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A generalization of a converse to Schur's theorem

In memory of Mrs. S. Devi

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Abstract. Niroomand (Arch. Math. **94** (2010) 401–404) proved a converse to a theorem of Schur in the following sense. He proved that if G is a group such that $[G, G]$ is finite and $G/Z(G)$ is finitely generated, then $G/Z(G)$ is finite, of order bounded above by $[G, G]^k$ where k is the minimal number of generators required for $G/Z(G)$. Here, we give a completely elementary short proof of a further generalization.

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1. Main theorem We give a simple proof of a generalization of a result due to Niroomand [1].

Theorem. *Let G be a group such that the set S of its commutators is finite. Then, $[G, G]$ is finite. Further, if $G/Z(G)$ can be generated by r elements $|G/Z(G)| \leq |S|^r$.*

Proof. Let $S = \{[x_i, y_i] : 1 \leq i \leq d\}$. Consider the finitely generated subgroup $H = \{x_1, y_1, \ldots, x_d, y_d\}$ of G. Note that S is also the set of commutators of H. Let $H/Z(H)$ be generated by the images of $g_1, g_2, \ldots, g_r \in H$. We may assume that $r \leq 2d$ but we do not need it here. Note that $g \in Z(H)$ if and only if g commutes with each of $g_1, \ldots, g_r$. That is, $Z(H) = \cap_{i=1}^r C_H(g_i)$. Consider the conjugacy class $\mathrm{cl}(g_i)$ in H of each g_i $(i \leq r)$. For each $g \in H$, $gg_ig^{-1} = sg_i$ for some $s \in S$. Thus, $|\mathrm{cl}(g_i)| \leq |S|$. Therefore, $[H : C_H(g_i)] \leq |S|$. Hence

$$|H/Z(H)| = [H : \cap_{i=1}^r C_H(g_i)] \leq \prod_{i-1}^{r} [H : C_H(g_i)] \leq |S|^r.$$

It follows from Schur's theorem that $[H, H]$ is finite. On the other hand, $[G, G] = \langle S \rangle \leq [H, H]$ which shows that $[G, G]$ is finite. The above argument to show that $|H/Z(H)| \leq |S|^r$ used only the fact that S is the set of commutators of H is finite and that $H/Z(H)$ is generated by r elements. Thus, applying it verbatim to G, one obtains $|G/Z(G)| \leq |S|^r$ where $G/Z(G)$ can be generated by r elements. $\square$

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Reference

[1] P. Niroomand, The converse of Schur's theorem, Arch. Math. **94** (2010), 401–404.

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