Project Report on Classification Theorem and Fundamental Group of a Surface

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September 13, 2016

Abstract

The sphere, torus, Klein bottle, and the projective plane are the classical examples of orientable and non-orientable surfaces. As with much of mathematics, it is natural to ask the question: are these all possible surfaces, or, more generally, can we classify all possible surfaces? In the first chapter, we examine a result originally due to Seifert and Threlfall that all compact surfaces are homeomorphic to the sphere, the connect sum of tori, or the connect sum of projective planes; for this report, we follow a modern proof from Lee [3]. For the 2nd chapter, we will study a much more global property of a space using some abstract algebraic notion.

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Chapter 1

Classification of Compact Surfaces

1.1 Introduction

In our daily life we identify Surface as a continuous set of points that has length and breadth but no thickness. Common examples are sphere, torus, cone and so forth. In mathematics, we generalise the notion of Surfaces in $\mathbb{R}^3$ to $\mathbb{R}^n$ for any $n \in \mathbb{N}, n \geq 2$. The analogue of Surface in $\mathbb{R}^n$ is called "Manifold". We note that locally a Surface looks like a plane. This motivates the definition of a Manifold, which can be defined as an object which is locally Euclidean. We will formally define the notion of a "n-Manifold", but in this chapter we will restrict ourselves to the Surface, specially Compact Surfaces. A Compact Surface is nothing but a surface which is also compact in $\mathbb{R}^3$. Common examples of compact surfaces are spheres, torus, projective plane. There are many more compact surfaces, but later in this chapter we will show that they are topologically equivalent to the above three or some combination of them.

1.2 Surfaces and Orientability

We begin this section by giving a formal definition of a n-Manifold.

Definition 1.1. *Let $n \in \mathbb{N}$, A* ***n dimensional Manifold(in short n-Manifold)*** *is a Hausdorff space such that each point has an open neighbourhood homeomorphic to the open n dimensional disk $U^n (= x \in \mathbb{R}^n : |x| < 1)$.*

We now look at some common examples of *n-Manifold*.

Example 1.2. Clearly the Euclidean n-space is trivially a n-Manifold.

Example 1.3. The sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is a n-Manifold. Let $p = (1, 0, ..., 0)$, then $S^n \backslash \{p\}$ is homeomorphic to $\mathbb{R}$. For the point p, the set $(x_1, x_2, ...x_{n+1}) \in S^n : x_1 > 0$ is a neighbourhood homeomorphic to U^n. This is easily seen by just projecting the points of this set on $\mathbb{R}^n$ with x_1 goes to zero.

Example 1.4. If M^n is a n-Manifold, then any open subset of it is also a n-Manifold.

Definition 1.5. *A* ***Surface*** *is a 2-Manifold, by this we mean that it is a Hausdorff space which is locally homeomorphic to the plane $\mathbb{R}^2$.*

The classic examples are the sphere, the torus, the Klein bottle and the projective plane.

The torus $\mathbb{T}^2$ is a subset of $\mathbb{R}^3$ which is obtained by rotating a circle of radius 1 centred at $(2, 0, 0)$ around the z-axis.

Equivalently, we see that the torus is homeomorphic to the quotient space of $I \times I$ (where I denotes the closed unit interval) modulo the equivalence relation given by $(x, 0) \sim (x, 1)$ for all $x \in I$ and $(0, y) \sim (1, y)$ for all $y \in I$.

In general given any even sided polygon, identifying the edges pairwise by Quotient Topology will result in a Surface. But the question is whether the converse is true or not. We will show that the converse is also true and to achieve this goal we will have to first prove that every surface can be covered by triangles *(in some sense)*. But before going into that we will define what is called orientability of a surface.

Definition 1.6. *A Surface is called* ***orientable*** *if given any two coordinate system of a neighbourhood of a given point, the change of coordinate has a positive Jacobian.*

Common examples are the sphere, the torus and the cylinder.

The above definition is equivalent to saying that there exists a well defined Normal map for the whole surface. This immediately gives us an idea of non-orientable surfaces. Surfaces which doesn't have a well defined Normal map are called non-orientable.

Lemma 1.7. *Let S be a connected surface and N_1 and N_2 are two unit normals defined on S. Then either $N_1 = N_2$ or $N_1 = -N_2$.*

Proof. Consider $A = \{p \in S : N_1(p) = N_2(p)\}$ and $B = \{p \in S : N_1(p) = -N_2(p)\}$. Then A and B are closed subsets of S as N_1 and N_2 are continuous. Also $S = A \cup B$. But as S is connected, either $A = S$ and $B = \phi$ or $A = \phi$ and $B = S$. This proves the lemma. □

The Mobius strip is a topological space that is described by the Quotient space of the rectangle $\{(x, y) \in \mathbb{R}^2 : -10 \leq x \leq 10, -1 \leq y \leq 1\}$ by identifying points $(10, y)$ and $(-10, -y)$ for $-1 < y < 1$. This is a non-orientable surface. Intuitively it is clear. Mobius strip is connected and so have only two normal direction. If we choose a specific direction(outward) for the normal at a point and move along the surface, we will end up at the same point but with normal directing in the opposite direction.

1.3 Triangulation

Definition 1.8. *Let $v_0, v_1, ..., v_k$ be in $\mathbb{R}^n$ be such that $\{v_0 - v_1, v_0 - v_2, ..., v_0 - v_k\}$ is linearly independent. The **Simplex** spanned by these points is the set $\sigma = \{x \in \mathbb{R}^n : x = \sum_{i=0}^{k} t_i v_i$ s.t. $0 \leq t_i \leq 1$ and $\sum_{i=0}^{k} t_i = 1\}$ with the subspace topology.*

Each point v_i is a vertex of σ and the dimension of σ is k.

Definition 1.9. *Let $\{v_0, ..., v_k\}$ be vertices of a simplex σ. The simplex spanned by each non-empty subset of $\{v_0, ..., v_k\}$ is a face of σ. The simplex spanned by a proper subset of vertices is a **proper face**. The $(k-1)$-dimensional faces are called **boundary faces**.*

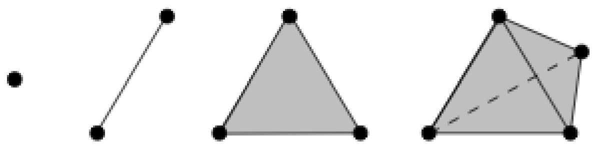


FIGURE 1. From left to right: 0-simplex, 1-simplex, 2-simplex, 3-simplex

Definition 1.10. *A **Euclidean Simplicial complex** K is a collection of simplices in $\mathbb{R}^n$ satisfying the following conditions:*

(1) If $\sigma \in K$, then every face of σ is in K.
(2) The intersection of any two simplices in K is either empty or a face of each.
(3) Every point in a simplex of K has a neighbourhood that intersects finitely many simplices of K.

The dimension of K is the maximum dimension of any simplex in K.
The following is an example of a valid simplicial complex.

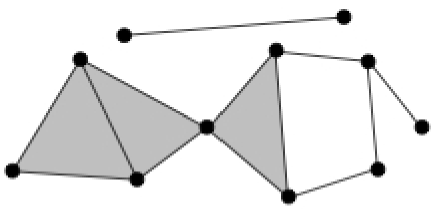


FIGURE 2. A Two dimensional simplicial complex

For 2-dimensional simplicial complexes, like those pictured above, condition 2 means that simplicies intersect at either vertices or edges. The following is an example of condition 2 being broken:

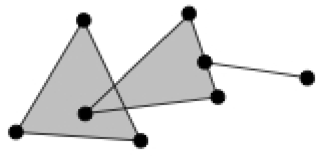


FIGURE 3. Not a simplicial complex

Given a Euclidean simplicial complex, the union of all simplices is a topological space with the subspace topology induced from $\mathbb{R}^n$.
In this section we will not go further in details of simplices, but rather we will now define triangulation of a surface using the concept of simplices. The concept of triangulation will be the major tool in proving the classification theorem for compact surfaces.

Definition 1.11. *A **polyhedron** is a topological space that is homeomorphic to a Euclidean simplicial complex.*

Definition 1.12. *A **triangulation** is a particular homeomorphism between a topological space and a Euclidean simplicial complex.*

Note that there can be multiple triangulations for a given surface. Recall that $I \times I/\sim$ with the equivalence relation given by $(x, 0) \sim (x, 1)$ for all $x \in I$ and $(0, y) \sim (1, y)$ for all $y \in I$ is homeomorphic to a torus. We can make $I \times I$ into a simplicial complex K as pictured below:

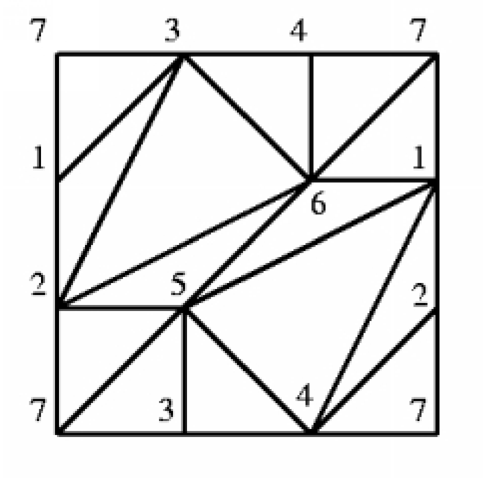


FIGURE 4. Minimal triangulation of a torus

The homeomorphism between this simplicial complex with the equivalence rela- tion $\sim$ from above and the torus is a triangulation of the torus. We are only interested in triangulation of compact surfaces. We know that every surface is locally homeomorphic to $\mathbb{R}^2$ and triangles in $\mathbb{R}^2$ is a 2-simplex. Keeping these two facts in mind we give this definition.

Definition 1.13. *A **triangulation of a compact surface** S consists of finite family of closed subsets $\{T_1, T_2, ..., T_n\}$ that cover S, and a family of homeomorphisms $\{\phi_i : T_i' \to T_i : i = 1, 2, ..., n\}$, where each T_i' is a triangle in $\mathbb{R}^2$. The subsets T_i are called 'triangles'. The subsets of T_i that are the images of the vertices and edges of the triangle T_i' under ϕ_i are also called 'vertices' and 'edges', respectively. Finally, it is required that any two distinct triangles, T_i and T_j, either be disjoint, have a single vertex in common, or have one entire edge common.*

Given any compact surface, it seems possible that it can be covered by some 'triangles'. The next theorem establishes this fact.

Theorem 1.14. ***(Triangulation Theorem for 2-Manifolds)**. Every 2-Manifold is homeomorphic to the polyhedron of a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.*

Proof. The proof of this result is long and intricate, and, thus, we shall not present it here. The basic approach is to cover the manifold with regular coordinate disks and show that each disk can be triangulated compatibly.

The main lemma that is needed is the Schonies Theorem, which states that a topological embedding of the circle into $\mathbb{R}^2$ can be extended to an embedding of the closed disk. A proof of the Schonies Theorem and the triangulation theorem for surfaces can be obtained in Mohar and Thomassen [1]. $\square$

1.4 Polygonal Representation and Connected Sum

Definition 1.15. *A subset P of the plane is a* ***polygonal region*** *if it is a compact subset whose boundary is a one dimensional Euclidean simplicial complex satisfying the following conditions:*

(1) Each point q of an edge that is not a vertex has a neighbourhood $U \subset \mathbb{R}^2$ such that $P \cap U$ is equal to the intersection of U with a closed half-plane $\{(x, y) : ax + by + c \geq 0\}$

(2) Each vertex v has a neighbourhood $V \subset \mathbb{R}^2$ such that $P \cap V$ is equal to the intersection of V with two closed half-planes whose boundaries intersect at v.

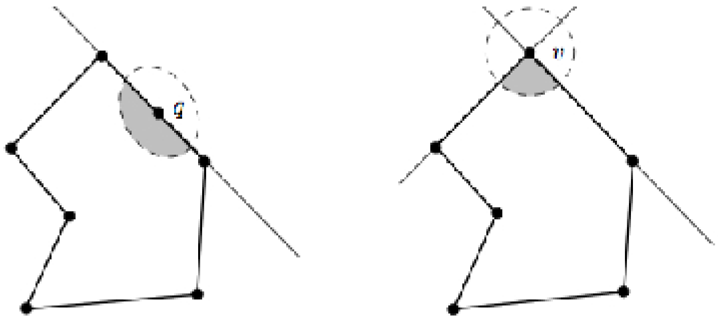


FIGURE 5. Left: Condition 1. Right: Condition 2.

Theorem 1.16. *Let P be a polygonal region in the plane with an even number of edges and suppose we are given an equivalence relation that identifies each edge with exactly one other edge by means of a Euclidean simplicial isomorphism. Then the resultant quotient space is a compact surface.*

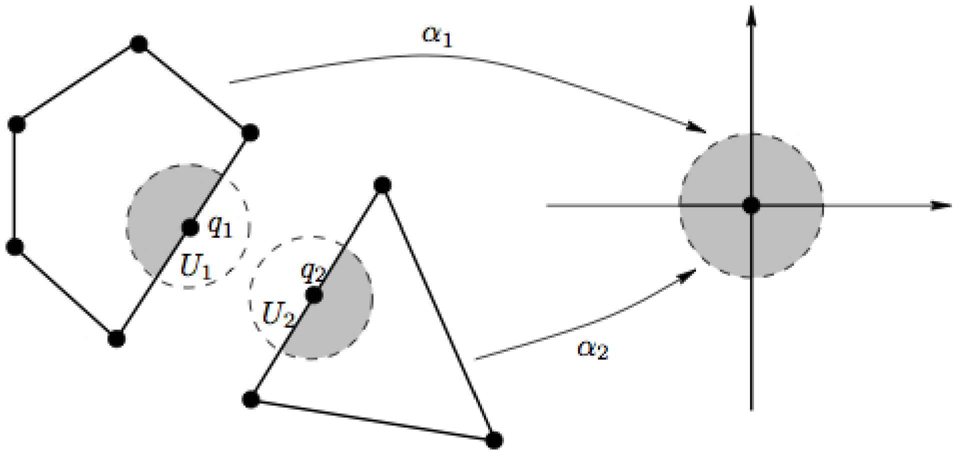
Proof. Let M be the quotient space $P/\sim$ and let $\pi : P \to M$ denote the quotient map. Since P is compact, $\pi(P) = M$ is compact. The equivalence relation identifies only edges with edges and vertices with vertices so the points of M are either:

(1) face points - points whose inverse image in P are in $Int(P)$.
(2) edge points - points whose inverse images are on edges but not vertices.
(3) vertex points - points whose inverse images are vertices.

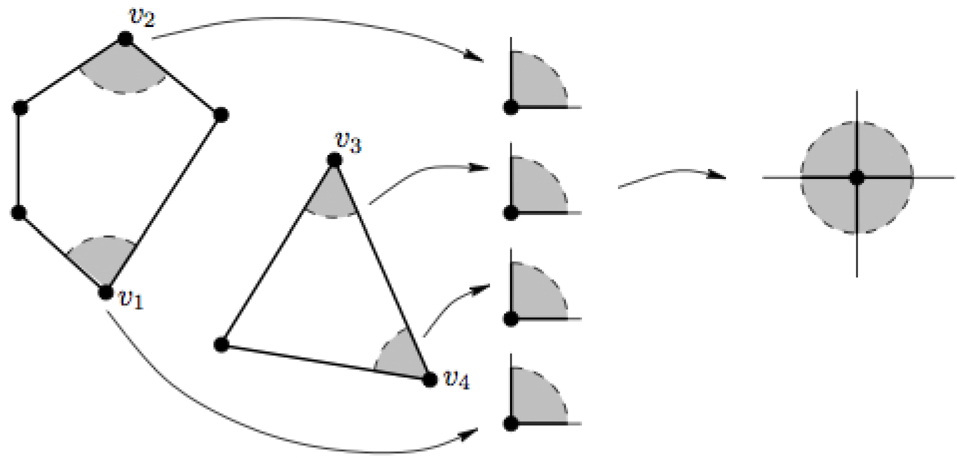
To prove that M is locally Euclidean, it suffices to consider the three types of points.

Face points - Because π is injective on $Int(P)$ and π, being a quotient map is onto, π is bijective on $Int(P)$. So by the closed map lemma, π is a homeomorphism on $Int(P)$. Since $Int(P) \subset \mathbb{R}^2$ is a open set, $\mathbb{R}^2 \cong Int(P) \cong \pi(Int(P))$, so every face point is in a locally Euclidean neighbourhood, namely $\pi(Int(P))$.

Edge points - For any edge point q, pick a sufficiently small neighbourhood N such that there is no vertex points in N. By the definition of a polygonal region, q has two inverse images, q_1 and q_2 with neighbourhoods U_1 and U_2 such that $V_1 = U_1 \cap P$ and $V_2 = U_2 \cap P$ are disjoint half-planes. Furthermore, notice that $\pi|_{V_1 \cap V_2}$ is also a quotient map. We construct affine homeomorphism f_1 and f_2 such that f_1 maps V_1 to a half disk on the upper half plane and f_2 maps V_2 to the lower disk on the lower half plane. We can shrink V_1 and V_2 until they are saturated open sets in P; i.e., for every boundary point of V_1, the corresponding boundary point is in V_2 and vice versa. We can now define another quotient map $\psi : V_1 \cap V_2 \to \mathbb{R}^2$ such that $\psi = f_1$ on V_1 and $\psi = f_2$ on V_2. Modulus the equivalence relation $r_1 \sim r_2$, where r_1 and r_2 are edge points in V_1 and V_2 respectively, whenever $\psi(r_1) = \psi(r_2)$. Notice that ψ is a quotient map onto a Euclidean ball centred at the origin and makes the same identifications as π. By the uniqueness of the quotient map, the quotient spaces are homeomorphic, so edge points are locally Euclidean.



Vertex points - Repeat the same process as the edge points, but this time there will be multiple pieces of the polygon that are identified in $\mathbb{R}^2$. The resultant quotient space is homeomorphic to an open ball, so we may conclude by appealing to the uniqueness of the quotient map. Therefore, we know that M is locally Euclidean.



To show that M is Hausdorff, simply pick sufficient small balls. Since M is the quotient space of the quotient map from the polygonal region P, the pre-image of any pair of points in M can be separated into disjoint open sets by picking sufficiently small open balls; the image of these open balls will be open sets in M that separate the two points in M. □

The converse of this theorem is also true. But we will come to that later. Now we look at some examples.

Example 1.17. The sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ is homeomorphic to the square region $S = \{(x,y) \in \mathbb{R}^2 : |x|+|y| \leq 1\}$ modulo the equivalence relation $(x,y) \sim (-x,y)$ for $(x,y) \in \partial S$.

Example 1.18. The torus $\mathbb{T}^2$ is homeomorphic to the square region $S = \{(x,y) \in \mathbb{R}^2 : |x|+|y| \leq 1\}$ modulo the equivalence relation $(x,y) \sim (-x,-y)$ for $(x,y) \in \partial S$.

Example 1.19. The Klein bottle $\mathbb{K}^2$ is homeomorphic to the square region S (as defined in **Example 1.17**) modulo the equivalence relation $(x,y) \sim (-x,-y)$ for $(x,y) \in \partial S$ such that $0 \leq x, y \leq 1$ or $-1 \leq x, y \leq 0$, and another equivalence relation $(x,y) \sim (-y,-x)$ for $(x,y) \in \partial S$ such that $-1 \leq x \leq 0 \leq y \leq 1$ or $-1 \leq y \leq 0 \leq x \leq 1$.

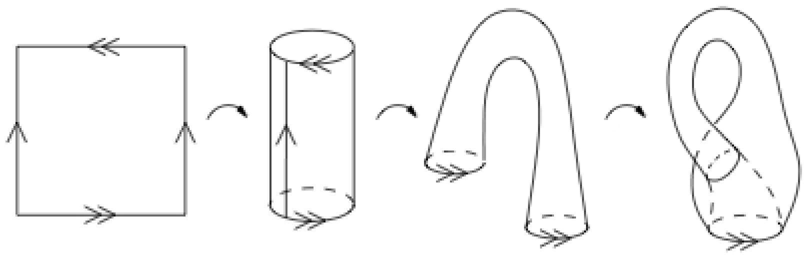


FIGURE 6. Klein Bottle

Example 1.20. The projective plane $\mathbb{P}^2$ is homeomorphic to the square region modulo the equivalence relation $(x,y) \sim (-x,-y)$ for $(x,y) \in \partial S$.

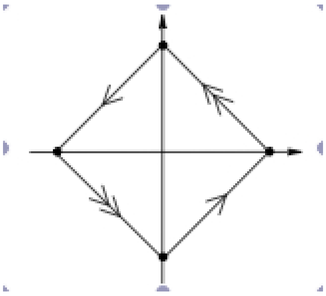


FIGURE 7. Identification that yields a Projective Plane

We want to join two surfaces such that the resulting space is also a surface. For that we define the notion of connected sum of surfaces. As an example the connected sum of two tori is a two hole torus.

Definition 1.21. *Let S_1 and S_2 be disjoint surfaces. Their* ***connected sum****, denoted by $S_1\#S_2$, is formed by cutting a small circular hole in each surface and then gluing the surfaces together along the holes.*

The above definition informally defines the notion of connected sums. To be more precise, we give a rigorous mathematical construction of the connected sum $S_1\#S_2$.
construction: We choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 and D_2 are homeomorphic to the closed disk. Let $S_i' = S_i \backslash int(D_i)$ for $i = 1$ and 2. We choose a homeomorphism h of ∂D_1 onto ∂D_2. Then $S_1\#S_2$ is the quotient space of $S_1' \cup S_2'$ obtained by identifying the points x and $h(x)$ for all $x \in \partial D_1$.
Next we state a very elementary lemma concerning the connected sum of two surfaces.

Lemma 1.22. *The connect sum $M_1\#M_2$ of two connected surfaces M_1 and M_2 is a connected surface.*

Proof. The proof of this lemma is similar to **Theorem 1.16**. The main idea is to identify the surface as a quotient space of a polygon and then study the image of the neighbourhoods of the three types of points described in **Theorem 1.16** under the quotient map. □

Definition 1.23. *Given a set S, a* ***word*** *in S is an ordered k-tuple of symbols of the form a or a^{-1} where $a \in S$. The* ***length*** *of a word is the number of elements in the word, where a and a^{-1} count as distinct elements.*

Definition 1.24. *A* ***polygonal presentation*** *is a finite set S with finitely many words $W_1, ..., W_k$, where W_i is a word in S of length 3 or longer. We denote a polygonal presentation $P = \{S|W_1, ..., W_k\}$.*

Example 1.25. Suppose $S = \{a, b\}$. $W_1 = \{aba^{-1}b^{-1}\}$ and $W_2 = \{aa\}$. Then $P = \{a, b|aba^{-1}b^{-1}, aa\}$.

Definition 1.26. *In the special case where W_i is a word of length 2, we define P_i to be a* ***sphere*** *if the word is aa^{-1} and the* ***projective plane*** *if the word is aa.*

Definition 1.27. *A* ***surface presentation*** *is a polygonal presentation such that each symbol $a \in S$ occurs only exactly twice in $W_1, ..., W_k$, counting each a or a^{-1} as one occurrence.*

Example 1.28. The common surfaces $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{K}$ and $\mathbb{P}^2$ all have presentations:
(1) The sphere: $\{a|aa^{-1}\}$ or $\{a, b|abb^{-1}a^{-1}\}$
(2) The torus: $\{a, b|aba^{-1}b^{-1}\}$
(3) The projective plane: $\{a|aa\}$ or $\{a, b|abab\}$
(4) The Klein Bottle: $\{a, b|abab^{-1}\}$

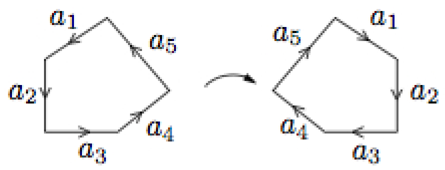
We will now state the converse of **Theorem 1.16**. This theorem will be used to prove the classification theorem.

Theorem 1.29. *Every compact surface admits a polygonal presentation.*

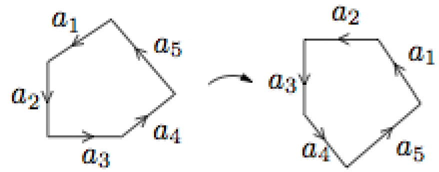
Proof. We shall not prove this theorem here. For a detailed proof of the theorem we refer to [2]. □

From now on we will use the following notations:
(1) e denotes any symbol not in S.
(2) W_1W_2 denotes the word formed by concatenating W_1 and W_2.
(3) $(a^{-1})^{-1} = a$.

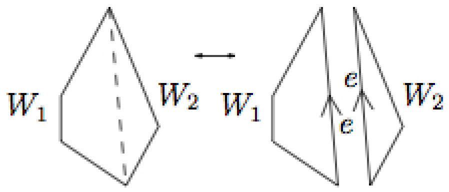
Definition 1.30. *The folowing are* ***Elementary Transformation*** *of a polygonal representation:*
a) ***Reflecting:*** $\{S|a_1, ..., a_n\} \to \{S|a_m^{-1}, ..., a_1^{-1}\}$.



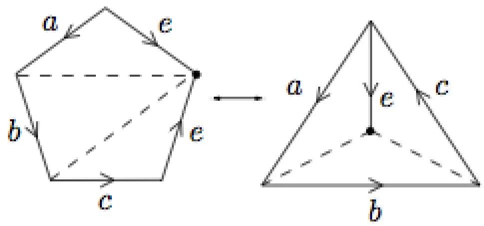
b) ***Rotating:*** $\{S|a_1, ..., a_n\} \to \{S|a_2, ...a_m, a_1\}$.



c) ***Cutting:*** $\{S|W_1w_2\} \to \{S|W_1ee^{-1}W_2\}$.
d) ***Pasting:*** $\{S|W_1e, e^{-1}W_2\} \to \{S|W_1W_2\}$.



W_1 and W_2 must have length atleast 2.
e) ***Folding:*** $\{S|W_1ee^{-1}, W_2\} \to \{S|W_1, W_2\}$. *$W_1$ must have length atleast 3.*
f) ***Unfolding:*** $\{S|W_1, W_2\} \to \{S|W_1ee^{-1}, W_2\}$



There are two important results which we will state next.
1) Elementary transformation of a polygonal representation gives rise to topologically invariant spaces.
2) If S_1 and S_2 are represented by the words W_1 and W_2 then $S_1\#S_2$ is represented by the word W_1W_2.

1.5 The Classification Theorem

We begin this section by proving some lemmas.

Lemma 1.31. *The Klein bottle is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2$.*

Proof. Klein bottle is represented by $\{a, b|abab^{-1}\}$. By a sequence of elementary transformation we get
$\{a, b|abab^{-1}\} \cong \{a, b, c|abc, c^{-1}ab^{-1}\}$ (cut along c)
$\cong \{a, b, c|bca, b^{-1}c^{-1}a\}$ (rotate)
$\cong \{a, b, c|bca, a^{-1}cb\}$ (reflect)
$\cong \{a, b, c|bcbc\}$ (paste along a and rotate)
This the representation of $\mathbb{P}^2\#\mathbb{P}^2$. □

Lemma 1.32. *The connected sum $\mathbb{T}^2\#\mathbb{P}^2$ is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$.*

Proof. By the previous lemma, $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2 \cong \mathbb{K}^2\#\mathbb{P}^2$.
$\mathbb{K}^2\#\mathbb{P}^2 = \{a, b, c|abab^{-1}cc\}$.
$\cong \{a, b, c|cabab^{-1}c\}$. (rotate)
$\cong \{a, b, c, d|cabd^{-1}, dab^{-1}c\}$. (cut along d)
$\cong \{a, b, c, d|abd^{-1}c, c^{-1}ba^{-1}d^{-1}\}$. (rotate 1st word and reflect 2nd word)
$\cong \{a, b, d, e|a^{-1}d^{-1}abe, e^{-1}d^{-1}b\}$. (paste along c and cut along e)
$\cong \{a, b, d, e|ea^{-1}d^{-1}ab, b^{-1}de\}$. (rotate and reflect)

$$\cong \{a, d, e | a^{-1}d^{-1}adee\} = \mathbb{T}^2 \# \mathbb{P}^2.$$

□

We now have all the necessary tools to prove the main theorem of this chapter. This theorem was first proved in 1907 by Max Dehn and Poul Heegaard.

Theorem 1.33. ***Classification Theorem of Compact Surface:*** *Every non-empty, compact, connected 2-manifold is homeomorphic to either a sphere or a connected sum of one or more torus or a connected sum of one or more projective plane.*

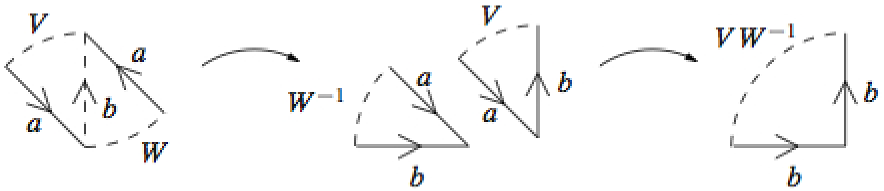
Proof. Given any compact surface M, this proof will show that by a sequence of elementary transformations on its polygonal representation, we get a surface that has a polygonal representation homeomorphic to the sphere, the connected sum of torus, or the connected sum of projective planes.

Step 1. without loss of generality we can assume that M admits a polygonal representation with one word. For since M is connected, each word must have a letter in common with another word, so by repeated pasting, rotation and reflection transformations, we get a polygonal presentation with only one word, which admits a presentation with one face.

Step 2. If there is an adjacent complementary pair(i.e. a pair like $(a...a^{-1})$, we may remove it by folding. The only time, when an adjacent complementary pair cannot be removed is if it has length less than 3 i.e. aa^{-1}. in which case, we have a sphere. Now we assume that the surface is not a sphere.

Step 3. Suppose we have a non-adjacent twisted pair(i.e. a pair like $a...a$). Then the word will take the form $UaVa$, where U and V are non-empty words. By a sequence of elementary transformations we get:

$$\begin{aligned}\{a, U, V | UaVa\} &\cong \{a, b, U, V | Uab, b^{-1}Va\} \text{ (Cutting)}\\ &\cong \{a, b, U, V | bUa, a^{-1}V^{-1}b\} \text{ (rotate 1st word and reflect 2nd word)}\\ &\cong \{b, U, V | bbUV^{-1}\} \text{ (pasting and then rotate)}\end{aligned}$$

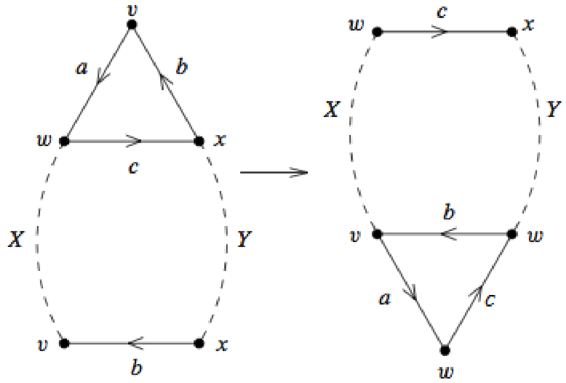


We may have introduced new non-adjacent twisted pairs in the process. However, as that the set of symbols S is finite, by repeating the same process as above, we can transform each non-adjacent twisted pair into adjacent pairs without affecting the bb pair. So after a finite number of iterations, we get a word with no non-adjacent twisted pairs and a string of adjacent complementary pairs. The complementary pairs can be removed by repeating step 2, which does not increase the total number of non-adjacent twisted pairs.

Step 4. We have seen that the edges of the polygon must be identified in pairs. But the vertices may be identified in sets of two, three, four and so on. Let us call two vertices of the polygon to be *equivalent* if and only if they are to be identified. Clearly this is an equivalence relation. Let us choose some equivalence class of vertices $[v]$. Suppose that there are vertices not in the equivalence class $[v]$. Then there must be some edge a that connects $[v]$ to some other vertex class $[w]$. Since this is a polygonal surface, the other edge that touches a at $[v]$ cannot be a^{-1}, or else we would have got rid of it in step 2. The other edge cannot be a, because, if it were, then the initial and terminal ends would be identified under the quotient map, which is not the case. So we label this other edge b and the other vertex x.

Somewhere else in the polygon, there is another edge labelled either b or b^{-1}. Without loss of generality, assume that it is b^{-1}. The proof for b is similar except for an extra reflection. Thus the presentation is of the form $baXb^{-1}Y$. By elementary transformations:

$$\begin{aligned}\{a, b, X, Y | baXb^{-1}Y\} &\cong \{a, b, c, X, Y | bac, c^{-1}Xb^{-1}Y\} \text{ (cutting)}\\ &\cong \{a, b, c, X, Y | acb, b^{-1}Yc^{-1}X\} \text{ (rotate)}\\ &\cong \{a, c, X, Y | acYc^{-1}X\} \text{ (paste along } b)\end{aligned}$$

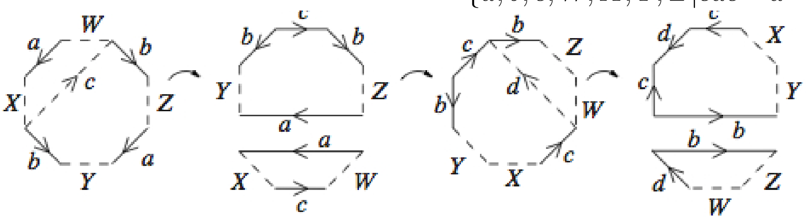


By pasting the edges labelled b, we have reduced the number of distinct vertices in the polygon labelled v. We may have increased the number of vertices labelled w and we may have introduced new complementary pairs. To repair the latter, perform step 2 again. Note that step 2 does not increase the number of vertices labelled v. Thus, by repeating this process finitely many times, we can eliminate the vertex class $[v]$. Repeating this procedure for each vertex class, we can get the desired result.

Step 5.Now we claim that if the presentation has any complementary pairs a, a^{-1}, then it has another complementary pair b, b^{-1} that occurs intertwined with the first. i.e., $a, ..., b, ..., a^{-1}, ..., b^{-1}$. On the contrary let us assume that the presentation is of the form $aXa^{-1}Y$, where X and Y only contain matched complementary pairs or adjacent twisted pair.(complementary pairs remain exclusively within X or Y .) Recall that non-adjacent twisted pairs and adjacent complementary pairs are not possible by step 2 and 3. Thus each edge in X is identified with another edge in Y and similarly for Y . This means the terminal vertices of a and a^{-1} both touch vertices in X and the initial vertices are identified with only vertices in Y. This is a contradiction, since all vertices are within one equivalence class by Step 4.

Step 6. Now the presentation is given $WaXbYa^{-1}Zb^{-1}$. By a sequence of elementary transformations, we get the following:

$$\begin{aligned}
\{a, b, W, X, Y, Z|WaXbYa^{-1}Zb^{-1}\} &\cong \{a, b, c, W, X, Y, Z|WaXc, c^{-1}bYa^{-1}Zb^{-1}\}\\
&\cong \{a, b, c, W, X, Y, Z|XcWa, a^{-1}Zb^{-1}c^{-1}bY\}\\
&\cong \{a, b, c, W, X, Y, Z|XcWZb^{-1}c^{-1}bY\}\\
&\cong \{a, b, c, W, X, Y, Z|c^{-1}bYXcWZb^{-1}\}\\
&\cong \{a, b, c, W, X, Y, Z|c^{-1}bYXcd, d^{-1}WZb^{-1}\}\\
&\cong \{a, b, c, W, X, Y, Z|YXcdc^{-1}b, b^{-1}d^{-1}WZ\}\\
&\cong \{a, b, c, W, X, Y, Z|YXcdc^{-1}d^{-1}WZ\}\\
&\cong \{a, b, c, W, X, Y, Z|cdc^{-1}d^{-1}WZYX\}
\end{aligned}$$



So, by repeating this process M admits a presentation in which all intertwined complementary pairs occur together with no other edges in between.

Step 7. By the previous steps we have seen that all twisted pairs occurs adjacent to each other, i.e. aa, which is a projective plane. Also all complementary pairs occur like $aba^{-1}b^{-1}$, which is a torus. If the presentation consists exclusively of either case, then we are done, since we would either have the connect sum of torus or connect sum of projective planes. If the presentation contains both twisted and complementary pairs, then the presentation must be one of the following forms: $aabcb^{-1}c^{-1}X$ or $bcb^{-1}c^{-1}aaX$. In either case by the previous lemma, $\mathbb{T}^2\#\mathbb{P}^2 \cong \mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. So, if both $\mathbb{P}^2$ and $\mathbb{T}^2$ occurs in the presentation, then we can eliminate $\mathbb{T}^2$ and get a connected sum of $\mathbb{P}^2$ by the above relation.

□

we proved that all compact surfaces are homeomorphic to the sphere, the connect sum of torus, or the connect sum of projective planes, but we have yet to prove that the surfaces are topologically distinct. e.g., a sphere is not homeomorphic to a torus. The answer to this non-trivial question lies with other topological invariants such as the Euler Characteristic and orientibility.

1.6 Euler Characteristic of a Surface

Definition 1.34. *Let M be a compact surface with triangulation $\{T_1, T_2, ..., T_n\}$. Let v be the number of vertices, e be the number of edges and f be the number of faces. Then The* ***Euler Characteristics*** *of the surface M, denoted by $\chi(M)$, is given by $\chi(M) = v - e + f$.*

The Euler Characteristics remains invariant if we choose some different triangulation. The proof is tedious and involved and can be proved by means of homology theory. For now we will use Euler Characteristics to distinguish between compact surface.

Lemma 1.35. *Let S_1 and S_2 be two compact surface. Then,*
$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$

Proof. Let the number of vertices, edges and faces of S_1 and S_2 is v_1, e_1, f_1 and v_2, e_2, f_2 respectively. We form their connected sum by removing the interior of a triangle and identifying the vertices and edges of the removed triangles. Now the number of vertices in the triangulation of $S_1 \# S_2$ is $v_1 + v_2 - 3$, edges is $e_1 + e_2 - 3$ and faces is $f_1 + f_2 - 2$. So by definition, $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$. □

The Euler Characteristics of the three fundamental surfaces can be calculated from their triangulations. The Euler Characteristics of the Sphere, Torus and Projective plane is $2, 0$ and 1 respectively.From this and the above lemma the Euler characteristics of a connected sum of n torus is $2 - 2n$, connected sum of n Projective Planes is $2 - n$. Note that the Euler Characteristics of an orientable surface is always even, but for non-orientable surfaces it can be even or odd.
Assuming the topological invariance of Euler characteristics and using the classification theorem, we have the following result:

Theorem 1.36. *Two compact surfaces are homeomorphic if and only if their Euler characteristics are equal and they are both orientable or both non-orientable.*

Such a classification of topological spaces is very rare. There is no such theorem for compact 3-Manifolds yet.

Chapter 2

Fundamental Group

2.1 Introduction

This chapter introduces one of the simplest and most important functors of algebraic topology, the fundamental group, which creates an algebraic image of a space from the loops in the space, the paths in the space starting and ending at the same point. This chapter begins with the basic definitions and constructions, and then proceeds quickly to an important calculation, the fundamental group of the circle. Then we show some application of this fact and also how continuous maps are interpreted in terms of homomorphisms of the fundamental group.

2.2 Paths and Homotopy

The fundamental group will be defined in terms of loops and deformations of loops. Sometimes it will be useful to consider more generally paths and their deformations.

Definition 2.1. *A **path** in a space X is a continuous map $f : I \to X$ where I is the unit interval* $[0, 1]$.

The idea of continuously deforming a path, keeping its endpoints fixed, is made precise by the following definition.

Definition 2.2. *A **homotopy** of paths in X is a family $f_t : I \to X$, $0 \leq t \leq 1$, such that*
(1) The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t .
(2) The associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.
When two paths f_0 and f_1 are connected in this way by a homotopy f_t , they are said to be homotopic. The notation for this is $f_0 \simeq f_1$.

Example 2.3. Any two paths f_0 and f_1 in $\mathbb{R}^n$ having the same endpoints x_0 and x_1 are homotopic via the homotopy $f_t(s) = (1 - t)f_0(s) + tf_1(s)$. During this homotopy each point $f_0(s)$ travels along the line segment to $f_1(s)$ at constant speed. This is called **Linear Homotopy**. More generally that for a convex subspace $X \subset \mathbb{R}^n$, all paths in X with given endpoints x_0 and x_1 are homotopic, since if f_0 and f_1 lie in X then, by definition of convex set, the homotopy f_t also lies in X.

Theorem 2.4. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

Proof. Reflexivity is evident since $f \simeq f$ by the constant homotopy $f_t = f$. Symmetry is also easy since if $f_0 \simeq f_1$ via f_t, then $f_1 \simeq f_0$ via the inverse homotopy $f_{1-t}(i.e.F_{-1}(s, t) = F(s, 1-t)$. For transitivity, if $f_0 \simeq f_1$ via f_t and if $f_1 = g_0$ with $g_0 \simeq g_1$ via g_t, then $f_0 \simeq g_1$ via the homotopy h_t that equals f_{2t} for $0 \leq t \leq \frac{1}{2}$ and g_{2t-1} for $\frac{1}{2} \leq t \leq 1$. These two definitions agree for $t = 1/2$ since we assume $f_1 = g_0$. Continuity of the associated map $H(s, t) = h_t(s)$ comes from the elementary fact from topology that a function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately. In the case at hand we have $H(s, t) = F(s, 2t)$ for $0 \leq t \leq \frac{1}{2}$ and $H(s, t) = G(s, 2t - 1)$ for $\frac{1}{2} \leq t \leq 1$ where F and G are the maps from the unit square $I \times I$ to X associated to the homotopies f_t and g_t. Since H is continuous on $I \times [0, \frac{1}{2}]$ and on $I \times [\frac{1}{2}, 1]$, it is continuous on $I \times I$. □

Definition 2.5. *The equivalence class of a path f under the equivalence relation of homotopy will be denoted $[f]$ and called the homotopy class of f.*

Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, there is a composition or product path $f \cdot g$ that traverses first f and then g, defined by the formula

$f \cdot g(s) = f(2s)$ if $0 \leq s \leq \frac{1}{2}$ and $f \cdot g(s) = g(2s-1)$ if $\frac{1}{2} \leq s \leq 1$

Thus f and g are traversed twice as fast in order for $f \cdot g$ to be traversed in unit time. This product operation respects homotopy classes since if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies f_t and g_t, and if $f_0(1) = g_0(0)$ so that $f_0 \cdot g_0$ is defined, then $f_t \cdot g_t$ is defined and provides a homotopy $f_0 \cdot g_0 \simeq f_1 \cdot g_1$. In particular, suppose we restrict attention to paths $f : I \to X$ with the same starting and ending point $f(0) = f(1) = x_0 \in X$. Such paths are called loops and the common starting and ending point x_0 is referred to as the base-point. The set of all homotopy classes $[f]$ of loops at the base-point x_0 is denoted $\pi(X, x_0)$.

Theorem 2.6. *$\pi(X, x_0)$ is a group with respect to the product $[f] \cdot [g] = [f \cdot g]$.*

The proof of this fact is rather long and tedious and involves only to show certain homotopies. For a detailed proof we refer to [1] Chapter-2, Section-3.

Definition 2.7. *The group $\pi(X, x_0)$ is called **Fundamental Group** of X at the base-point x_0.*

Example 2.8. For a convex set $X \subset \mathbb{R}^n$ with base-point $x_0 \in X$, we have $\pi(X, x_0) = \{0\}$, the trivial group, since any two loops f_0 and f_1 based at x_0 are homotopic via the linear homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$.

Now we look at the dependence of $\pi(X, x_0)$ on the choice of the base-point x_0. Since $\pi(X, x_0)$ involves only the path-component of X containing x_0, we can expect a relation between $\pi(X, x_0)$ and $\pi(X, x_1)$ for two base-points x_0 and x_1 only if x_0 and x_1 lie in the same path-component of X. So let $h : I \to X$ be a path from x_0 to x_1, with the inverse path $h(s) = h(1-s)$ from x_1 back to x_0. We can then associate to each loop f based at x_1 the loop $h \cdot f \cdot \overline{h}$ based at x_0. we define a general n-fold product $f_1 \cdot f_2 \cdots f_n$ in which the path f_i is traversed in the time interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and a change of base-point map $\psi_h : \pi(X, x_1) \to \pi(X, x_0)$ by $\psi_h([f]) = [h \cdot f \cdot \overline{h}]$. This is well-defined since if f_t is a homotopy of loops based at x_1 then $h \cdot f_t \cdot \overline{h}$ is a homotopy of loops based at x_0.

Lemma 2.9. *The map $\psi_h : \pi(X, x_1) \to \pi(X, x_0)$ is an isomorphism.*

Proof. ψ_h is a homomorphism since $\psi_h([f \cdot g]) = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = \psi_h([f]) \cdot \psi_h([g])$. Further ψ_h is an isomorphism with inverse $\psi_{\overline{h}}$ as $\psi_h \psi_{\overline{h}}([f]) = \psi_h([\overline{h} \cdot f \cdot h]) = [h \cdot \overline{h} \cdot f \cdot \overline{h} \cdot h] = [f]$. Similarly $\psi_{\overline{h}} \psi_h([f]) = [f]$. $\square$

Thus if X is path-connected, the group $\pi(X, x_0)$ is, up to isomorphism, independent of the choice of base-point x_0. In this case the notation $\pi(X, x_0)$ is often abbreviated to $\pi(X)$. In general, a space is called simply-connected if it is path-connected and has trivial fundamental group.

Lemma 2.10. *A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

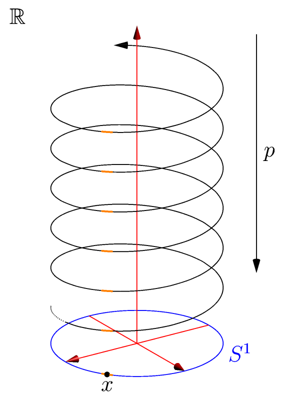
Proof. We need to be concerned only with the uniqueness of connecting paths. The existence of paths, connecting two points, follows directly from path-connectedness. Suppose $\pi(X) = \{0\}$. If f and g are two paths from x_0 to x_1, then $f \simeq f \cdot \overline{g} \cdot g \simeq g$ since the loops $\overline{g} \cdot g$ and $f \cdot \overline{g}$ are each homotopic to constant loops, using the assumption $\pi(X) = \{0\}$ in the latter case. Conversely, if there is only one homotopy class of paths connecting a base-point x_0 to itself, then all loops at x_0 are homotopic to the constant loop and $\pi(X) = \{0\}$ $\square$

2.3 Fundamental Group of a Circle

In this section we will calculate the Fundamental group of a circle. Then we will also see some applications of this theorem. But before that we introduce the notion of a Covering space.

Definition 2.11. *Given a space X, a covering space of X consists of a space $\overline{X}$ and a map $p : \overline{X} \to X$ satisfying the following condition:*
For each point $x \in X$ there is an open neighbourhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Such a U is called evenly covered. For example define $p(s) = (\cos 2\pi s, \sin 2\pi s)$. This map can be visualized geometrically by embedding $\mathbb{R}$ in $\mathbb{R}^3$ as the helix parametrized by $s \to (\cos 2\pi s, \sin \pi s, s)$, and then p is the restriction to the helix of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$, $(x, y, z) \to (x, y)$.



To prove the theorem we will need just the following two facts about covering spaces.

Lemma 2.12. *Let $\overline{X}$ be a covering space of X, $p : \overline{X} \to X$. Then*
(a) For each path $f : I \to X$ starting at a point $x_0 \in X$ and each $\overline{x_0} \in p^{-1}(x_0)$ there is a unique lift $\overline{f} : I \to \overline{X}$ starting at $\overline{x_0}$.
(b) For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\overline{x_0} \in p^{-1}(x_0)$ there is a unique lifted homotopy $\overline{f_t} : I \to \overline{X}$ of paths starting at $\overline{x_0}$.

This lemma is called the **Lifting Lemma**. This can be proved from the following much more general statement: Given a map $F : Y \times I \to X$ and a map $\overline{F} : Y \times \{0\} \to \overline{X}$ lifting $F|Y \times \{0\}$, then there is a unique map $\overline{F} : Y \times I \to \overline{X}$ lifting F and restricting to the given $\overline{F}$ on $Y \times \{0\}$. We will not prove this here, but we will now prove our main theorem in this chapter.

Theorem 2.13. *$\pi(\mathbb{S}^1)$ is an infinite cyclic group generated by the homotopy class of the loop $w(s) = cos2\pi s, sin2\pi s)$ based at $(1, 0)$.*

Proof. Note that $[w]^n = [w_n]$ where $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in Z$. The theorem is therefore equivalent to the statement that every loop in $\mathbb{S}^1$ based at $(1, 0)$ is homotopic to w_n for a unique $n \in Z$. To prove this the idea will be to compare paths in $\mathbb{S}^1$ with paths in $\mathbb{R}$ via the map $p : \mathbb{R} \to \mathbb{S}^1$ given by $p(s) = (cos2\pi s, sin2\pi s)$ defined before. Observe that the loop w_n is the composition $p\overline{w_n}$ where $\overline{w_n} : I \to \mathbb{R}$ is the path $\overline{w_n}(s) = ns$, starting at 0 and ending at n, winding around the helix $|n|$ times, upward if $n > 0$ and downward if $n < 0$. The relation $w_n = p\overline{w_n}$ is expressed by saying that $\overline{w_n}$ is a lift of w_n. We will prove the theorem by studying how paths in $\mathbb{S}^1$ lift to paths in $\mathbb{R}$.

Let $f : I \to \mathbb{S}^1$ be a loop at the base-point $x_0 = (1, 0)$, representing a given element of $\pi(\mathbb{S}^1, x_0)$. By part (a) of lemma 2.12 there is a lift $\overline{f}$ starting at 0. This path $\overline{f}$ ends at some integer n since $p\overline{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$. Another path in $\mathbb{R}$ from 0 to n is $\overline{w_n}$, and $\overline{f} \simeq \overline{w_n}$ via the linear homotopy. Composing this homotopy with p gives a homotopy $f \simeq w_n$ so $[f] = [w_n]$.

To show that n is uniquely determined by $[f]$, suppose that $f \simeq w_n$ and $f \simeq w_m$, so $w_m \simeq w_n$. Let f_t be a homotopy from $w_m = f_0$ to $w_n = f_1$. By part (b) of lemma 2.12 this homotopy lifts to a homotopy $\overline{f_t}$ of paths starting at 0. The uniqueness part of (a) implies that $\overline{f_0} = \overline{w_m}$ and $\overline{f_1} = \overline{w_n}$. Since $\overline{f_t}$ is a homotopy of paths, the endpoint $\overline{f_t}(1)$ is independent of t. For $t = 0$ this endpoint is m and for $t = 1$ it is n, so $m = n$. □

We now look at some applications of this theorem. Our first application is the Brouwer fixed point theorem in dimension 2. But before that we give a couple of definitions.

Definition 2.14. *A subset A of a topological space X is said to be a* ***retract*** *of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. The map r is called a* ***retraction****.*

Definition 2.15. *A subset A of a topological space X is said to be a* ***deformation retract*** *of X if there exists a continuous map $r : X \to A$ and a homotopy f_t such that*

$$\begin{aligned} f_0(x) &= x & x &\in X \\ f_1(x) &= r(x) & x &\in X \\ f_t(a) &= a & a &\in A, t \in I \end{aligned}$$

Theorem 2.16. *Every continuous map $h : D^2 \to D^2$ has a fixed point.*

Proof. Suppose on the contrary that there is no fixed point, i.e. there is no $x \in D^2$ such that $h(x) = x$. Define a map $r : D^2 \to \mathbb{S}^1$ by letting $r(x)$ be the point of $\mathbb{S}^1$ where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x intersects boundary of D^2. Continuity of r is clear since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points. The crucial property of r, besides continuity, is that $r(x) = x$ if $x \in \mathbb{S}^1$. Thus r is a retraction of D^2 onto $\mathbb{S}^1$. We will show that no such retraction can exist. Let f_0 be any loop in $\mathbb{S}^1$. In D^2 there is a homotopy of f_0 to a constant loop, for example the linear homotopy $f_t(s) = (1-t)f_0(s) + tx_0$ where x_0 is the base-point of f_0. Since the retraction r is the identity on $\mathbb{S}^1$, the composition rf_t is then a homotopy in $\mathbb{S}^1$ from $rf_0 = f_0$ to the constant loop at x_0. But this contradicts the fact that $\pi(\mathbb{S}^1)$ is non zero. □

The techniques used to calculate $\pi(\mathbb{S}^1)$ can be applied to prove the Borsuk–Ulam theorem in dimension two:

Theorem 2.17. *For every continuous map $f : \mathbb{S}^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in $\mathbb{S}^2$ with $f(x) = f(-x)$.*

The Borsuk–Ulam theorem holds more generally for maps $\mathbb{S}^n \to \mathbb{R}^n$. The proof for $n = 1$ is easy since the difference $f(x) - f(-x)$ changes sign as x goes halfway around the circle, hence this difference must be zero for some x. For $n \geq 2$ the theorem is certainly less obvious. The theorem says in particular that there is no one-to-one continuous map from $\mathbb{S}^2$ to $\mathbb{R}^2$, so $\mathbb{S}^2$ is not homeomorphic to a subspace of $\mathbb{R}^2$, an intuitively obvious fact that is not easy to prove directly.

Proof. If the conclusion is false for $f : \mathbb{S}^2 \to \mathbb{R}^2$, we can define a map $g : \mathbb{S}^2 \to \mathbb{S}^1$ by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Define a loop η circling the equator of $\mathbb{S}^2 \subset \mathbb{R}^3$ by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$, and let $h : I \to \mathbb{S}^1$ be the composed loop $g\eta$. Since $g(-x) = -g(x)$, we have the relation $h(s + \frac{1}{2}) = -h(s)$ for all $s \in [0, \frac{1}{2}]$. As we showed in the calculation of $\pi(\mathbb{S}^1)$, the loop h can be lifted to a path $h' : I \to \mathbb{R}$. The equation $h(s + \frac{1}{2}) = -h(s)$ implies that $h'(s + \frac{1}{2}) = h'(s) + \frac{q}{2}$ for some odd integer q which is independent of s since by solving the equation $h'(s + \frac{1}{2}) = h'(s) + \frac{q}{2}$ for q we see that q depends continuously on $s \in [0, \frac{1}{2}]$, so q must be a constant since it can only take integer values. In particular, we have $h'(1) = h'(\frac{1}{2}) + \frac{q}{2} = h'(0) + q$. This means that h represents q times a generator of $\pi(\mathbb{S}^1)$. Since q is odd, we conclude that h is not null-homotopic. But h was the composition $g\eta : I \to \mathbb{S}^2 \to \mathbb{S}^1$, and η is obviously null-homotopic in $\mathbb{S}^2$, so $g\eta$ is null-homotopic in $\mathbb{S}^1$ by composing a null-homotopy of η with g. Thus we have arrived at a contradiction. □

An obvious corollary of this theorem is as follows:

Corollary 2.18. *Whenever $\mathbb{S}^2$ is expressed as the union of three closed sets A_1, A_2 and A_3, then atleast one of these sets must contain a pair of antipodal points $\{x, -x\}$.*

Lemma 2.19. *$\pi(X \times Y)$ is isomorphic to $\pi(X) \times \pi(Y)$ if X and Y are path connected.*

Proof. A basic property of the product topology is that a map $f : Z \to X \times Y$ is continuous if and only if the maps $g : Z \to X$ and $h : Z \to Y$ defined by $f(z) = (g(z), h(z))$ are both continuous. Hence a loop f in $X \times Y$ based at (x_0, y_0) is equivalent to a pair of loops g in X and h in Y based at x_0 and y_0 respectively. Similarly, a homotopy f_t of a loop in $X \times Y$ is equivalent to a pair of homotopies g_t and h_t of the corresponding loops in X and Y. Thus we obtain a bijection $\pi(X \times Y, (x_0, y_0) \cong \pi(X, x_0) \times \pi(Y, y_0)$ given by $[f] \mapsto ([g], [h])$. This is obviously a group homomorphism, and hence an isomorphism. □

2.4 Induced Homomorphisms

Suppose $\phi : X \to Y$ is a map taking the base-point $x_0 \in X$ to the base-point $y_0 \in Y$. We write $\phi : (X, x_0) \to (Y, y_0)$ in this situation. Then ϕ induces a homomorphism $\phi_* : \pi(X, x_0) \to \pi(Y, y_0)$, defined by composing loops $f : I \to X$ based at x_0 with ϕ, that is, $\phi_*[f] = [\phi f]$. This induced map ϕ_* is well-defined since a homotopy f_t of loops based at x_0 yields a composed homotopy ϕf_t of loops based at y_0, so $\phi_*[f_0] = [\phi f_0] = [\phi f_1] = \phi_*[f_1]$. Furthermore, ϕ_* is a homomorphism since $\phi(fg) = (\phi f)(\phi g)$, both functions having the value $\phi f(2s)$ for $0 \leq s \leq \frac{1}{2}$ and the value $\phi g(2s-1)$ for $\frac{1}{2} \leq s \leq 1$.

Two basic properties of induced homomorphisms are:
1) $(\phi\psi)_* = \phi_*\psi_*$ for a composition $(X, x_0) \to (Y, y_0) \to (Z, z_0)$.
2) $\mathbf{1}_* = \mathbf{1}$, which is a concise way of saying that the identity map $\mathbf{1} : X \to X$ induces the identity map $\mathbf{1} : \pi(X, x_0) \to \pi(X, x_0)$.
The first of these follows from the fact that composition of maps is associative, so $(\phi\psi)f = \phi(\psi f)$, and the

second is obvious. So, clearly if ϕ is a homeomorphism then ϕ_* is an isomorphism.

Lemma 2.20. *If a space X retracts onto a subspace A, then the homomorphism $i_* : \pi(A, x_0) \to \pi(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.*

Proof. Let $r : X \to A$ be a retraction. Then note that $ri = \mathbb{1}$. Hence $r_* i_* = \mathbb{1}$. This forces i_* to be injective. So, $\pi(A, x_0)$ is a subgroup of $\pi(X, x_0)$. Now take $[f] \in \pi(X, x_0)$ Then $r \circ f$ is a loop in A and $f \cong r \circ f$. Note that a A deformation retracts to X, we have map $H : X \times I \to X$ such that $H(x, 0) = x$; $H(x, 1) = r(x)$ and $H(a, t) = a; \forall a \in A, t \in I, x \in X$. So, define $F : I \times I \to X$ by $F(t, s) = (H(f(t), s)$. Then clearly $F(t, 0) = f(t)$ and $F(t, 1) = r(f(t) = r \circ f(t)$. So, $[f] = [r \circ f] \in \pi(A, x_0)$. Hence they are same upto isomorphism. □

2.5 Free Groups

Before proceeding to the Van Kampen's Theorem, we introduce the notion of a free group briefly. The detail are avoided here. We will only need some simple results regarding the free group.

Definition 2.21. *As a set, the **Free Product** $*_a G_a$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{a_i} and is not the identity element of G_{a_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_a, that is, $a_i \neq a_{i+1}$. Words satisfying these conditions are called reduced, the idea being that unreduced words can always be simplified to reduced words by writing adjacent letters that lie in the same G_{a_i} as a single letter and by cancelling trivial letters. The empty word is allowed, and will be the identity element of $*_a G_a$. The group operation in $*_a G_a$ is juxtaposition, $(g_1 g_2 \cdots g_m) h_1 h_2 \cdots h_n = g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n$. This product may not be reduced, however if g_m and h_1 belong to the same G_a, they should be combined into a single letter $(g_m h_1)$ according to the multiplication in G_a, and if this new letter happens to be the identity of G_a, it should be cancelled from the product. This may allow g_{m-1} and h_2 to be combined, and possibly cancelled too. Repetition of this process eventually produces a reduced word. Inverse of $g_1 g_2 \cdots g_m$ is $g_m^{-1} \cdots g_1^{-1}$.*

A basic property of the free product $*_a G_a$ that we will require is that any collection of homomorphisms $\phi_a : G_a \to H$ extends uniquely to a homomorphism $\Phi : *_a G_a \to H$. Namely, the value of Φ on a word $g_1 \cdots g_n$ with $g_i \in G_{a_i}$ must be $\phi_{a_1}(g_1) \cdots \phi_{a_n}(g_n)$, and using this formula to define Φ gives a well-defined homomorphism since the process of reducing an unreduced product in $*_a G_a$ does not affect its image under Φ. For example, for a free product $G * H$ the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism from $G * H$ to $G \times H$.

2.6 Van Kampen's Theorem

Suppose a space X is decomposed as the union of a collection of path-connected open subsets A_a, each of which contains the base-point $x_0 \in X$. By the remarks in the preceding paragraph, the homomorphisms $j_a : \pi(A_a) \to \pi(X)$ induced by the inclusions $A_a \hookrightarrow X$ extend to a homomorphism $\Phi : *_a \pi(A_a) \to \pi(X)$. The van Kampen's theorem will say that Φ is very often surjective, but we can expect Φ to have a non-trivial kernel in general. For if $i_{ab} : \pi(A_a \cap A_b) \to \pi(A_a)$ is the homomorphism induced by the inclusion $A_a \cap A_b \hookrightarrow A_a$ then $j_a i_{ab} = j_b i_{ba}$, both these compositions being induced by the inclusion $A_a \cap A_b \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{ab}(w) i_{ba}(w)^{-1}$ for $w \in \pi(A_a \cap A_b)$. Van Kampen's theorem asserts that under fairly broad hypotheses this gives a full description of Φ.

Theorem 2.22. *If X is the union of path-connected open sets A_a each containing the base-point $x_0 \in X$ and if each intersection $A_a \cap A_b$ is path-connected, then the homomorphism $\Phi : *_a \pi(A_a) \to \pi(X)$ is surjective. If in addition each intersection $A_a \cap A_b \cap A_c$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{ab}(w) i_{ba}(w)^{-1}$ for $w \in \pi(A_a \cap A_b)$, and hence Φ induces an isomorphism $\pi(X) \cong \dfrac{*_a \pi(A_a)}{N}$.*

We will not prove the theorem here. For a detailed proof we refer to [2] page-44.
We now turn our attention to calculate the Fundamental Group of some spaces using the above theorem and the theory developed earlier in this chapter.

Theorem 2.23. $\pi(\mathbb{S}^n) = 1$ *for* $n \geq 2$.

Proof. Take A_1 and A_2 to be the complements of two antipodal points in $\mathbb{S}^n$. Choose a base-point x_0 in $A_1 \cap A_2$. If $n \geq 2$ then $A_1 \cap A_2$ is path-connected. Then Van Kampen's Theorem applies to say that every loop in $\mathbb{S}^n$ based at x_0 is homotopic to a product of loops in A_1 or A_2. Both $\pi(A_1)$ and $\pi(A_2)$ are zero since A_1 and A_2 are homeomorphic to R^n. Hence every loop in $\mathbb{S}^n$ is null-homotopic and hence the claim follows. $\square$

Definition 2.24. *Given a collection of spaces $\{X_a | a \in J\}$ with base-points $x_a \in X_a$, the **Wedge Sum** of the X_a's is the quotient space of the disjoint union $\amalg_a X_a$ obtained by identifying the points x_a. It is denoted as $\vee_a X_a$.*

Theorem 2.25. *The wedge sum of two circle has the Fundamental group $\mathbb{Z} * \mathbb{Z}$.*

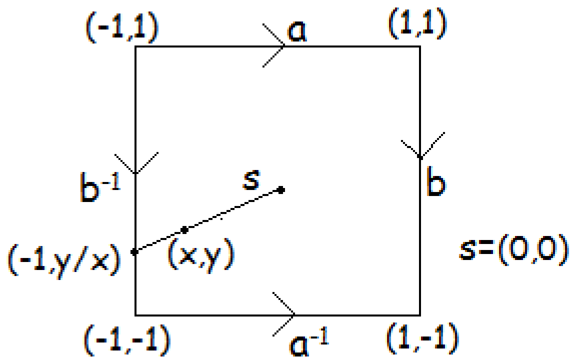
Proof. Let $S = \mathbb{S}^1 \vee \mathbb{S}^1$. Let s be point which is identified. We want to calculate $\pi(S, s)$. Let us take $U = \mathbb{S}^1 \vee \mathbb{S}^1 \setminus \{p\}$ and $V = \mathbb{S}^1 \setminus \{q\} \vee \mathbb{S}^1$, where p, q belongs to different copies of $\mathbb{S}^1$ and not equal to s. Note that, both U and V deformation retracts to $\mathbb{S}^1$ and $U \cap V$ is simply connected. So, by Van Kampen's Theorem $\pi(S, s) = \pi(U, s) * \pi(V, s)$, as the normal subgroup N is trivial as it is generated by identity. Hence $\pi(S) = \mathbb{Z} * \mathbb{Z}$. $\square$

Inductively, it is clear that the fundamental group of a wedge sum of n copies of circles is the free product of n copies of $\mathbb{Z}$.

We know that the fundamental group of a torus is $\mathbb{Z} \times \mathbb{Z}$. We will now calculate the fundamental group of two more surfaces namely the 2-holed torus and the punctured torus.

1) **Fundamental Group of Punctured Torus:**

The punctured torus is a torus with one point removed. Let S be the polygonal representation:



S represents the punctured torus.

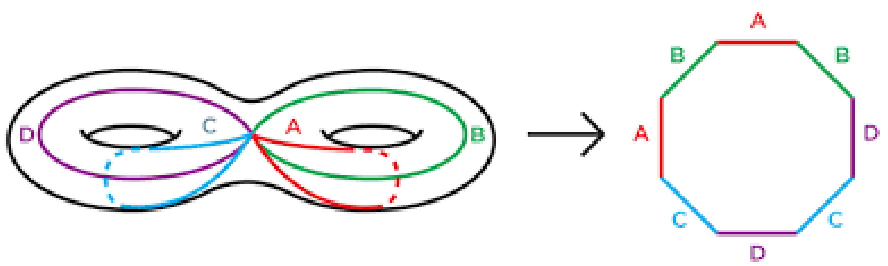
claim: S deform retracts to the wedge of two circles.

Define the retraction $r : S \to bd(S)$ by, sending (x, y) to the intersection of the boundary of S with the line joining (x, y) and $(0, 0)$. We will divide the square in 4 parts by joining the corners with $(0, 0)$. Then depending on which part (x, y) lies, the value of $r(x, y)$ changes. For example in the above picture, $r(x, y) = (-1, \frac{y}{x})$. Then r is continuous and if $(a, b) \in bd(S)$, then $r(a, b) = (a, b)$. So r is a retraction. Now, define $H : S \times I \to S$ by, $H((x, y), t) = (1-t)(x, y) + tr(x, y)$. Then, $H((x, y), 0) = (x, y)$, $H((x, y), 1) = r(x, y)$ and $H((a, b), t) = (1-t)(a, b) + tr(a, b) = (1-t)(a, b) + t(a, b) = (a, b)$ for all $(x, y) \in S$, $(a, b) \in bd(S)$, $t \in I$. Hence H is a deformation retraction, which proves the claim.

So, by Lemma 2.20, we have that $\pi(S, (1, 1)) \cong \pi(bd(S), (1, 1))$. Now by the identification in the boundary we see that $bd(S)$ is nothing but wedge of two circles. Hence by Theorem 2.25, we have that $\pi(S, (1, 1)) \cong \pi(bd(S), (1, 1)) \cong \pi(\mathbb{S}^1 \vee \mathbb{S}^1, (1, 1)) \cong \mathbb{Z} * \mathbb{Z}$.

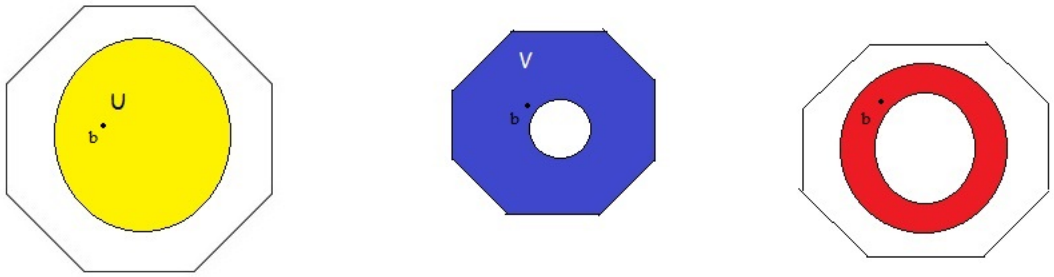
2) **Fundamental Group of 2-holed Torus:**

Let X denote the 2-holes torus. Then X has the following polygonal representation:



First we need to define open sets $U, V \subset \mathbb{T}^2$ that meet the conditions of the Seifert-van Kampen Theorem. Let

U, V be the subsets of $\mathbb{T}^2$ as pictured here:



This choice of U and V gives what we need. The fundamental group of U is the trivial group, since U is contractible. Also, the fundamental group of $U \cap V$ is $\mathbb{Z}$ since it has $\mathbb{S}^1$ as a deformation retract.
Now we need to consider the fundamental group of V. One way to think about this is to consider V as a wire (i.e. we only consider the edges of the octagon). Because V deform retracts onto the boundary. One can define the retraction map r to be the map sending (x, y) to the intersection of the boundary with the line joining (x, y) and the centre of the octagon. It is clear from the last problem that it is indeed a retraction. And we again define the deformation retraction as the linear homotopy between identity map and r. Hence $\pi(V) \cong \pi(\vee_{i=1}^{4}\mathbb{S}^1) \cong *_{i=1}^{4}\mathbb{Z}$, as the boundary of the polygon with the identification is nothing but a wedge sum of 4 circles. We now apply the Van Kampen's theorem. So, $\pi(X) \cong \frac{\pi(U)*\pi(V)}{N}$. Where N is the normal subgroup generated by elements of the form $i_1(w)$ and $i_2(w)$, for $w \in \pi(U \cap V) \cong \mathbb{Z}$. It is enough to look at the image of $1 \in \mathbb{Z}$ under i_1 and i_2 as $Z\mathbb{Z}$ is a cyclic group generated by 1. As, U is trivial, we have that $i_1(1) = Id_U$. Now we need to trace the loop of $U \cap V$ that generates $\pi(U \cap V, b)$ around V to see what we get. From the picture before we see that this loop is homotopic to to the element $ABA^{-1}B^{-1}DCD^{-1}C^{-1}$ We now take these two elements, and form the normal subgroup N. Hence $\pi(X) \cong \{A, B, C, D | ABA^{-1}B^{-1}DCD^{-1}C^{-1} = 1\}$.

We can generalise this to g-holed torus (a genus g surface). If X is a surface of genus g, then $\pi(X) \cong \{A_1, B_1, \cdots A_g, B_g | [A_1, B_1] \cdots [A_g, B_g] = 1\}$. Where $[A, B] = ABA^{-1}B^{-1}$.
The study of fundamental groups is just a beginning of a whole set of groups, called the Homotopy groups. Fundamental group is just the first order homotopy group, which is determined by the structure of loops in the space. In higher dimensional Homotopy groups, we will basically study how the higher dimensional loops behave in a space and then try to study the properties of the space using this.

Acknowledgements:
Simply, I could not have done this work alone without the help of my supervisor. I would specially like to thank Dr. B. Sury for providing the nice topics to study upon. Not only advising about my project work he has also helped me out of some problems on several occasions and given me problems to work on, which really triggers the thought process.

Reference:
1) A Basic Course in Algebraic Topology, by Willam S. Massey.
2) Algebraic Topology, by Alan Hatcher.
3) Introduction to Topological Manifolds, by J. M. Lee