), (5.2) and Lemma 5.5 to get

\[
\tau_G(g) = |G/G_2| - |Z(G)/G_3|p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n.
\]

If $g \in Z(G) \setminus G_3$, then

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) = 0 + \sum_{\chi \in \text{Irr}(G/G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)}
\]

\[
= 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5)}.
\]
If \( g \in G_2 \setminus Z(G) \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g) = |G/G_2| + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g) = |G/G_2| + |Z(G)/G_3|p^n \quad \text{(use (5.1), (5.2) and Lemma 5.5)}.
\]

If \( g \in G_2Z(G) \) but neither in \( G_2 \) nor in \( Z(G) \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g) = 0 + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)}
\]
\[
= 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5)}.
\]

Finally, if \( g \in G \setminus G_2Z(G) \), then by Lemma 2.3, (5.1) and (5.2), we get
\[
\tau_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) = 0.
\]
This completes the proof. \( \square \)

We can now characterize nilpotent, generalized Camina groups (Theorem C).

**Proof of Theorem C.** Let \( G \) be a nilpotent, generalized Camina group with associated prime \( p \). If \( p = 2 \), then \( G \) has nilpotency class 2 and for \( p \) odd, \( G \) has nilpotency class at most 3 (see [13, Theorem 2]). Now if nilpotency class is 2, then \( (G, Z(G)) \) is a generalized Camina pair and hence the result follows from Theorem 1.1.

Next suppose \( G \) has nilpotency class 3. If \( Z(G) \) is not cyclic then by Lemma 2.1, \( G \) has no faithful irreducible character. Therefore from Proposition 2.5, \( G \) has no Johnson polynomial. Now suppose \( Z(G) \) is cyclic. Therefore \( G \) has a faithful irreducible character \( \chi \) (say). Let \( f(x) \) be a Johnson polynomial and let \( f(\chi) = \tau_G \). From (5.1) and (5.2), it is clear that \( \chi \in \text{Irr}(G|G_3) \). Then, in view of Proposition 2.4 and 5.6, \( G \) has no Johnson polynomial.

This completes the proof. \( \square \)

6. \( p \)-groups of order \( p^5 \)

In this final section, we completely classify the groups of order \( p^5 \) (for \( p \) odd) which admit a Johnson polynomial. Throughout this section \( p \) always denotes an odd prime. We will use not only the results of the previous sections but, more crucially, also use the classification of groups of order \( p^5 \) by R. James ([8, Section 4.5]).

We begin by recalling some well known results which we will use.
Theorem 6.1. [1] Theorem 22.5] If $G$ is a nonabelian $p$-group with $cd(G) = \{1, p\}$, then exactly one of the following holds:

1. $G$ has an abelian subgroup of index $p$,
2. $G/Z(G)$ is of order $p^3$ and exponent $p$.

Lemma 6.2. [7] Lemma 2.9] Let $H$ be a subgroup of $G$. Suppose $\chi$ is a character of $G$. Then

$$\langle \chi \downarrow_H, \chi \downarrow_H \rangle \leq |G/H| \langle \chi, \chi \rangle$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

Lemma 6.3. [1] Theorem 20] If $G$ is a $p$-group, then for each $\chi \in \text{Irr}(G)$, $\chi(1)^2$ divides $|G/Z(G)|$.

Here is an easy consequence of the above lemma.

Lemma 6.4. Let $G$ be a non-abelian group of order $p^4$. Then $cd(G) = \{1, p\}$.

Proof. Since $Z(G) \neq 1$, $|Z(G)| = p$ or $p^2$. Therefore $|G/Z(G)| = p^3$ or $p^2$. So by Lemma 6.3, the result follows. $\square$

Theorem 6.5. [7] Theorem 6.15] Let $H$ be an abelian normal subgroup of $G$. Then $\chi(1)$ divides $|G/H|$ for all $\chi \in \text{Irr}(G)$.

As mentioned at the outset of this section, we will use the classification of groups of order $p^5$ by R. James ([8] Section 4.5]). More particularly, we will use the list of polycyclic presentations of these groups that the author compiled, and divided the non-abelian ones into families denoted by $\Phi_1, \cdots, \Phi_{10}$, according to isoclinism.

Lemma 6.6. If $G \in X = \{\Phi_2(41), \Phi_2(311)b, \Phi_5(2111), \Phi_5(1^5)\}$ (see [8] Section 4.5]), then $G$ has a Johnson polynomial.

Proof. First we consider the isoclinism family $\Phi_2$. There are two type of groups in this family with $Z(G)$ cyclic namely,

$$G = \Phi_2(41) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^3} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle \quad \text{and} \quad H = \Phi_2(311)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha_1^p = \alpha_1^p = \alpha_2^p = 1 \rangle.$$

Here $|Z(G)| = |\langle \alpha^{p^3} \rangle| = p^3$, $|G_2| = |\langle \alpha^{p^3} \rangle| = p, |Z(H)| = |\langle \gamma^{p^2} \rangle| = p^3, |H_2| = |\langle \gamma^{p^2} \rangle| = p$. By Lemma 6.3, we have $cd(G) = \{1, p\}$ and $cd(H) = \{1, p\}$. Now by Lemma 6.2 it is clear that $(G, Z(G))$ and $(H, Z(H))$ are generalized Camina pair. Hence by Theorem 1.1 $G$ and $H$ have a Johnson polynomial.

Now we discuss the isoclinism family $\Phi_5$. There are only two type of groups in this family and both have cyclic center. Here are the groups:

1. $\Phi_5(2111) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$;
2. $\Phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$. 


Note that both $\Phi_5(2111)$ and $\Phi_5(1^5)$ are extra-special $p$-groups. Therefore for these two groups $(G, Z(G))$ is a GCP (see [9, Theorem 2.18]). Since $G$ is an extra-special $p$-group, $Z(G) = G_2$ and $|Z(G)| = p$. Therefore by Theorem 1.1, the polynomial

$$f(x) = p^n \sum_{j=1}^{p-1} (x/p^n)^j + p^{2n}(x/p^n)^p$$

is a Johnson polynomial of $G$ and $f(\chi) = \tau_G$ for every $\chi \in \text{nl}(G)$, where $G \in \Phi_5$. □

**Lemma 6.7.** If $G$ in the isoclinism family $\Phi_3$, then $G$ has no Johnson polynomial.

**Proof.** Let $G \in \Phi_3$. There are two type of groups in this family with $Z(G)$ cyclic namely, $\Phi_3(2111)c$ and $\Phi_3(311)b_r$ (see [8, Section 4.5]). For $p = 3$ and $p \geq 5$, we define these groups separately.

1. $G = \Phi_3(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_1^p \alpha_3 = \alpha_2^p \alpha_3 = 1 \rangle$ for $p = 3$;

2. $H = \Phi_3(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_1^p = 1 \rangle$ for $p \geq 5$;

3. $K = \Phi_3(311)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha^p = \alpha_1^p = 1 \rangle$ for $r = 1$ or $\nu$, where $\nu$ is a fixed quadratic non-residue mod $p$ and $p \geq 3$.

Observe that $|Z(G)| = |\langle \gamma \rangle| = p^2$, $|Z(H)| = |\langle \gamma \rangle| = p^2$ and $|Z(K)| = |\langle \alpha_1^p \rangle| = p^2$.

First we will deal with $H$. Consider a normal abelian subgroup

$$N = \langle \alpha, \alpha_1, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p, \alpha^p = \gamma^p = 1 \ (i = 1, 2) \rangle$$

of $H$ of index $p$. By Theorem 6.3, $cd(H) = \{1, p\}$. Since $N$ is a normal abelian subgroup of index $p$, every nonlinear irreducible characters of $H$ must be induced from $N$ and hence $\chi(H \setminus N) = 0$ for all $\chi \in \text{nl}(H)$. Now

$$\overline{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = 1 \ (i = 1, 2) \rangle$$

is an extra-special $p$-group of order $p^3$. Therefore, $\overline{H}$ has $p - 1$ nonlinear irreducible characters of each of degree $p$ which vanish out side $Z(\overline{H})$ in $\overline{H}$ and on $Z(\overline{H})$ it is $p\lambda$, where $\lambda \in \text{Irr}(Z(\overline{H})) \setminus \{1_{Z(\overline{H})}\}$. In particular, $H$ has $p - 1$ nonlinear irreducible characters which contains $Z(H)$ in their kernel.

Take $Q = \langle \gamma^p \rangle$. Then $\text{Irr}(H/Z(H)) = \text{Irr}(H/Q|Z(H)/Q) \cup \text{Irr}(H/Q)$. Now, suppose $\chi \in \text{Irr}(H|Q)$. Then $\chi$ is faithful. Let $\phi$ be an irreducible constituent of $\chi|_M^H$, where $M = \langle \alpha_2, \gamma \rangle$. Since $\chi$ is faithful, $\phi$ is not $H$-invariant. Therefore, by Clifford’s theorem $\chi|_M^H = \sum_{i=1}^p \phi_i$, where $\phi_i = \phi$ and $p$ is the index of the inertia group $N$ of $\phi$ in $H$. Now $\phi_1|_{Z(H)}^M = \lambda$, where $\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$ for each $1 \leq i \leq p$.

Therefore, by [7, Corollary 6.17], we have

$$\chi|_M^H = \sum_{\sigma \in \text{Irr}(M/Z(H))} \sigma \phi_1 = \rho_{M/Z(H)} \phi_1,$$

where $\rho_{M/Z(H)}$ is the regular character of $M/Z(H)$. Hence for each $\chi \in \text{Irr}(H|Q)$, we have $\chi(M \setminus Z(H)) = 0$. 

Let $\Phi$ be the inertia group $N$ and $\Phi = \text{Irr}(N)$.

Then $F(\Phi, \Phi_3(2111)) = \langle \phi \rangle$.

Observe that $|\Phi_3(2111)| = p^3$, $|\Phi_3(311)b_r| = p^5$, and $|\Phi_3(311)b_r| = p^3$.

Therefore, $\Phi_3(2111)$ is a GCP (see [9, Theorem 2.18]). Since $G$ is an extra-special $p$-group, $Z(G) = G_2$ and $|Z(G)| = p$. Therefore by Theorem 1.1, the polynomial

$$f(x) = p^n \sum_{j=1}^{p-1} (x/p^n)^j + p^{2n}(x/p^n)^p$$

is a Johnson polynomial of $G$ and $f(\chi) = \tau_G$ for every $\chi \in \text{nl}(G)$, where $G \in \Phi_5$. □
Next, we consider \( \chi \in \text{Irr}(H/Q|Z(H)/Q) \), where \( H/Q = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \gamma^p = \alpha_i^p = 1 \ (i = 1, 2) \rangle \) and \( Z(H/Q) = M/Q \). Since \( (H/Q, Z(H/Q)) \) is a generalized Camina pair, \( \chi(\alpha_2) = p\lambda(\alpha_2) \), where \( \lambda \in \text{Irr}(Z(H/Q)) \setminus \text{Irr}(Z(H/Q)/(H/Q)_2) \) (see [19] Theorem 3.1).

But then
\[
\tau_H(\alpha_2) = \sum_{\chi \in \text{lin}(H)} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(H/Z(H))} \chi(\alpha_2) = |H/H_2| + \sum_{\lambda \in Z(H) \setminus \{1_{Z(H)}\}} p\lambda(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Q)} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Q|Z(H)/Q)} \chi(\alpha_2) = p^3 - p + 0 - p^2 + p
\]
(6.2)
\[
= p^3 - p^2.
\]

Now suppose \( H \) has a Johnson polynomial \( f(x) \) such that \( f(\chi) = \tau_H \), where \( \chi \in \text{nl}(H) \). Therefore \( \chi \) is faithful and \( \chi \in \text{Irr}(H/Q) \). Now \( f(0) = f(\chi(\alpha_2)) = \tau_H(\alpha_2) = p^3 - p^2 \) and \( f(0) = f(\chi(\alpha)) = \tau_H(\alpha) = 0 \). The resultant contradiction proves that \( H \) can have no Johnson polynomial.

One can use a very similar argument to show that neither \( G \) nor \( K \) can have a Johnson polynomial. \( \square \)

**Lemma 6.8.** If \( G \) in the isoclinism family \( \Phi_7 \) or \( \Phi_8 \), then \( G \) has no Johnson polynomial.

**Proof.** Suppose \( G \) is in the isoclinism family \( \Phi_7 \). For \( p = 3 \) and \( p \geq 5 \), we will define these groups separately.

For \( p = 3 \):

1. \( G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3 = \alpha^3, \alpha_1^3 \alpha_3 = \alpha_3^3 = \beta^3 = 1 \ (i = 1, 2) \rangle \);
2. \( G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3, \alpha_1^3 = \alpha_3^3 = \beta^3 = 1 \ (i = 1, 2) \rangle \);
3. \( G = \Phi_7(2111)b_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3, \alpha_1^3 = \alpha_3^3 = \beta^3 = 1 \ (i = 1, 2) \rangle \);
4. \( G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3, \alpha_3^3 = \alpha_1^3 \alpha_3 = \alpha_3^3 \beta^3 = 1 \ (i = 1, 2) \rangle \);
5. \( G = \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3, \alpha_3^3 = \alpha_1^3 \alpha_3 = \alpha_3^3 \beta^3 = 1 \ (i = 1, 2) \rangle \).

For \( p \geq 5 \):

1. \( G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3 = \alpha^p, \alpha_1^p = \alpha_3^p = \beta^p = 1 \ (i = 1, 2) \rangle \);
2. \( G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3 = \alpha_1^p = \alpha_3^p = \beta^p = 1 \ (i = 1, 2) \rangle \);
3. \( G = \Phi_7(2111)b_\nu = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_i+1, [\alpha_1, \beta] = \alpha_3 = \alpha_1^p = \alpha_3^p = \beta^p = 1 \ (i = 1, 2) \rangle \) where \( \nu \) is a fixed quadratic non-residue \( \mod p \) and \( 2 \leq \nu \leq p - 1 \);
(4) $G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^p, \alpha^p = \alpha^p_1 = \alpha^p_{i+1} = 1 \ (i = 1, 2) \rangle$;

(5) $G = \Phi_7(15) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha^p_1 = \alpha^p_{i+1} = \beta^p = 1 \ (i = 1, 2) \rangle$.

It is clear that $|Z(G)| = |(\alpha_3)| = p$ and

$$G/Z(G) = H \times K = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha^p_1 = \alpha^p_2 = 1 \rangle \times \langle \beta \rangle$$

is of order $p^4$ for all $G \in \Phi_7$, where $H$ is extra-special $p$-group of order $p^3$ and $K$ is a cyclic group of order $p$. Hence, by Lemma 6.4 we have $cd(G/Z(G)) = \{1, p\} \subseteq cd(G)$. Since $G$ has no abelian subgroup of index $p$ for all $G \in \Phi_7$, from Theorem 6.1 and Lemma 6.3 we get $cd(G) = \{1, p, p^2\}$. From Lemma 6.2 it is easy to observe that if $\chi(1) = p^2$, then $\chi(g) = 0$ for all $g \in G \setminus Z(G)$. Hence $(G, Z(G))$ is a Camina pair. Since $H$ is an extra-special $p$-group, every nonlinear irreducible character $\phi$ of $H$ vanishes outside $Z(H) = \langle \alpha_2 \rangle$ in $H$ and $\phi|_{Z(H)} = p\lambda$ for some $\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$, where $1_{Z(H)}$ is the trivial character of $Z(H)$. Hence

$$\text{nl}(G/Z(G)) = \{ \phi \times \psi \mid \phi \in \text{nl}(H), \psi \in \text{Irr}(K) \}.$$ 

Now if $g = \alpha_2$, then

$$\tau_G(\alpha_2) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)$$

$$= |G/G_2| + 0 + \sum_{\phi \in \text{nl}(H)} (\phi \times \psi)(\alpha_2) \quad ((G, Z(G)) \text{ is a Camina pair})$$

$$= p^3 + \sum_{\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}, \psi \in \text{Irr}(K)} (p\lambda \times \psi)(\alpha_2) \quad (H \text{ is an extra-special group})$$

$$= p^3 + p \sum_{\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}} p\lambda(\alpha_2)$$

(6.3)

$$= p^3 - p^2.$$ 

Since $(G, Z(G))$ is a Camina pair and $H$ is an extra-special group,

$$\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g) = 0$$

for all $g \in H \setminus Z(H)$. Now suppose $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G/Z(G))$. Since $\chi \in \text{Irr}(G/Z(G))$, $\chi(g) = 0$ for all $g \in G \setminus Z(G)$. In particular, $\chi(\alpha_1) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha_2) = p^3 - p^2$ whereas $f(0) = f(\chi(\alpha_1)) = \tau_G(\alpha_1) = 0$, which is a contradiction. Thus, $G$ cannot have a Johnson polynomial.

Next suppose $G$ is in the isoclinism family $\Phi_8$;

$G := \Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha^p_1, \beta^p = \alpha^p_2 = 1 \rangle$. Here $|Z(G)| = |(\alpha^p_1)| = p$ and

$$G/Z(G) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \alpha^p_1, \alpha^p_1 = \alpha^p_2 = 1 \rangle.$$
is of order $p^4$. To show that $cd(G) = \{1, p, p^2\}$, we may use the same argument as we do for the groups in the family $\Phi_7$; hence we skip the details. Now one can observe that $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair. Therefore, by Theorem B, $G$ has no Johnson polynomial.

\begin{proof}
Observe that $Z(H) = \langle \alpha_3 \rangle$ and $H_2 = \langle \alpha_2, \alpha_3 \rangle$. Since $H$ has a normal abelian subgroup $N = \langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \langle \alpha_3 \rangle$ of order $p^3$, by Theorem 6.5, $cd(H) = \{1, p\}$. Now, if we consider the group

$$
\mathcal{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle.
$$

we see that it is an extra-special $p$-group of order $p^3$. Therefore, $\mathcal{H}$ has $p - 1$ nonlinear irreducible characters of degree $p$ which vanish outside $Z(\mathcal{H}) = \langle \alpha_2 \rangle$ and, for $\chi \in \text{nl}(\mathcal{H})$, we have

$$
\chi \downarrow_{Z(\mathcal{H})} = p\lambda
$$

for some $\lambda \in \text{Irr}(Z(\mathcal{H})) \setminus \{1_{Z(\mathcal{H})}\}$. In particular, we have all the nonlinear irreducible characters of $H$ having $Z(H)$ in their kernel. Now, let $\psi \in \text{Irr}(H|Z(H))$. Since $|Z(H)| = p$, $\psi$ is faithful and hence $\phi$ is not $H$-invariant, where $\phi$ is an irreducible constituent of $\psi^H_{H_2}$. Therefore, by Clifford's theorem $\psi^H_{H_2} = \sum p_i \phi_i$, where $\phi_1 = \phi$ and $p$ is the index of the inertia group $N$ of $\phi$ in $H$. Now $\phi_1^H_{H_2} = \lambda$, where $\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$ for each $1 \leq i \leq p$. Therefore, by [7] Corollary 6.17, we have

$$
\psi^H_{H_2} = \sum_{\beta \in \text{Irr}(H_2/Z(H))} \beta \phi_1 = \rho_{H_2/Z(H)} \phi_1,
$$

where $\rho_{H_2/Z(H)}$ is the regular character of $H_2/Z(H)$. Hence for each $\psi \in \text{Irr}(H|Z(H))$, we have

$$
\psi(H_2 \setminus Z(H)) = 0.
$$

Now

$$
\sum_{\chi \in \text{nl}(H)} \chi(\alpha_2) = \sum_{\chi \in \text{nl}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2)
$$

$$
= \sum_{\lambda \in \text{Irr}(Z(K)) \setminus \{1_{Z(K)}\}} p \lambda(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2) \quad \text{(Use (6.5))}
$$

$$
= -p + 0 = -p
$$

(6.7)
and
\[
\sum_{\chi \in \text{nl}(H)} \chi(\alpha_3) = \sum_{\chi \in \text{nl}(H/Z(H))} \chi(\alpha_3) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_3)
\]
\[
= p(p-1) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_3) \quad \text{(Since } \chi(\alpha_3) = p \text{ for all } \chi \in \text{nl}(H))
\]
\[
(6.8) \quad = p(p-1) - p^2 = -p
\]

This completes the proof of the lemma. \qed

**Lemma 6.10.** If \( G \in \Phi_9 \), then \( G \) has no Johnson polynomial.

**Proof.** Suppose \( G \) is in the isoclinism family \( \Phi_9 \); these are defined as follows. **For** \( p = 3 \):

1. \( G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_4, \alpha_3^3 \alpha_3 = \alpha_3^3 = \alpha_4^3 = 1 (i = 1, 2, 3) \rangle; \)
2. \( G = \Phi_9(2111)b_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^3 \alpha_3 = \alpha_4, \alpha^3 = \alpha_2^2 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 (i = 1, 2, 3) \rangle; \)
3. \( G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_1^3 \alpha_3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 (i = 1, 2, 3) \rangle. \)

**For** \( p \geq 5 \):

1. \( G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, \alpha_1^p = \alpha_3^p = 1 (i = 1, 2, 3) \rangle; \)
2. \( G = \Phi_9(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^p = \alpha_4^k, \alpha^p = \alpha_3^p = 1 (i = 1, 2, 3) \rangle \)

where \( k = g^r \) for \( r = 1, 2, \cdots, (p-1, 3) \);
3. \( G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_3^p = 1 (i = 1, 2, 3) \rangle. \)

Here \( |Z(G)| = |\langle \alpha_4 \rangle| = p \) and
\[
G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_3^p = 1 (i = 1, 2) \rangle
\]
is of order \( p^4 \) for all \( G \in \Phi_9 \). Note that \( G \) has an abelian normal subgroup \( N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) of index \( p \) for all \( G \in \Phi_9 \). Therefore, by Theorem 6.5 we have \( cd(G) = \{1, p\} \) for all \( G \in \Phi_9 \).

Now consider \( p \geq 5 \). In this case
\[
G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_3^p = 1 (i = 1, 2) \rangle.
\]

Since \( N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is a normal abelian subgroup of index \( p \), every nonlinear irreducible character of \( G \) must be induced from \( N \) and hence \( \chi(G \setminus N) = 0 \) for all \( \chi \in \text{nl}(G) \). Let \( K = \langle \alpha_3, \alpha_4 \rangle \). Now, let \( \chi \in \text{Irr}(G/Z(G)) \). Since \( |Z(G)| = p \), \( \chi \) is faithful. Let \( \phi \) be an irreducible constituent of \( \chi_{|K}^G \). Since \( \chi \) is faithful, \( \phi \) is not \( G \)-invariant. And hence by Clifford’s theorem, we have \( \chi_{|K}^G = \sum \phi \), where \( \phi_1 = \phi \) and \( p \) is the index of the inertia group \( N \) of \( \phi \) in \( G \). Now \( \phi_1_{|Z(G)}^K = \lambda \), where \( \lambda \in \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \) for each \( 1 \leq i \leq p \). Therefore, by [17, Corollary 6.17], we have
\[
(6.9) \quad \chi_{|K}^G = \sum_{\gamma \in \text{Irr}(K/Z(G))} \gamma \phi_1 = \rho_{K/Z(G)} \phi_1,
\]
where $\rho_{K/Z(G)}$ is the regular character of $K/Z(G)$. Hence for each $\chi \in \text{Irr}(G|Z(G))$, we have $\chi(K \setminus Z(G)) = 0$. Therefore,

$$
\tau_G(\alpha_3) = \sum_{\chi \in \text{lin}(G)} \chi(\alpha_3) + \sum_{\chi \in \text{nl}(G)} \chi(\alpha_3) \\
= |G/G_2| + \sum_{\chi \in \text{Irr}(G|Z(G))} \chi(\alpha_3) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_3) \quad (\text{Since } \alpha_3 \in G_2) \\
= p^2 + 0 - p \quad (\text{Use Lemma 6.9}) \\
(6.10) = p^2 - p
$$

Now suppose $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G|Z(G))$. By (6.9), we have $\chi(\alpha_3) = 0$ for all $\chi \in \text{Irr}(G|Z(G))$. Now by (6.10), we have $f(0) = f(\chi(\alpha_3)) = \tau_G(\alpha_3) = p^2 - p$. and since $\chi(\alpha) = 0$ for all $\chi \in \text{nl}(G)$, $f(0) = f(\chi(\alpha)) = \tau_G(\alpha) = 0$, which is a contradiction. Hence in the case of $p \geq 5$, $G$ has no Johnson polynomial.

Very similarly, for $p = 3$, one can show that $G$ has no Johnson polynomial. This completes the proof of this lemma.

**Lemma 6.11.** If $G \in \Phi_{10}$, then $G$ has no Johnson polynomial.

**Proof.** Suppose that $G \in \Phi_{10}$; these are defined as follows.

For $p = 3$:

1. $\Phi_{10}(2111)a_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 = \alpha^3, \alpha_3^3 = \alpha_2^2 \alpha_4 = \alpha_3 = \alpha_4 = 1 (i = 1, 2, 3) \rangle$;

2. $\Phi_{10}(2111)a_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^2 = \alpha_4^2 = \alpha^3, \alpha_3^2 \alpha_3 = \alpha_2^2 \alpha_4 = \alpha_3 = \alpha_4^3 = 1 (i = 1, 2, 3) \rangle$;

3. $\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_3^3 = \alpha_3^2 \alpha_4 = \alpha_3 = \alpha_4 = 1 (i = 1, 2, 3) \rangle$.

For $p \geq 5$:

1. $\Phi_{10}(2111)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^p = \alpha_{i+1}^p = 1 (i = 1, 2, 3) \rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 4)$ and $g$ is the smallest positive integer which is primitive root $\text{mod } p$;

2. $\Phi_{10}(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^p, \alpha_3^p = \alpha_{i+1}^p = 1 (i = 1, 2, 3) \rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 3)$ and $g$ is the smallest positive integer which is primitive root $\text{mod } p$;

3. $\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_3^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i = 1, 2, 3) \rangle$.

Here $|Z(G)| = |\langle \alpha_4 \rangle| = p$ and

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1 (i = 1, 2) \rangle$$

is of order $p^4$ for all $G \in \Phi_{10}$. Note that $G$ has no abelian subgroup of index $p$ for all $G \in \Phi_{10}$. By Lemma 6.4, we have $cd(G/Z(G)) = \{1, p\} \subseteq cd(G)$. Therefore from Theorem 6.1 and Lemma 6.3 we
get $cd(G) = \{1, p, p^2\}$. If $\chi(1) = p^2$, then by Lemma 6.2 $\chi$ vanish outside $Z(G)$ in $G$ for all $G$ in $\Phi_{10}$. This shows that $(G, Z(G))$ is a Camina pair for all $G$ in $\Phi_{10}$. Now consider the group $G/Z(G)$ for $p \geq 5$.

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_4^p = \alpha_5^p = 1 \rangle.$$ 

Then

$$
\tau_G(\alpha_2) = \sum_{\chi \in \text{lin}(G)} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G)} \chi(\alpha_2) \\
= |G/G_2| + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } \alpha_2 \in G_2) \\
= p^2 + 0 + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } (G, Z(G)) \text{ is a Camina pair}) \\
= p^2 - p \quad \text{(Use Lemma 6.9)} 
$$

Next suppose that $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G/Z(G))$. Since $(G, Z(G))$ is a Camina pair, $\chi(g) = 0$ for all $g \in G \setminus Z(G)$ and $\chi \in \text{Irr}(G/Z(G))$. In particular, $\chi(\alpha) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha) = p^2 - p$ and $f(0) = f(\chi(\alpha)) = \tau_G(\alpha) = 0$, which is a contradiction. Hence in this case, $G$ has no Johnson polynomial.

Next, for $p = 3$, the group $G/Z(G)$ is

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^3 = \alpha_1^3 \alpha_3 = \alpha_2^3 = \alpha_3^3 = 1 \rangle,$$

and one can use a very similar argument to show that $G$ has no Johnson polynomial in this case also. This completes the proof of the lemma. □

Finally, we may summarize the results of this section in the form of Theorem D.

**Proof of Theorem D.** As above, we use the list of nonabelian $p$-group of order $p^5$ given by R. James [8 Section 4.5]. From Lemma 2.1 and Proposition 2.5, it is clear that if $G$ has a Johnson polynomial then $Z(G)$ must be cyclic. The nonabelian $p$-groups of order $p^5$ with $Z(G)$ cyclic occur in the isoclinism family $\Phi_2, \Phi_3, \Phi_5, \Phi_7, \Phi_8, \Phi_9$, and $\Phi_{10}$ (see [8 pages 620-621]). Therefore, the result follows from Lemmata 6.6, 6.7, 6.8, 6.10 and 6.11.

In view of the above results, it seems reasonable to pose the following conjecture for $p$-groups:

**Conjecture:** A $p$-group (with $p$ odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

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